Why is hydrogen so degenerate?

Q: WHY ARE THERE n^2 LEVELS WITH E_n

A1: "HIDDEN SYMMETRY"

A2: "ACCIDENTAL SYMMETRY"

A3: "DYNAMICAL SYMMETRY"

A4: "SUPER SYMMETRY"

1970), 2nd ed., p. 404.

[6]The partition function sum over all configurations of $r(\tau)$ can be cast in the language of an integral over all paths taken by a quantum particle. For the latter approach to quantum mechanics, see, e.g., R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[7]B. D. Sullivan and M. M. Telo da Gama, in *Fluid Interfacial Phenomena*, edited by C. A. Croxton (Wiley, New York, 1986).

Supersymmetry, shape invariance, and exactly solvable potentials

Ranabir Dutt,[a] Avinash Khare,[b] and Uday P. Sukhatme
Department of Physics, University of Illinois at Chicago, Chicago, Illinois 60680

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It is well known that the harmonic oscillator potential can be solved by using raising and lowering operators. This operator method can be generalized with the help of supersymmetry and the concept of "shape-invariant" potentials. This generalization allows one to calculate the energy eigenvalues and eigenfunctions of essentially all known exactly solvable potentials in a simple and elegant manner.

I. INTRODUCTION

Most textbooks on nonrelativistic quantum mechanics show how the harmonic oscillator potential can be elegantly solved by the raising and lowering operator method.[1] The purpose of this article is to describe a generalization of the operator method[2] that can be used to handle many more potentials of physical interest. The generalization is based on two main concepts: supersymmetry and shape invariant potentials. For quantum-mechanical purposes, the main implication of supersymmetry is simply stated. Given any potential $V_-(x)$, supersymmetry allows one to construct a partner potential $V_+(x)$ with the same energy eigenvalues (except for the ground state).[3,4] Furthermore, if $V_-(x)$ and $V_+(x)$ have similar shapes, they are said to be "shape invariant." This concept was introduced three years ago by Gendenshtein.[5] He calculated the energy eigenvalue spectrum and pointed out that essentially all known solvable potentials[6] (Coulomb, harmonic oscillator, Morse, Eckart, Pöschl–Teller, etc.) are shape invariant.[7] This work has been extended by us [2] to a calculation of all the bound state wavefunctions from the ground state in a manner analogous to the harmonic oscillator operator method.

The whole development is very elegant, appealing, and yet rather simple, so that any student of quantum mechanics should be able to understand and appreciate it. Indeed, we strongly feel that the material presented here can be profitably included in future quantum mechanics courses and textbooks. Accordingly, we have kept this article at a pedagogical level and made it as self-contained as possible. In Sec. II, we give a quick review of the standard operator method for solving the one-dimensional simple harmonic oscillator potential in nonrelativistic quantum mechanics. Section III contains a summary of the main ideas of supersymmetric quantum mechanics. Section IV is the heart of this article. In it, we precisely define the notion of shape invariant potentials, and then show how one can simply obtain the energy eigenvalues [Eq. (36)] and eigenfunctions [Eqs. (47) and (48)] by a generalized operator method. A useful table of all known shape-invariant potentials and their eigenstates is given. A discussion of related problems and concluding remarks are contained in Sec. V.

II. OPERATOR METHOD FOR THE HARMONIC OSCILLATOR

The one-dimensional harmonic oscillator Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2. \tag{1}$$

In terms of the raising and lowering operators a^+ and a defined by

$$a = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\frac{d}{dx} + \frac{1}{2}\left(\frac{2m\omega}{\hbar}\right)^{1/2}x,$$
$$a^+ = -\left(\frac{\hbar}{2m\omega}\right)^{1/2}\frac{d}{dx} + \frac{1}{2}\left(\frac{2m\omega}{\hbar}\right)^{1/2}x, \tag{2}$$

the Hamiltonian takes the form

$$H = (a^+a + \tfrac{1}{2})\,\hbar\omega. \tag{3}$$

The following commutation relations are easily derived.

$$[a,a^+] = 1,\quad [a,H] = a\hbar\omega,\quad [a^+,H] = -a^+\hbar\omega. \tag{4}$$

The utility of operators a and a^+ comes from their ability to generate new eigenstates from a given one. In particular, if ψ_n is an eigenfunction of H with eigenvalue E_n, then $a\psi_n$ and $a^+\psi_n$ are also eigenfunctions with eigenvalues $E_n - \hbar\omega$ and $E_n + \hbar\omega$, respectively. Since the operator a^+a in H is positive semidefinite, all eigenvalues $E_n \geqslant \frac{1}{2}\hbar\omega$. Therefore, the successive lowering of eigenstates by the operator a must eventually stop at the ground-state wavefunction ψ_0, by requiring

$$a\psi_0(x) = 0. \tag{5}$$

Operating with $\hbar\omega a^+$ yields

$$\hbar\omega a^+a\psi_0(x) = (H - \tfrac{1}{2}\hbar\omega)\psi_0(x) = 0, \tag{6}$$

which corresponds to a ground-state energy $E_0 = \frac{1}{2}\hbar\omega$. Also, using the definition of the lowering operator a [Eq.

Table I. All known shape invariant potentials and their properties are given. Unless otherwise specified, the range of these potentials is $-\infty < x < \infty$, $0 \leqslant r < \infty$.

