LECTURE 17 7-21-09

FINISH SQUARE WELL

START FREE PARTICLE

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}|E_2\rangle$$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right) \qquad \sin(k_n x)$$

$$\psi(p,t) = \frac{1}{\sqrt{2}}\left(\hat{\psi}_1(p) + \hat{\psi}_2(p)\right)$$

$\langle x(t)\rangle$

$\langle p(t)\rangle$

VISUAL / INTUITIVE ASPECTS TO CONCLUDE,

BUT FIRST INTRODUCE THE PROBABILITY CURRENT

Conclude Angular Momentum

$\vec{L}$ mechanical gyroscope moving mass

$\vec{\mu}$ magnetic moment moving charge

Q: So what is moving?

A: The probability is moving!

PROBABILITY CURRENT (FLUX)

$$|\psi(\vec{r})|^2 = \psi^*(\vec{r})\,\psi(\vec{r}) = \text{PROB DENSITY}$$

1d	L^{-1}	$\rho(x)\,dx$
2d	L^{-2}	$\rho(x,y)\,dA$
3d	L^{-3}	$\rho(\vec{r})\,dV$

$$\rho(\vec{r},t) = \psi^*(\vec{r},t)\,\psi(\vec{r},t)$$

WANT A CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

$$\vec{j} = \left(\frac{-i\hbar}{2m}\right)\left[\psi^*\nabla\psi - \nabla\psi^*\psi\right]$$

$$= \frac{\hbar}{m}\,\mathrm{Im}\left[\psi^*\nabla\psi\right] = \mathrm{Re}\left[\psi^*\,\frac{\hbar}{im}\,\nabla\psi\right]$$

ALL

$\Rightarrow$ IF ψ is real, $\vec{j} = 0$

" " " imaginary, $\vec{j} = 0$

$\vec{j} = 0$ unless there is a spatial gradient in the phase

PROB CONSERVATION $\Leftrightarrow$ $H = H^{+}$

NOW, TURN THIS AROUND

$$\psi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}\ \exp\left(i\,\varphi(\vec{r},t)\right)$$

$\rho > 0$

φ real

WHAT DOES φ MEAN PHYSICALLY?

$$\psi^{*}\nabla\psi = \sqrt{\rho}\,\nabla\left(\sqrt{\rho}\right) + \frac{i}{\hbar}\rho\,\nabla\varphi$$

$$\Rightarrow \quad \vec{j}(\vec{r},t) = \frac{\rho\,\nabla\varphi}{m}$$

SPATIAL VARIATION OF PHASE $\Rightarrow$ PROB CURRENT

PROB FLUX

APPLY TO HYDROGEN

$$\vec{j}_{nlm}(\vec{r}) = \frac{\hbar}{2mi} \Phi^*_{nlm}(\vec{r}) \nabla \Phi_{nlm}(\vec{r}) + cc$$

FOR ANY WAVEFCN $\psi(\vec{r})$

$$\Downarrow \psi(\vec{r}) = A(\vec{r})\, e^{i\phi(\vec{r})}$$

$$A(\vec{r}) \geq 0$$

$$0 \leq \varphi(\vec{r}) \leq 2\pi$$

$$\rho(\vec{r})\, dV = A^2(\vec{r})\, dV$$

PROB DENSITY DEPENDS ONLY ON AMPLITUDE

$$\vec{j} = \frac{\hbar}{m} A^2(\vec{r})\, \vec{\nabla} \varphi(\vec{r})$$

PROB CURRENT DEPENDS ON AMP AND PHASE

Definition

[edit]

In non-relativistic quantum mechanics, the **probability current** $\vec{j}$ of the wave function Ψ is defined as

$$\vec{j} = \frac{\hbar}{2mi}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) = \frac{\hbar}{m}\mathrm{Im}(\Psi^*\vec{\nabla}\Psi) = \mathrm{Re}(\Psi^*\frac{\hbar}{im}\vec{\nabla}\Psi)$$

in the position basis and satisfies the quantum mechanical continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

with the probability density ρ defined as

$$\rho = |\Psi|^2.$$

The divergence theorem implies the continuity equation is equivalent to the integral equation

$$\frac{\partial}{\partial t}\int_V |\Psi|^2 dV + \int_S \vec{j}\cdot d\vec{A} = 0$$

where the V is any volume and S is the boundary of V. This is the conservation law for probability in quantum mechanics.

In particular, if Ψ is a wavefunction describing a single particle, the integral in the first term of the preceding equation (without the time derivative) is the probability of obtaining a value within V when the position of the particle is measured. The second term is then the rate at which probability is flowing out of the volume V. Altogether the equation states that the time derivative of the change of the probability of the particle being measured in V is equal to the rate at which probability flows into V.

Examples [edit]

Plane wave [edit]

The probability current associated with the (three dimensional) plane wave

$$\Psi = Ae^{i\vec{k}\cdot\vec{r}}e^{i\omega t}$$

is

$$\vec{j} = \frac{\hbar}{2mi}|A|^2\left(e^{-i\vec{k}\cdot\vec{r}}\vec{\nabla}e^{i\vec{k}\cdot\vec{r}} - e^{i\vec{k}\cdot\vec{r}}\vec{\nabla}e^{-i\vec{k}\cdot\vec{r}}\right) = |A|^2\frac{\hbar\vec{k}}{m}.$$

This is just the square of the amplitude of the wave times the particle's velocity,

$$\vec{v} = \frac{\vec{p}}{m} = \frac{\hbar\vec{k}}{m}.$$

Note that the probability current is nonzero despite the fact that plane waves are stationary states and hence

$$\frac{d|\Psi|^2}{dt} = 0$$

everywhere. This demonstrates that a particle may be in motion even if its spatial probability density has no explicit time dependence.

Particle in a box

[edit]

The energy eigenstates of a particle in a box of one spatial dimension and of length L are, for $0 < x < L$,

$$\Psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

and zero elsewhere. The associated probability currents are

$$j_n = \frac{\hbar}{2mi}\left(\Psi_n^* \frac{\partial \Psi_n}{\partial x} - \Psi_n \frac{\partial \Psi_n^*}{\partial x}\right) = 0$$

since $\Psi_n = \Psi_n^*$.

Not the Wind, Not the Flag

Two monks were arguing about a flag.
One said: "The flag is moving."
The other said: "The wind is moving."
The sixth patriarch happened to be passing by.
He told them:
"Not the wind, not the flag; mind is moving."

As the seventeen monks were walking toward her, Miaoxin said, it's not the wind moving, it's not the flag moving, it's not the mind moving."

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A. SQUARE WELL

(1) WITH MAG AND PHASE
PHASE CLOCKS — X-SPACE

GROUND STATE $|1\rangle$
FIRST EXCITED STATE $|2\rangle$
50:50 SUPERPOSITION $|1\rangle + |2\rangle$

EXPECTATION VALUES
UNCERTAINTIES

(2) ADD MOMENTUM VIEW
CONTRAST W POSITION VIEW — P-SPACE

(3) ADD PROB CURRENT — ADD $\vec{j}$

VIEW MAG & PHASE

NOT PROB & PHASE

B. SHO

C. COULOMB

D. OTHER SUPERPOSITIONS