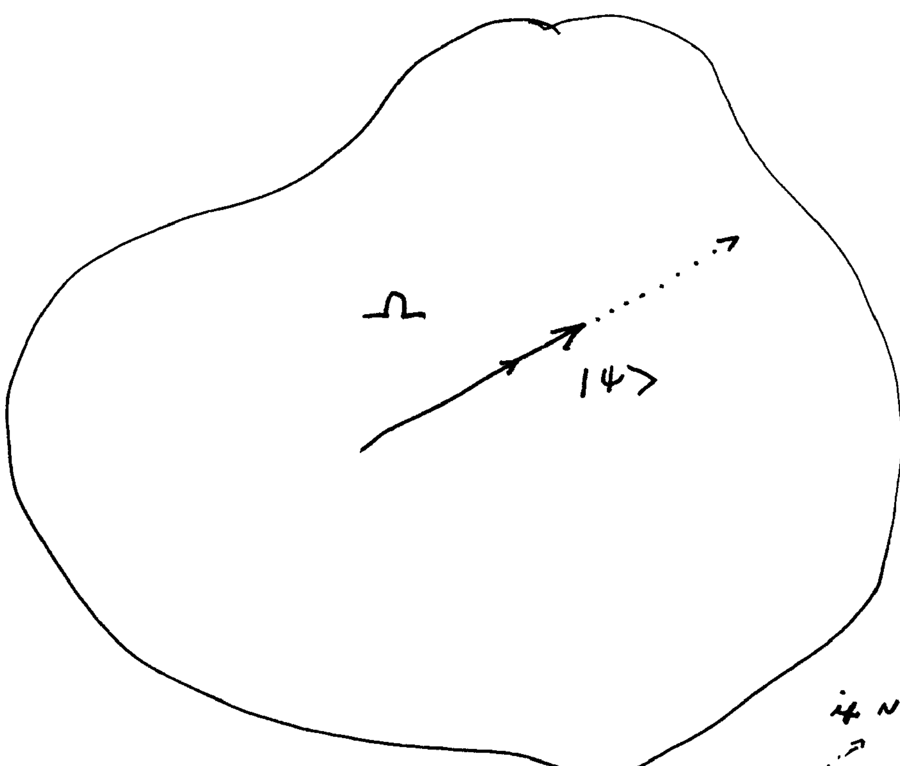
LECTURE 3

TODAY: EIGENVALUES AND EIGENVECTORS

" " EIGENFUNCTIONS

" " EIGENSTATES
BRAS
KETS

Q: WHEN DOES AN OPERATOR JUST STRETCH OR SHRINK A VECTOR WITHOUT CHANGING ITS DIRECTION?



if NOT HERMITEAN
STRETCH $|\alpha|$
PHASE $e^{i\varphi}$

$$\Omega |\psi\rangle = \alpha |\psi\rangle$$

$\alpha = a + bi = |\alpha| e^{i\varphi}$

general op $\Rightarrow$ α is complex

Hermitian operator $\Rightarrow$ α is real

THE IMPORTANCE OF ev AND e$\vec{v}$:

TIME EVOLUTION IS GIVEN BY TDSE:

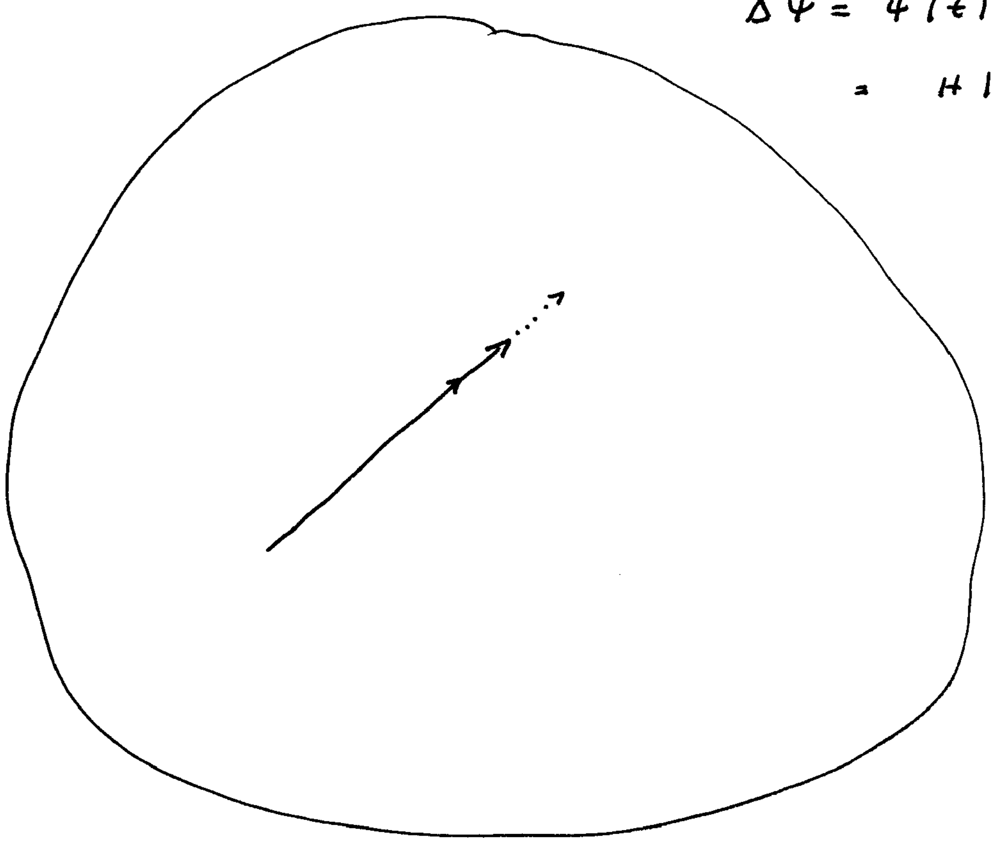
$$H\,|\psi(t)\rangle = i\hbar \frac{d}{dt}\,|\psi(t)\rangle$$