Name of potential	Superpotential $W(x)$	Potential $V_-(x;a_0)$	$\{a_0\}$	$\{a_1\}$	$R(a_1)$	Eigenvalues $E_n^{(-)}$	Ground-state wavefunction $\psi_0^{(-)}$
Shifted oscillator	$\sqrt{\frac{m}{2}}\omega x - b$	$\frac{1}{2}m\omega^2\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2 - \frac{\hbar\omega}{2}$	ω	ω	$\hbar\omega$	$n\hbar\omega$	$\exp\left[-\frac{m\omega}{2\hbar}\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2\right]$
Three-dimensional oscillator	$\sqrt{\frac{m}{2}}\omega r - \frac{(l+1)\hbar}{\sqrt{2m}\,r}$	$\frac{1}{2}m\omega^2 r^2 + \frac{l(l+1)\hbar^2}{2mr^2} - (l+\frac{3}{2})\hbar\omega$	l	$l+1$	$2\hbar\omega$	$2n\hbar\omega$	$r^{l+1}\exp\left(-\frac{m\omega r^2}{2\hbar}\right)$
Coulomb	$\sqrt{\frac{m}{2}}\frac{e^2}{(l+1)\hbar} - \frac{(l+1)\hbar}{\sqrt{2m}\,r}$	$-\frac{e^2}{r} + \frac{l(l+1)\hbar^2}{2mr^2} - \frac{me^4}{2(l+1)^2\hbar^2}$	l	$l+1$	$\frac{me^4}{2\hbar^2}\left[\frac{1}{(a_0+1)^2} - \frac{1}{(a_1+1)^2}\right]$	$\frac{me^4}{2\hbar^2}\left[\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2}\right]$	$r^{l+1}\exp\left(-\frac{me^2 r}{(l+1)\hbar^2}\right)$
Morse	$A - Be^{-\alpha x}$	$A^2 + B^2 e^{-2\alpha x} - 2B\left(A + \frac{\alpha\hbar}{2\sqrt{2m}}\right)e^{-\alpha x}$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$\exp\left[-\frac{\sqrt{2m}}{\hbar}\left(Ax + \frac{B}{\alpha}e^{-\alpha x}\right)\right]$
	$A\tanh\alpha x + B\,\mathrm{sech}\,\alpha x$	$A^2 + \left(B^2 - A^2 - \frac{A\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha x + B\left(2A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}\,\alpha x\tanh\alpha x$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$(\mathrm{sech}\,\alpha x)^{\sqrt{2m}A/\alpha\hbar} \times \exp\left[-\frac{2\sqrt{2m}B}{\alpha\hbar}\tan^{-1}(e^{\alpha x})\right]$
Rosen–Morse	$A\tanh\alpha x + \frac{B}{A}$	$A^2 + \frac{B^2}{A^2} + 2B\tanh\alpha x - A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha x$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2 + B^2\left[\frac{1}{a_0^2} - \frac{1}{a_1^2}\right]$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 + B^2\left[\frac{1}{A^2} - \frac{1}{\left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}\right]$	$(\mathrm{sech}\,\alpha x)^{\sqrt{2m}A/\alpha\hbar} \times \exp\left(-\frac{\sqrt{2m}Bx}{\hbar A}\right)$
	$A\coth\alpha r - B\,\mathrm{csch}\,\alpha r$ $(A<B)$	$A^2 + \left(B^2 + A^2 + \frac{A\alpha\hbar}{\sqrt{2m}}\right)\mathrm{csch}^2\alpha r - B\left(2A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\coth\alpha r\,\mathrm{csch}\,\alpha r$	A	$A - \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$\frac{(\sinh\alpha r)^{(\sqrt{2m}/\alpha\hbar)(B-A)}}{(1+\cosh\alpha r)^{\sqrt{2m}B/\alpha\hbar}}$
Eckart	$-A\coth\alpha r + \frac{B}{A}$ $(B>A^2)$	$A^2 + \frac{B^2}{A^2} - 2B\coth\alpha r + A\left(A - \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{csch}^2\alpha r$	A	$A + \frac{\alpha\hbar}{\sqrt{2m}}$	$a_0^2 - a_1^2 + B^2\left[\frac{1}{a_0^2} - \frac{1}{a_1^2}\right]$	$A^2 - \left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 + B^2\left[\frac{1}{A^2} - \frac{1}{\left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}\right]$	$(\sinh\alpha r)^{\sqrt{2m}A/\alpha\hbar} \times \exp\left(-\frac{\sqrt{2m}Br}{\hbar A}\right)$
	$-A\cot\alpha x + B\csc\alpha x$ $(0\leqslant\alpha x\leqslant\pi;\ A>B)$	$-A^2 + \left(A^2 + B^2 - \frac{A\alpha\hbar}{\sqrt{2m}}\right)\csc^2\alpha x - B\left(2A - \frac{\alpha\hbar}{\sqrt{2m}}\right)\cot\alpha x\csc\alpha x$	A	$A + \frac{\alpha\hbar}{\sqrt{2m}}$	$a_1^2 - a_0^2$	$\left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 - A^2$	$(\sin\alpha x)^{\sqrt{2m}(A-B)/\alpha\hbar}$ $(1+\cos\alpha x)^{\sqrt{2m}B/\alpha\hbar}$
Pöschl–Teller I	$A\tan\alpha x - B\cot\alpha x$ $(0\leqslant\alpha x\leqslant\pi/2)$	$-(A+B)^2 + A\left(A - \frac{\alpha\hbar}{\sqrt{2m}}\right)\sec^2\alpha x + B\left(B - \frac{\alpha\hbar}{\sqrt{2m}}\right)\csc^2\alpha x$	(A,B)	$\left(A + \frac{\alpha\hbar}{\sqrt{2m}},\ B + \frac{\alpha\hbar}{\sqrt{2m}}\right)$	$\left(A + B + \frac{2\alpha\hbar}{\sqrt{2m}}\right)^2 - (A+B)^2$	$\left(A + B + \frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2 - (A+B)^2$	$(\sin\alpha x)^{\sqrt{2m}B/\alpha\hbar} \times (\cos\alpha x)^{\sqrt{2m}A/\alpha\hbar}$
Pöschl–Teller II	$A\tanh\alpha r - B\coth\alpha r$ $(B<A)$	$(A-B)^2 - A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha r + B\left(B - \frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{csch}^2\alpha r$	(A,B)	$\left(A - \frac{\alpha\hbar}{\sqrt{2m}},\ B + \frac{\alpha\hbar}{\sqrt{2m}}\right)$	$(A-B)^2 - \left(A - B - \frac{2\alpha\hbar}{\sqrt{2m}}\right)^2$	$(A-B)^2 - \left(A - B - \frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2$	$\frac{(\sinh\alpha r)^{\sqrt{2m}B/\alpha\hbar}}{(\cosh\alpha r)^{\sqrt{2m}A/\alpha\hbar}}$

tentials

$$\psi_n^{(+)}(x;a_0) = \psi_n^{(-)}(x;a_1). \tag{49}$$

Repeated application of Eq. (48) for $n = 0,1,2,3,\ldots$, gives all the eigenfunctions. The procedure for successively obtaining higher-energy eigenfunctions stops if any wavefunction is not normalizable. Of course, this corresponds to the case, where a potential can only hold a finite number of bound states.

Note that for $A^+(x;a_0)$ in Eq. (48), one can either use Eq. (16) in terms of the superpotential $W(x)$ or alternatively use Eq. (12) in terms of the ground-state wavefunction $\psi_0 \equiv \psi_0^{(-)}(x;a_0)$. If the latter choice is made, one has yet another useful expression for the eigenfunctions.

$$\psi_{n+1}^{(-)}(x;a_0) \propto \frac{1}{\psi_0}\frac{d}{dx}\left[\psi_0\psi_n^{(-)}(x;a_1)\right]. \tag{50}$$

Again, as an illustration, we compute the low-lying

Supersymmetry

From Wikipedia, the free encyclopedia

In particle physics, **supersymmetry** (often abbreviated **SUSY**) is a symmetry that relates elementary particles of one spin to another particle that differs by half a unit of spin and are known as superpartners (or sparticles). In other words, in a supersymmetric theory, for every type of boson there exists a corresponding type of fermion, and vice-versa.

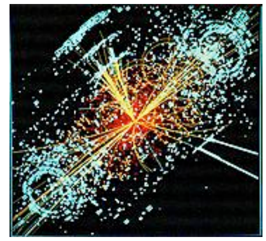
As of 2009, there is indirect evidence that supersymmetry exists. However, since the superpartners of the Standard Model particles have not been observed, supersymmetry, if it exists, must be a broken symmetry allowing the sparticles to be relatively heavy.

If supersymmetry exists close to the TeV energy scale, it allows the solution of two major puzzles in particle physics. One is the hierarchy problem - on theoretical grounds there are arguably huge expected (but not entirely necessary) corrections to the particles' masses, which without fine-tuning will make them appear much larger than they actually are relative to average natures. Another opportunity for possible development is the unification of the weak interactions, the strong interactions and electromagnetism.

Another advantage of supersymmetry is that supersymmetric quantum field theory can sometimes be solved. Supersymmetry is also a feature of most versions of string theory, though it can exist in nature even if string theory is incorrect.

The Minimal Supersymmetric Standard Model is one of the best studied candidates for physics beyond the Standard Model.

Beyond the Standard Model



Standard Model

Evidence

Hierarchy problem • Dark matter
Cosmological constant problem
Strong CP problem
Neutrino oscillation

Theories

Technicolor
Kaluza–Klein theory
Grand Unified Theory
Theory of everything
String theory

Supersymmetry

MSSM • Superstring theory
Supergravity

Quantum gravity

String theory
Loop quantum gravity
Causal dynamical triangulation
Canonical general relativity

Experiments

Gran Sasso • INO • LHC
SNO • Super-K • Tevatron

Unsolved problems in physics: *Is supersymmetry a symmetry of Nature? If so, how is supersymmetry broken, and why? Can the new particles predicted by supersymmetry be detected?*

Contents

History

In the early 1970s, Yu. A. Golfand and E.P. Likhtman in Moscow (in 1971), D.V. Volkov and V.P. Akulov in Kharkiv (in 1972) and J. Wess and B. Zumino in USA (in 1974) independently discovered supersymmetry, a radically new type of symmetry of spacetime and fundamental fields. It has allowed one to establish a relationship between elementary particles of different quantum nature, bosons and fermions, and to non-trivially unify spacetime and internal symmetries of the microscopic World. Supersymmetry first arose in the context of an early version of string theory by Pierre Ramond, John H. Schwarz and Andre Neveu, but the mathematical structure of supersymmetry has subsequently been applied successfully to other areas of physics; firstly by Wess, Zumino, and Abdus Salam and their fellow researchers to particle physics, and later to a variety of fields, ranging from quantum mechanics to statistical physics. It remains a vital part of many proposed theories of physics.

The first realistic supersymmetric version of the Standard Model was proposed in 1981 by Howard Georgi and Savas Dimopoulos and is called the Minimal Supersymmetric Standard Model or MSSM for short. It was proposed to solve the hierarchy problem and predicts superpartners with masses between 100 GeV and 1 TeV. As of 2009 there is no irrefutable experimental evidence that supersymmetry is a symmetry of nature. In 2009 the Large Hadron Collider at CERN is scheduled to produce the world's highest energy collisions and offers the best chance at discovering superparticles for the foreseeable future.

Supersymmetric quantum mechanics

From Wikipedia, the free encyclopedia

In theoretical physics, **supersymmetric quantum mechanics** is an area of research where mathematical concepts from high-energy physics are applied to the seemingly more prosaic field of quantum mechanics.

Contents

Introduction

Understanding the consequences of supersymmetry has proven mathematically daunting, and it has likewise been difficult to develop theories that could account for symmetry breaking, *i.e.*, the lack of observed partner particles of equal mass. To make progress on these problems, physicists developed *supersymmetric quantum mechanics*, an application of the supersymmetry (SUSY) superalgebra to quantum mechanics as opposed to quantum field theory. It was hoped that studying SUSY's consequences in this simpler setting would lead to new understanding; remarkably, the effort created new areas of research in quantum mechanics itself.

For example, as of 2004 students are typically taught to "solve" the hydrogen atom by a laborious process which begins by inserting the Coulomb potential into the Schrödinger equation. After a considerable amount of work using many differential equations, the analysis produces a recursion relation for the Laguerre polynomials. The final outcome is the spectrum of hydrogen-atom energy states (labeled by quantum numbers *n* and *l*). Using ideas drawn from SUSY, the final result can be derived with significantly greater ease, in much the same way that operator methods are used to solve the harmonic oscillator[1]. Oddly enough, this approach is analogous to the way Erwin Schrödinger first solved the hydrogen atom[2]. Of course, he did not *call* his solution supersymmetric, as SUSY was thirty years in the future—but it is still remarkable that the SUSY approach, both older and more elegant, is taught in so few universities.

The SUSY solution of the hydrogen atom is only one example of the very general class of solutions which SUSY provides to *shape-invariant potentials*, a category which includes most potentials taught in introductory quantum mechanics courses.

SUSY quantum mechanics involves pairs of Hamiltonians which share a particular mathematical relationship, which are called *partner Hamiltonians*. (The potential energy terms which occur in the Hamiltonians are then called *partner potentials*.) An introductory theorem shows that for every eigenstate of one Hamiltonian, its partner Hamiltonian has a corresponding eigenstate with the same energy (except possibly for zero energy eigenstates). This fact can be exploited to deduce many properties of the eigenstate spectrum. It is analogous to the original description of SUSY, which referred to bosons and fermions. We can imagine a "bosonic Hamiltonian", whose eigenstates are the various bosons of our theory. The SUSY partner of this Hamiltonian would be "fermionic", and its eigenstates would be the theory's fermions. Each boson would have a fermionic partner of equal energy—but, in the relativistic world, energy and mass are interchangeable, so we can just as easily say that the partner particles have equal mass.

SUSY concepts have provided useful extensions to the WKB approximation. In addition, SUSY has been applied to non-quantum statistical mechanics through the Fokker-Planck equation, showing that even if the original inspiration in high-energy particle physics turns out to be a blind alley, its investigation has brought about many useful benefits.

The SUSY QM superalgebra

In fundamental quantum mechanics, we learn that an algebra of operators is defined by commutation relations among those operators. For example, the canonical operators of position and momentum have the commutator $[x,p]=i$. (Here, we use "natural units" where Planck's constant is set equal to 1.) A more intricate case is the algebra of angular momentum operators; these quantities are closely connected to the rotational symmetries of three-dimensional space. To generalize this concept, we define an *anticommutator,* which relates operators the same way as an ordinary commutator, but with the opposite sign:

$$\{A,B\} = AB + BA.$$

If operators are related by anticommutators as well as commutators, we say they are part of a *Lie superalgebra*. Let's say we have a quantum system described by a Hamiltonian $\mathcal{H}$ and a set of N self-adjoint operators Q_i. We shall call this system *supersymmetric* if the following anticommutation relation is valid for all $i,j = 1,\ldots,N$:

$$\{Q_i, Q_j\} = \mathcal{H}\delta_{ij}.$$

If this is the case, then we call Q_i the system's *supercharges*.

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ed. V.K. Dobrev, Heron Press, Sofia, 2006

Algebraic Solution of the Supersymmetric Hydrogen Atom

Andreas Wipf[1], **Andreas Kirchberg**[2], **Dominique Länge**[3]

[1]FS-Univ. Jena, Max-Wien-Platz 1, Theor. Phys. Institut, D-07743 Jena
[2]Reichenberger Straße 9, 01129 Dresden
[3]LM-Univ. München, Sektion Physik, Theresienstr. 37, D-80333 München

Abstract

The $\mathcal{N} = 2$ supersymmetric extension of the SCHRÖDINGER-HAMILTONian with $1/r$-potential in d dimension is constructed. The system admits a supersymmetrized LAPLACE-RUNGE-LENZ vector which extends the rotational $SO(d)$ symmetry to a hidden $SO(d+1)$ symmetry. It is used to determine the discrete eigenvalues with their degeneracies and the corresponding bound state wave functions.

1 Classical motion in *Newton/Coulomb* potential

For a closed system of two non-relativistic point masses interacting via a central force the angular momentum $\boldsymbol{L}$ of the relative motion is conserved and the motion is always in the plane perpendicular to $\boldsymbol{L}$. If the force is derived from a $1/r$-potential, there is an additional conserved quantity: the LAPLACE-RUNGE-LENZ[1] vector,

$$\boldsymbol{C} = \frac{1}{m}\,\boldsymbol{p}\times\boldsymbol{L} - \frac{e^2}{r}\,\boldsymbol{r}.$$

This vector is perpendicular to $\boldsymbol{L}$ and points in the direction of the semi-major axis. For the hydrogen atom the corresponding Hermitian vector operator has the form

$$\boldsymbol{C} = \frac{1}{2m}\left(\boldsymbol{p}\times\boldsymbol{L} - \boldsymbol{L}\times\boldsymbol{p}\right) - \frac{e^2}{r}\boldsymbol{r} \tag{1}$$

with reduced mass m of the proton-electron system. By exploiting the existence of this conserved vector operator, PAULI calculated the spectrum of the hydrogen

[1] A more suitable name for this constant of motion would be HERMANN-BERNOULLI-LAPLACE vector, see [1].

atom by purely algebraic means [2,3]. He noticed that the angular momentum $\boldsymbol{L}$ together with the vector operator

$$\boldsymbol{K} = \sqrt{\frac{-m}{2H}}\,\boldsymbol{C}, \tag{2}$$

which is well-defined and Hermitian on bound states with negative energies, generate a hidden $SO(4)$ symmetry algebra,

$$\begin{array}{rcl} [L_a, L_b] & = & i\epsilon_{abc}L_c\,, \\ [L_a, K_b] & = & i\epsilon_{abc}K_c\,, \\ [K_a, K_b] & = & i\epsilon_{abc}L_c\,, \end{array} \tag{3}$$

and that the HAMILTON-Operator can be expressed in terms of $\mathcal{C}_{(2)} = \boldsymbol{K}^2 + \boldsymbol{L}^2$, one of the two second-order CASIMIR operators of this algebra, as follows

$$H = -\frac{me^4}{2}\frac{1}{\mathcal{C}_{(2)} + \hbar^2}. \tag{4}$$

One also notices that the second CASIMIR operator $\tilde{\mathcal{C}}_{(2)} = \boldsymbol{L}\cdot\boldsymbol{K}$ vanishes and arrives at the bound state energies by purely group theoretical methods. The existence of the conserved vector $\boldsymbol{K}$ also explains the accidental degeneracy of the hydrogen spectrum.