$$\left(\frac{dt}{i\hbar}\right)\left\{H\,|\psi(t)\rangle\right\} = d\,|\psi(t)\rangle$$

$$\frac{H|\psi(0)\rangle}{i\hbar}\int_0^t dt = \int_0^t d(\ )$$

$$\Delta\psi = \psi(t) - \psi(0)$$

$$= H\,|\psi(0)\rangle\,\frac{t}{i\hbar}$$



e$\vec{v}$ of H are called stationary states

for stationary states the only time evolution

is the phase of e$\vec{v}$ or ef

NORMAL MODE DEMO!

2 KINDS OF EVOLUTION: HAMILTONIAN

MEASURE E MEASUREMENT

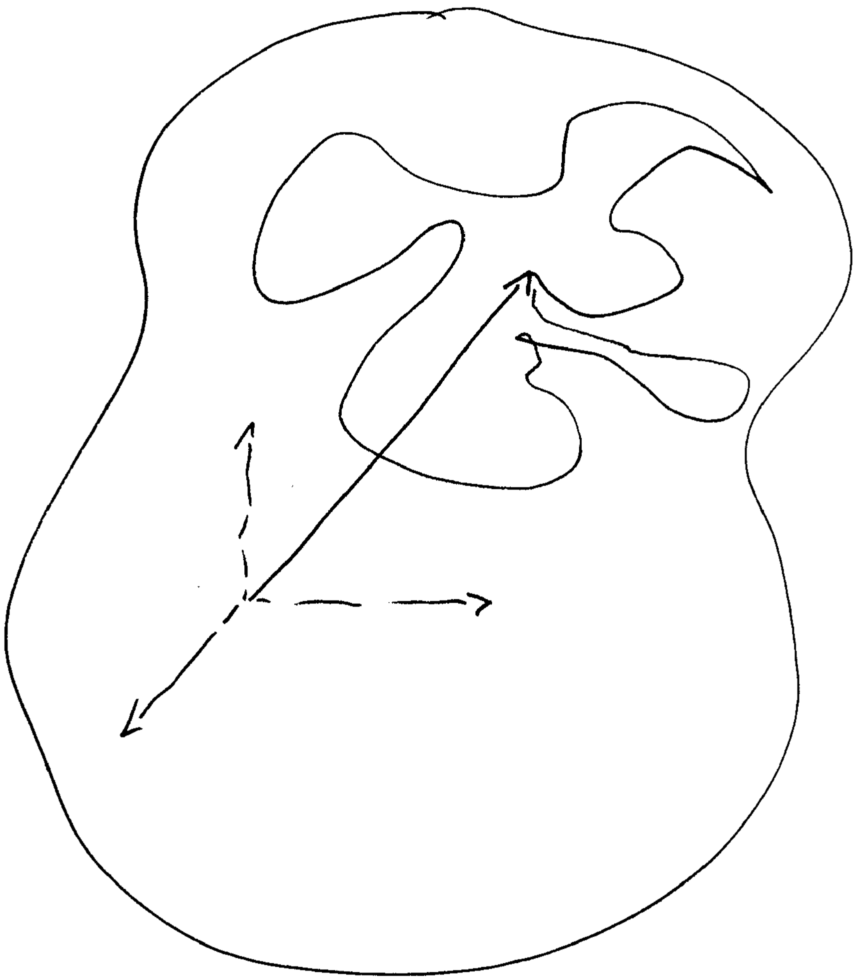
TALK ABOUT THIS

(1) jump is instantaneous

(2) everything controlled by H

=> "decoherence"

ORTHODOX COPENHAGEN INTERP



$\vec{v}$ jumps to $e\vec{v}$ of the thing you measure

measure E	eigenstate of H	energy eigenstate
measure X	eigenstate of X_{op}	position eigenstate
measure P	eigenstate of P_{op}	momentum eigenstate

I have taught graduate courses in quantum mechanics at Columbia, Stanford, Oxford, and Yale, and for almost all of them have dealt with measurement in the following manner. On beginning the lectures I told the students, "You must first learn the rules of calculation in quantum mechanics, and then I will discuss the theory of measurement and discuss the meaning of the subject." Almost invariably, the time allotted to the course ran out before I had to fulfill my promise.