We generalize the COULOMB-problem to d dimensions by keeping the $1/r$-potential. Distances are measured in units of the reduced COMPTON wavelength, such that the SCHRÖDINGER-operator takes the form

$$H = p^2 - \frac{\eta}{r}\,, \quad p_a = \frac{1}{i}\partial_a\,, \quad a = 1,\dots,d\,. \tag{5}$$

η is twice the fine structure constant. Energies are measured in units of $mc^2/2$.

The Hermitian generators $L_{ab} = x_a p_b - x_b p_a$ of the rotation group satisfy the familiar $so(d)$ commutation relations

$$[L_{ab}, L_{cd}] = i\left(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad}\right)\,. \tag{6}$$

It is not very difficult to guess the generalization of the LAPLACE-RUNGE-LENZ vector (1) in d dimensions [4],

$$C_a = L_{ab}p_b + p_b L_{ab} - \frac{\eta x_a}{r}\,. \tag{7}$$

These operators commute with H in (5) and form a $SO(d)$-vector,

$$[L_{ab}, C_c] = i(\delta_{ac}C_b - \delta_{bc}C_a)\,. \tag{8}$$

The commutator of C_a and C_b is proportional to the angular momentum,

$$[C_a, C_b] = -4iL_{ab}H\,. \tag{9}$$

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Supersymmetry in quantum mechanics

AVINASH KHARE
Institute of Physics, Bhubaneswar 751 005, India

Abstract. In the past ten years, the ideas of supersymmetry have been profitably applied to many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable. In this lecture I review the theoretical formulation of supersymmetric quantum mechanics and discuss many of its applications. I show that the well-known exactly solvable potentials can be understood in terms of a few basic ideas which include supersymmetric partner potentials and shape invariance. The connection between inverse scattering, isospectral potentials and supersymmetric quantum mechanics is discussed and multi-soliton solutions of the KdV equation are constructed. Further, it is pointed out that the connection between the solutions of the Dirac equation and the Schrödinger equation is exactly same as between the solutions of the MKdV and the KdV equations.

Keywords. Supersymmetry; shape invariant potentials; solvable potentials.

PACS No. 03.65

1. Introduction

Physicists have long strived to obtain a unified description of all basic interactions of nature, i.e. strong, electroweak, and gravitational interactions. Several ambitious attempts have been made in the last two decades, and it is now widely felt that supersymmetry (SUSY) is a necessary ingredient in any unifying approach. SUSY relates bosonic and fermionic degrees of freedom and has the virtue of taming ultraviolet divergences. One of the important predictions of SUSY theories is the existence of SUSY partners of quarks, leptons and gauge bosons. Despite the beauty of all these unified theories, there has so far been no experimental evidence of SUSY being realized in nature.

However, over the last 10 years, the ideas of SUSY have stimulated new approaches to other branches of physics [1] like nuclear, atomic, condensed matter, statistical physics as well as in quantum mechanics (QM). I have been fortunate to be involved in some of these developments in the area of supersymmetric quantum mechanics [2, 3]. Recently, Cooper, Sukhatme and myself have written an exhaustive Physics Reports on this topic where we have discussed many of these developments at length [4]. Today I would like to raise some of the issues in which SUSY has given us new insight in QM and discuss few of them in some detail.

1. It is well known that the infinite square well is one of the simplest exactly solvable problem in nonrelativistic QM and the energy eigenvalues are given by $E_n = c(n+1)^2$ where c is constant. Are there other potentials for which the energy eigenvalues have a similar form and is there a simple way of obtaining these potentials?

2. Free particle is obviously the simplest example in QM with no bound states, no reflection and transmission probability being unity. Are there other (nontrivial) potentials for which also there is no reflection and is it possible to easily construct such potentials?
3. Why is Schrödinger equation analytically solvable in the case of few potentials? Another question is if the one dimensional harmonic oscillator the only potential which can be solved by operator method? In this context, recall that the operator method of solving the one dimensional problem is in fact the whole basis of quantum field theory as well as many body theory.
4. It is well known that given a potential $V(x)$, the corresponding energy eigenvalues E_n, and the scattering matrix (the reflection and transmission coefficients $R(k)$ and $T(k)$ in the one dimensional case or phase shifts in the three dimensional case) are unique. Is the converse also true i.e. given E_n, $R(k)$ and $T(k)$ is the corresponding potential unique? If not then how does one construct the various potentials with the same E_n, R and T?
5. A related question is about the construction of the soliton solutions of the KdV and other nonlinear equations. Can these be easily constructed from the formalism of SUSY QM?
6. What is the connection between the Dirac and the Schrödinger equations? In particular, knowing the solution of the Schrödinger problem does there always exist a corresponding exactly solvable Dirac problem and what is the precise connection between the two?
7. Is there a unified treatment for constructing the bound states in the (classical) continuum?
8. Are there semiclassical approximations which do even better than the usual WKB approximation? For example is there an approximation scheme for which the lowest order is exact while all higher order corrections are zero?
9. Finally, can one also analytically solve few noncentral potentials by using operator method alone?

Before I discuss in some detail as to what SUSY QM has to say about these questions, I shall briefly discuss the formalism of SUSY QM and show that because of the underline SUSY, the energy eigenvalues, the eigenfunctions and the S-matrix of the two partner potentials are related in a very definite way.

2. Formalism

One of the key ingredients in solving exactly for the spectrum of one dimensional potential problems is the connection between the bound state wave functions and the potential. It is not usually appreciated that once one knows the ground state wave function (or any other bound state wave function) then one knows exactly the potential (up to a constant). Let us choose the ground state energy for the moment to be zero. Then one has from the Schrödinger equation that the ground state wave function $\psi_0(x)$ obeys [4]

$$H_1\psi_0(x) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_0}{\mathrm{d}x^2} + V_1(x)\psi_0(x) = 0, \tag{1}$$

Chapter 1

N=1 d=4 Supersymmetry

1.1 Why Supersymmetry?

Though originally introduced in early 1970's we still don't know how or if supersymmetry plays a role in nature. Why, then, have a considerable number of people been working on this theory for the last 25 years? The answer lies in the Coleman-Mandula theorem [2], which singles-out supersymmetry as the "unique" extension of Poincaré invariance in quantum field theory in more than two space-time dimensions (under some important but reasonable assumptions). Below I will give a qualitative description of the Coleman-Mandula theorem following a discussion in [3].

The Coleman-Mandula theorem states that *in a theory with non-trivial scattering in more than 1+1 dimensions, the only possible conserved quantities that transform as tensors under the Lorentz group (*i.e. *without spinor indices) are the usual energy-momentum vector P_μ, the generators of Lorentz transformations $J_{\mu\nu}$, as well as possible scalar "internal" symmetry charges Z_i which commute with P_μ and $J_{\mu\nu}$.* (There is an extension of this result for massless particles which allows the generators of conformal transformations.)

The basic idea behind this result is that conservation of P_μ and $J_{\mu\nu}$ leaves only the scattering angle unknown in (say) a 2-body collision. Additional "exotic" conservation laws would determine the scattering angle, leaving only a discrete set of possible angles. Since the scattering amplitude is an analytic function of angle (*assumption # 1*) it then vanishes for all angles.

We illustrate this with a simple example. Consider a theory of 2 free real bose fields ϕ_1 and ϕ_2:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2. \tag{1.1}$$

Such a free field theory has infinitely many conserved currents. For example, it follows http://www.sunclipse.org/?p=466

Last time, we found that the problem of the hydrogen atom could be split into a radial part and an angular part. Thanks to spherical symmetry, the angular part could be studied using angular momentum operators and spherical harmonics. We found that the 3D behavior of the electron could be reinterpreted as a 1D wavefunction of a particle in an effective potential which was the two-body interaction potential plus a “barrier” term which depended upon the angular momentum quantum number. Today, we’re going to solve the radial part of the problem and thereby find the eigenstates and eigenenergies of the hydrogen atom.

The technique we’ll employ has a certain charm, because we solved the first part, the angular dependence, using commutator relations, while as we shall see, the radial dependence can be solved with anticommutator relations.

THE FAMILY OF COULOMB HAMILTONIANS

We ended up with a family of Hamiltonians labeled by the angular momentum quantum number:

This is the way I learned to solve the hydrogenic atom in the misty days of my undergraduacy. The only textbook I know of which takes an approach like this is Ohanian’s Principles of Quantum Mechanics; other than a handful of universities, most schools attack the problem by plowing into Schrödinger’s second-order differential equation and eventually finding a recursion relation for the Laguerre polynomials. Prof. Rajagopal’s lecture notes call the standard method “much more painful,” and as for why most textbooks follow that route, “Go figure.” I suspect that too many teachers of quantum mechanics have been bitten by the Matrix Zombie and think that mathematics beyond differential equations is just too hard for introductory classes. Rather than making the time investment necessary to use “more advanced” techniques, they solve problems in laborious and rather unilluminating ways.

Unfortunately, MIT’s OpenCourseWare project doesn’t provide the lecture notes we used, or any later editions thereof; the site for 8.05 Quantum Physics II just lists the sections of textbooks which should be read, instead of providing actual juicy PDFs. This post, in particular, was based on the 8.05 material, while my earlier overview of the general superalgebra machinery mostly follows Fred Cooper, Avinash Khare and Uday Sukhatme’s review article, “Supersymmetry and Quantum Mechanics” (1994). As that review explains, Schrödinger himself solved for the hydrogen atom eigenstates with a method rather like this, in 1940; many years later, the supersymmetric context of that “factorization” method was discovered.

From here, we can go in several directions. After perhaps working a few examples, we can head towards the relativistic regime and find SUSY-based solutions to the Dirac Equation. Also, we can look back at classical mechanics and relate these ideas to the Laplace-Runge-Lenz vector, an avenue which will eventually lead us to superalgebras with central charge and BPS bounds. I’m also strongly tempted to look at the application of SUSY to diffusion problems via the Fokker-Planck Equation.

Supersymmetry on the WWW

supersymmetric quantum mechanics

supersymmetry primer

supersymmetry breaking

supersymmetric string theory

supersymmetry and morse theory

supersymmetry theory

supersymmetry algebra

supersymmetry for dummies

supersymmetric dark matter

supersymmetry angel
"Supersymmetry" is episode 5 of season 4 in the television show Angel. Co-written by Elizabeth Craft and Sarah Fain and directed by Bill L. Norton, it was originally broadcast on November 3, 2002 on the WB network.

Fred's article on superstring theory is published in an academic journal, and she is asked to present it at a physics symposium by her old college professor Seidel. Her presentation takes a sudden turn when a dimensional portal opens and snake-like creatures emerge to kill her. Angel had spied Lilah during the speech and at first thinks she is behind it, but she was simply keeping an eye on Wesley. Gunn and Angel suspect another member of the audience, a comic book fanatic, but it turns out he's just following stories of strange disappearances, as well as reading about Angel on internet forums. Fred learns that Professor Seidel is the one responsible and the one who had sent Fred into the Pylea dimension six years earlier. He felt Fred as well as other missing colleagues were competing for his job. Against Angel and Gunn's advice, Fred pursues vengeance against her former mentor and asks for Wesley's help. Meanwhile, Cordelia is staying with Connor at his vast empty loft. He trains her to slay vampires while a possible romance between them blossoms. Angel confronts Seidel but is attacked by a demon, and then Fred's revenge goes awry when Gunn unexpectedly kills the man. The two keep it a secret from the rest. Cordelia asks Angel whether or not they were ever in love.

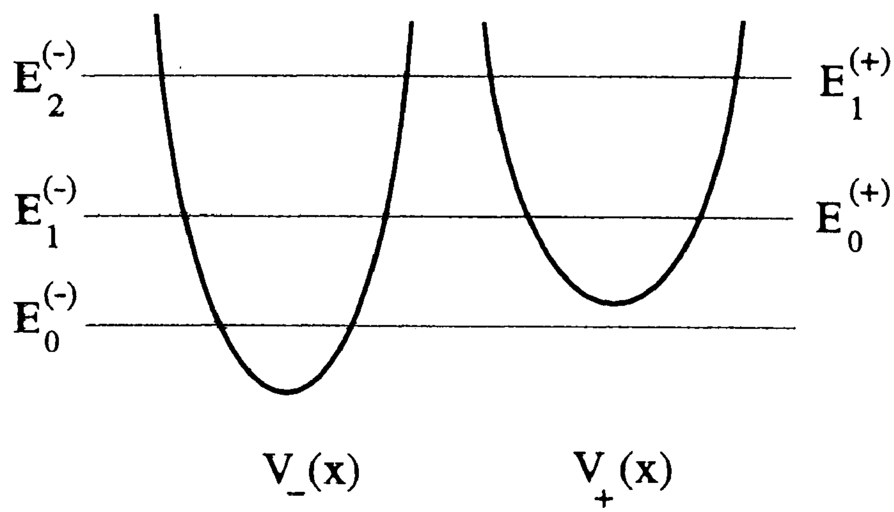


FIG. 1. A typical set of supersymmetric partner potentials with common eigenenergies.

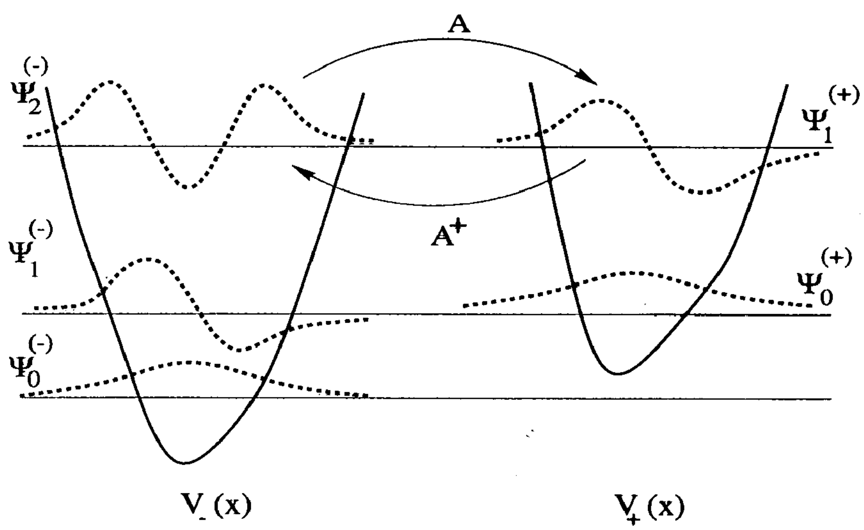


FIG. 2. Isospectrality of H_+ and $H_{\ddagger}$. Note that V_+ and $V_{\ddagger}$ have different shapes, as do various $\tilde{A}^+$ and $\tilde{A}^{\ddagger}$.