---Willis Lamb

general case

$$\Omega \, |v\rangle = |w\rangle$$

$|w\rangle$ is not parallel to $|v\rangle$

special case

$$\Omega \, |i\rangle = \underbrace{w_i}_{\text{eigenvalue}} \; \underbrace{|i\rangle}_{\text{eigenvector}}$$

When Ω is Hermitian $\Rightarrow$ all w_i's are real

$\Rightarrow$ $\{ |i\rangle \}$ is a basis for the Hilbert space

VERY SPECIAL CASE: IDENTITY OPERATOR

$$I \, |\psi\rangle = (+1) \, |\psi\rangle$$

every vector is an $e\vec{v}$ with ev = +1.

consider

EXAMPLE #1:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

unique ev => unique e$\vec{v}$

=> unique ~~ev~~ eigenbasis

EXAMPLE #2

or

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

degenerate eigenvalues => many eigenbases

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

This is very important for the physics!!!

We will want to find a complete set of commuting operators. CSCO

E, L^2, L_z for the hydrogen atom

$|n, l, m\rangle$

in CM find all of the CONSERVED QUANTITIES

in QM " " " " COMMUTING OBSERVABLES

commute with H

$$H\,\Omega \equiv \Omega\,H$$

(2) PROJECTION OPERATORS

$$\mathbb{P}_i \, | i \rangle = \big(| i \rangle \langle i | \big) | i \rangle = (+1) \, | i \rangle$$

$| i \rangle$ is eigenvector with eigenvalue $+1$

$$\mathbb{P}_i \, | j \rangle = \big(| j \rangle \langle j | \big) | i \rangle = (0) \, | j \rangle$$

$| j \rangle$ are eigenvector with eigenvalue 0

FINDING ev and $e\vec{v}$ OF Ω

(1) $\det(\Omega - \omega I) \overset{=0}{\Rightarrow}$ CHARACTERISTIC EQN (CE)

(2) SOLVE CE TO GET $ev = \{\omega_i\}$

(3) $\Omega \, | \omega_i \rangle = \omega_i \, | \omega_i \rangle$ SOLVE FOR $e\vec{v} = \{ | \omega_i \rangle \}$

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EXAMPLE 3

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & 2-\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = 0$$

$$(1-\omega)\ (2-\omega)\ (1-\omega)\ -\ 0\ +\ (1)\left[0 - (2-\omega)(1)\right] = 0$$

$$(2 - 3\omega + \omega^2)(1-\omega) \qquad -2 + \omega = 0$$

$$\cancel{2} - 3\omega + \omega^2 - \cancel{2}\omega + 3\omega^2 - \omega^3 \ -\cancel{2} + \cancel{\omega} = 0$$

$$-4\omega + 4\omega^2 - \omega^3 = 0 \qquad \text{CUBIC EQN}$$

$$\omega^3 - 4\omega^2 + 4\omega = 0$$

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$$\omega(\omega^2 - 4\omega + 4) = 0 \qquad \text{"CE"}$$

$$\omega(\omega-2)(\omega-2) = 0$$

SO ev's ARE $0, 2, 2$

FIND THE $e\vec{v}$'s

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a+c \\ 2b \\ a+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad b = 0$

$\Rightarrow \quad c = -a \qquad$ CHOOSE $a = +1$, $c = -1$

$$|\,\omega = 0\rangle = A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

NORMALIZATION CONSTANT

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NORMALIZE

$$\langle w=0 \mid w=0 \rangle = 1$$

$$A^* \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^* \quad A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1$$

$$AA^* \; (1+1) = 1$$

$$2\,|A|^2 = 1$$

"PHASE CONVENTION"

conventional to choose real positive value

$$A = \frac{1}{\sqrt{2}}$$

$$|w=0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

TO FIND THE OTHER TWO $e\vec{v}$'s

ASSOCIATED WITH $\omega = 2$ degeneracy $\Rightarrow$ no unique pair of $e\vec{v}$'s

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a+c \\ 2b \\ a+c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$$\left.\begin{array}{l} a+c = 2a \\ a+c = 2c \end{array}\right\} \Rightarrow \quad 2a = 2c$$

$a = c$

choose $a = c = 1$

$$2b = 2b \quad \} \Rightarrow \quad b = \text{anything}$$

INFINITELY MANY $e\vec{v}$'s

$$\begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}$$

~~SO WE CAN~~ LET'S CHOOSE ONE ARBITRARILY $\Rightarrow$ $b = 0$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

THEN CONSTRUCT SECOND $\perp$ TO FIRST

$$\begin{pmatrix} a & b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

any linear combination is an $e\vec{v}$ with $ev = 2$

$a = 0$ [redacted]

$b =$ anything

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

VERY IMPORTANT STEP!!!

CHECK YOUR ANSWER

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \checkmark$$

WHEW!!!