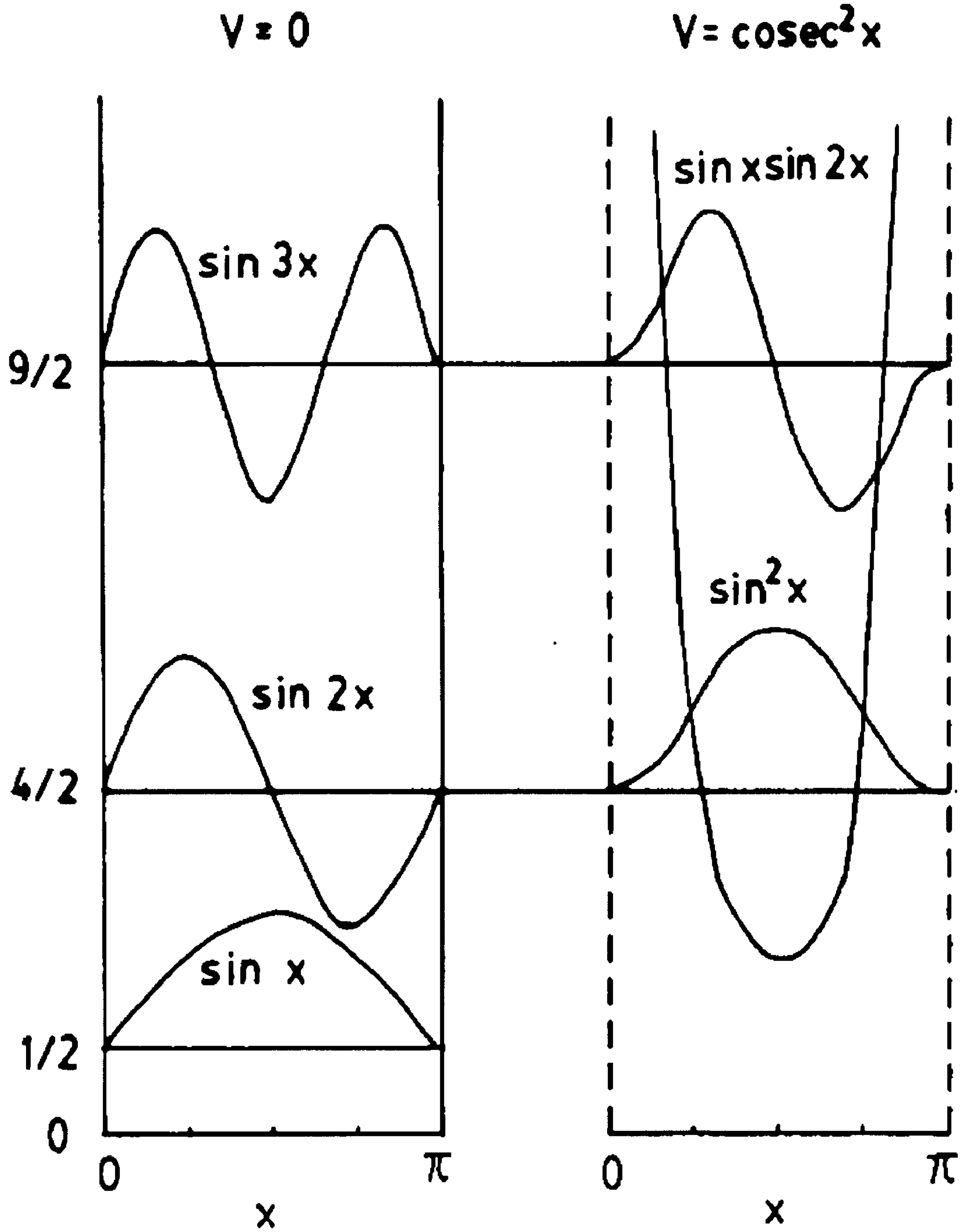


Fig. 2. The low-lying energy eigenstates of the infinite square well of width π and its supersymmetric partner potential $\csc^2 x$. The units used are $\hbar = m = 1$.

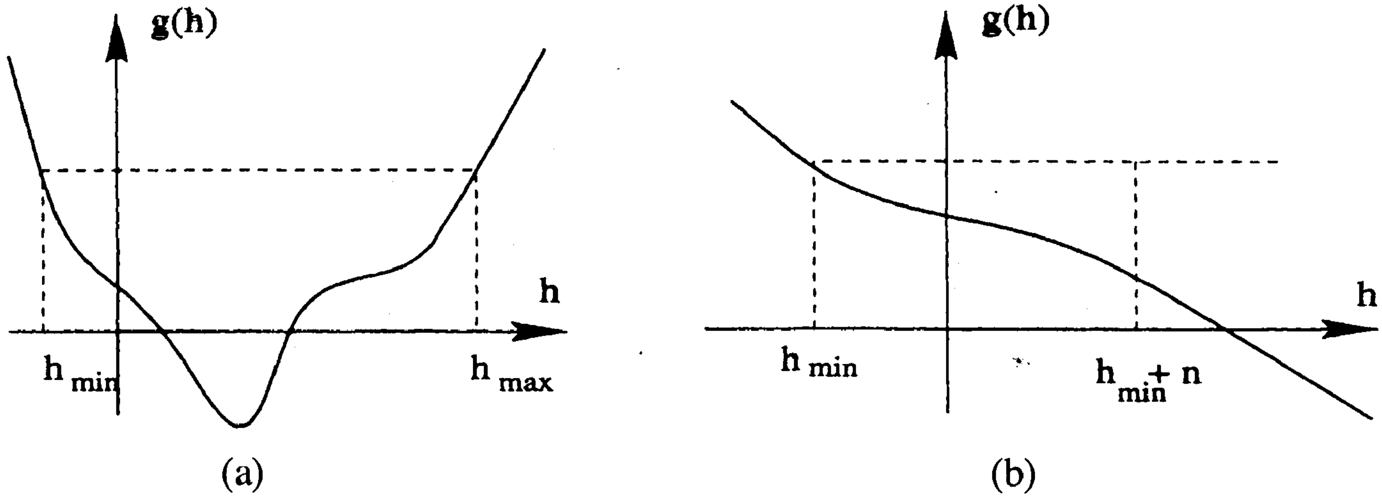


FIG. 4. Generic behaviors of $g(h)$.

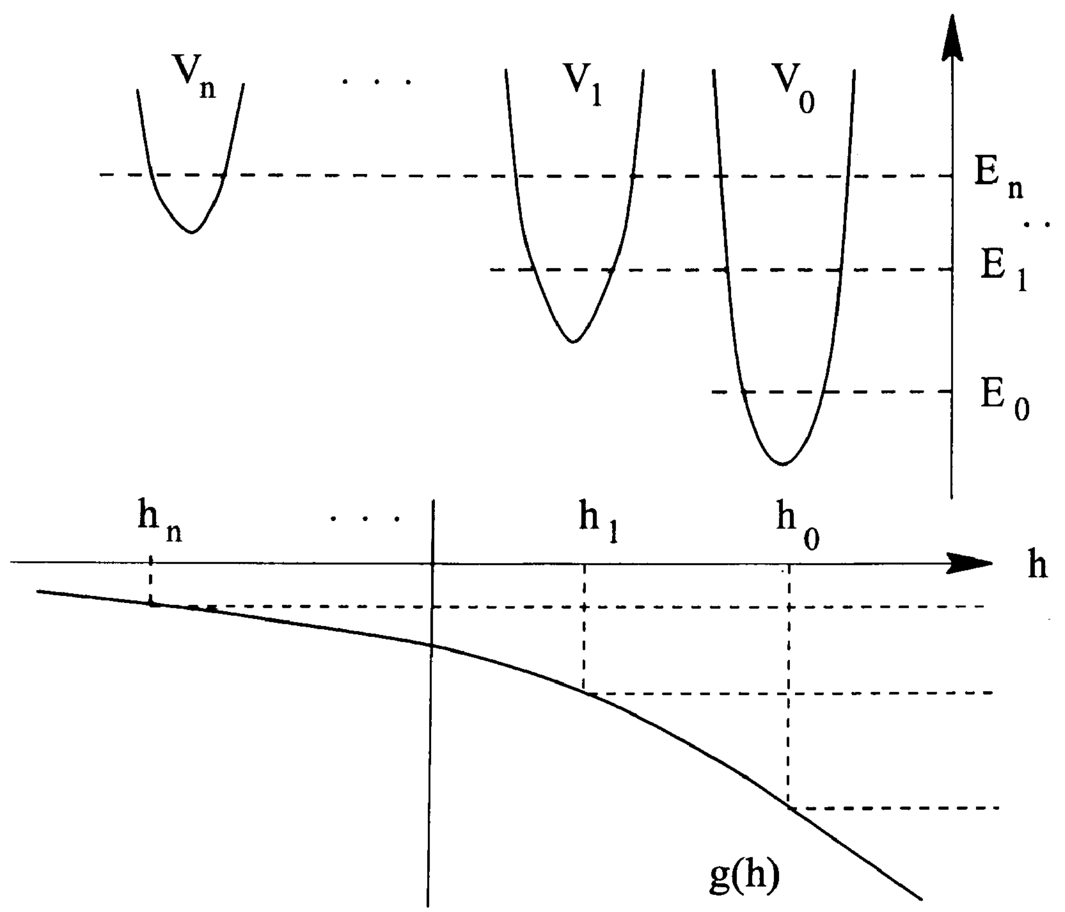


FIG. 5. Potential Algebra: Schematic of generation of SIP's by "hopping" of h.

CM: THE RUNGE-LENZ VECTOR R 1919
L 1924

$$\vec{A} = \vec{p} \times \vec{L} - \hat{r}$$

$\vec{A}$ POINTS TOWARDS APOCENTER or PERICENTER } convention

QM: RUNGE-LENZ-PAULI OPERATOR P 1929

$$\vec{A}_{op} = \frac{1}{2}\left[(\vec{p}_{op} \times \vec{L}_{op}) - (\vec{L}_{op} \times \vec{p}_{op})\right] - \hat{r}_{op}$$

analogous to $L_\pm$

$$L_\pm \,|n\, l\, m\rangle \sim |n\, l\, (m \pm 1)\rangle$$

$$A_+ \,|n\, l\, l\rangle \sim |n\, (l+1)\, (l+1)\rangle$$

$$A_z \,|n\, l\, l\rangle \sim |n\, (l+1)\, l\rangle$$

$$A_- \,|n\, l\, -l\rangle \sim |n\, (l+1)\, -(l+1)\rangle$$

$$A_+ = A_x + i A_y$$

$$A_- = A_x - i A_y$$

SUPERSYMMETRIC QM (SSQM)

$\vec{R} \propto \vec{A}$

COMMUTATION RELATIONS

$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$

$[L_i, K_j] = i\hbar \epsilon_{ijk} K_k$

$[K_i, K_j] = i\hbar \epsilon_{ijk} L_k$

$\Rightarrow$ CALLED SO_4 SYMMETRY

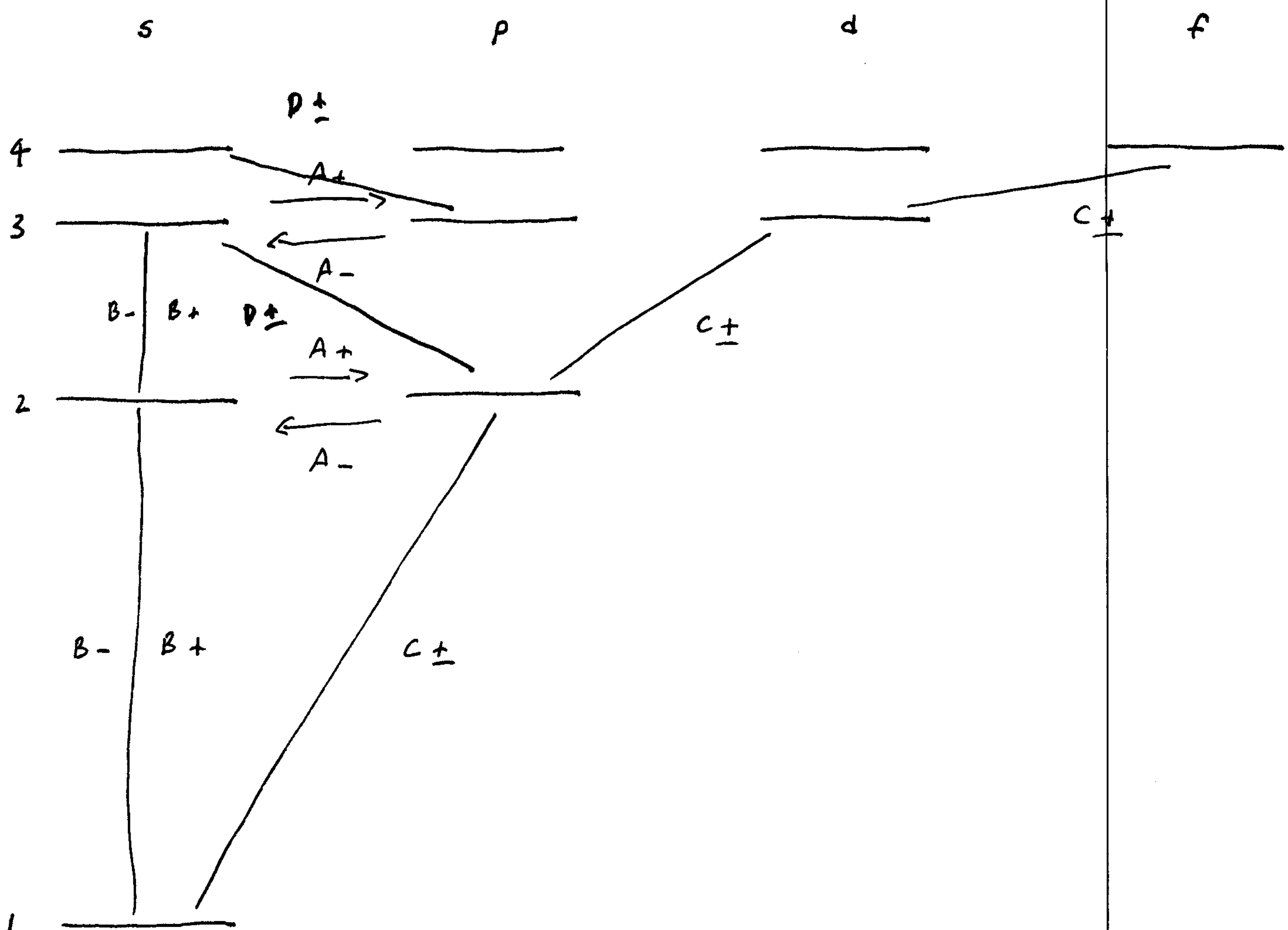
$\Rightarrow$ FOUR LADDER OPERATORS : A, B, C, D

$A_\pm$

$B_\pm$

$C_\pm$

$D_\pm$



PHYSICS LETTERS A 231 9 (1997)

Now, using the fundamental recurrence relations of confluent hypergeometric function [13], one may prove

$$\left((\gamma-\alpha-x)+x\frac{\mathrm{d}}{\mathrm{d}x}\right)F(\alpha,\gamma,x)=(\gamma-\alpha)F(\alpha-1,\gamma,x), \tag{25}$$

$$\left(\alpha+x\frac{\mathrm{d}}{\mathrm{d}x}\right)F(\alpha,\gamma,x)=\alpha F(\alpha+1,\gamma,x), \tag{26}$$

$$\left((\alpha+x)-(\gamma+x)\frac{\mathrm{d}}{\mathrm{d}x}\right)F(\alpha,\gamma,x)=\frac{(\gamma-\alpha)(\gamma+1-\alpha)}{\gamma(\gamma+1)}xF(\alpha,\gamma+2,x), \tag{27}$$

$$\left([(\gamma-1)(\gamma-2)+\alpha x]+x(\gamma-2+x)\frac{\mathrm{d}}{\mathrm{d}x}\right)F(\alpha,\gamma,x)=(\gamma-1)(\gamma-2)F(\alpha,\gamma-2,x), \tag{28}$$

$$\begin{aligned}&\left((\alpha-1)x+(\gamma-1-x)(\gamma-2-x)+(\gamma-2-x)x\frac{\mathrm{d}}{\mathrm{d}x}\right)F(\alpha,\gamma,x)\\&\quad=(\gamma-1)(\gamma-2)F(\alpha-2,\gamma-2,x),\end{aligned} \tag{29}$$

$$\left(-\alpha+(\gamma-x)\frac{\mathrm{d}}{\mathrm{d}x}\right)F(\alpha,\gamma,x)=\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}xF(\alpha+2,\gamma+2,x). \tag{30}$$

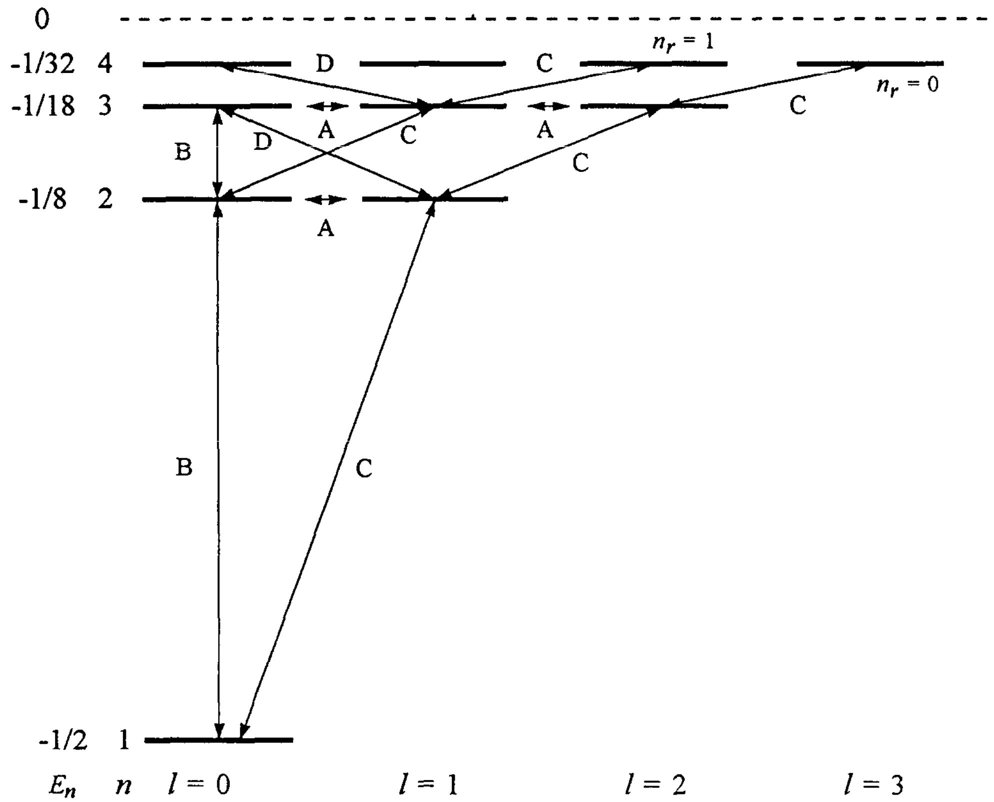


Fig. 1. The energy levels of a hydrogen atom and four kinds of raising and lowering operators. Operator A connects the nearest neighboring eigenstates with the same energy but different angular momenta. B connects the nearest neighboring eigenstates with the same angular momentum but different energy. C connects the nearest neighboring eigenstates with the same radial quantum number n_r and D connects the nearest neighboring eigenstates with the same $n+l$ (or $2l+n_r$).

Table 1
The selection rules and conserved quantum numbers of four kinds of raising and lowering operators of a 3D hydrogen atom

Raising and lowering operators	l	n_r	$n = l + n_r + 1$	Conserved quantum number
$A(l\uparrow, n)$	$l \to l+1$	$n_r \to n_r - 1$	$n \to n$	n
$A(l\downarrow, n)$	$l \to l-1$	$n_r \to n_r + 1$	$n \to n$	
$B(l, n\uparrow)$	$l \to l$	$n_r \to n_r + 1$	$n \to n+1$	l
$B(l, n\downarrow)$	$l \to l$	$n_r \to n_r - 1$	$n \to n-1$	
$C(l\uparrow, n\uparrow)$	$l \to l+1$	$n_r \to n_r$	$n \to n+1$	n_r
$C(l\downarrow, n\downarrow)$	$l \to l-1$	$n_r \to n_r$	$n \to n-1$	
$D(l\downarrow, n\uparrow)$	$l \to l-1$	$n_r \to n_r + 2$	$n \to n+1$	$n+l$
$D(l\uparrow, n\downarrow)$	$l \to l+1$	$n_r \to n_r - 2$	$n \to n-1$	

It is seen that these recurrence formulae concern with the relations of confluent hypergeometric functions with the same variable x. However, the variable of the confluent hypergeometric function in the radial wave function (24) is $\xi_n = 2\tau/n$. To connect the eigenstates with different quantum number n, we may define the operator $M(k)$,

$$M(k)f(x) = f(kx), \tag{31}$$

and using (25)–(30), we may derive other three kinds of raising and lowering operators, in addition to the operators $A_\pm(l)$ given in (14). To clearly indicate their effects, the angular momentum raising and lowering operators $A_\pm(l)$ are relabelled as $A(l\uparrow, n)$, and $A(l\downarrow, n)$. The four kinds of raising and lowering operators of a hydrogen atom are summarized in (32) through (35) and are graphically illustrated in Fig. 1. The corresponding selection rules and conserved quantum numbers are given in Table 1.

$$A(l\uparrow, n) = \frac{d}{dr} - \frac{l+1}{r} + \frac{1}{l+1}, \tag{32}$$

$$A(l\downarrow, n) = \frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \quad (l > 0)$$

$$B(l, n\uparrow) = \left(r\frac{d}{dr} - \frac{r}{n+1} + n\right) M\left(\frac{n}{n+1}\right), \tag{33}$$

$$B(l, n\downarrow) = \left(r\frac{d}{dr} + \frac{r}{n-1} - n\right) M\left(\frac{n}{n-1}\right) \quad (n > 1)$$

$$C(l\uparrow, n\uparrow) = \left([(l+1)(n+1) + r]\frac{d}{dr} - \frac{r}{n+1} - \frac{(l+1)^2(n+1)}{r} + (n-l-1)\right) M\left(\frac{n}{n+1}\right), \tag{34}$$

$$C(l\downarrow, n\downarrow) = \left([l(n-1) + r]\frac{d}{dr} + \frac{r}{n-1} + \frac{l^2(n-1)}{r} - (n-l)\right) M\left(\frac{n}{n-1}\right) \quad (n > 1)$$

$$D(l\downarrow, n\uparrow) = \left([l(n+1) - r]\frac{d}{dr} + \frac{r}{n+1} + \frac{l^2(n+1)}{r} - (n+l)\right) M\left(\frac{n}{n+1}\right), \tag{35}$$

$$D(l\uparrow, n\downarrow) = \left([(l+1)(n-1) - r]\frac{d}{dr} - \frac{r}{n-1} - \frac{(l+1)^2(n-1)}{r} + (n+l+1)\right) M\left(\frac{n}{n-1}\right)$$

$$(n > 1).$$