SO FAR, SINGLE PARTICLE WAVEFCNS

$$\psi(\vec{r}, t)$$

HYDROGEN ATOM HAS 2 PARTICLES

$$\psi(\underbrace{\vec{r}_1}_{\text{where proton is}}, \underbrace{\vec{r}_2}_{\text{where electron is}}, t)$$

H ATOMS COME IN LUMPS

$$\int |\psi(\vec{r}_1, \vec{r}_2, t)|^2 \, d^3r_1 \, d^3r_2 = 1$$

CALLED SINGLE PARTICLE BASIS

$$\langle \vec{r}_1, \vec{r}_2 | \psi(t) \rangle = \psi(\vec{r}_1, \vec{r}_2, t)$$

need operators

$$\vec{r}_{1op} \rightarrow \vec{r}_1 \qquad \vec{p}_{1op} \rightarrow -i\hbar\nabla_1$$

$$\vec{r}_{2op} \rightarrow \vec{r}_2 \qquad \vec{p}_{2op} \rightarrow -i\hbar\nabla_2$$

CAN GENERALIZE TO N PARTICLES

$\rightarrow$ 3N degrees of freedom + time

SCHRODINGER EQUATION TDSE

$$H \mid \psi \rangle = i\hbar \frac{d}{dt} \mid \psi \rangle$$

$$H = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + V(\vec{r}_1, \vec{r}_2)$$

$$\left[-\frac{\hbar^2}{2m} \vec{\nabla}_1^2 - \frac{\hbar^2}{2m} \vec{\nabla}_2^2 + V(\vec{r}_1, \vec{r}_2) \right] \psi(\vec{r}_1, \vec{r}_2, t)$$

$$= i\hbar \frac{d}{dt} \psi(\vec{r}_1, \vec{r}_2, t)$$

$\Rightarrow$ 7 DIMENSIONAL DIFF EQ

$(3N+1)$ DIM

I DON'T WANT TO ~~SOLVE SOLV~~ SOLVE 7d DIFF EQ!

BREAK PROBLEM INTO PIECES ...

WHOLE PROBLEM $\rightarrow$ CENTER-OF-MASS $\rightarrow$ x, y, z

WHOLE PROBLEM $\rightarrow$ RELATIVE MOTION $\rightarrow$ r, θ, φ

CHANGE VARIABLES

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \text{relative}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \text{center of mass}$$

$$\frac{m_1 m_2}{m_1 + m_2} = \mu \qquad \text{REDUCED MASS}$$

$$m_1 + m_2 = M \qquad \text{TOTAL MASS}$$

$$\left[-\frac{\hbar^2}{2\mu} \nabla_r^2 - \frac{\hbar^2}{2M} \vec{\nabla}_R^2 + V(\vec{r}) \right] \varphi(\vec{r}, \vec{R}, t)$$

$$= i\hbar \frac{d}{dt} \varphi(\vec{r}, \vec{R}, t)$$

SEPARATE THIS EQN: $\varphi(\vec{r}, \vec{R}, t) = f(\vec{R}, t)\, g(\vec{r}, t)$

$\Rightarrow$ POSSIBLE BECAUSE $\vec{R}$ does not appear in V

$$\left[-\frac{\hbar^2}{2\mu} \vec{\nabla}_r^2 - \frac{\hbar^2}{2M} \vec{\nabla}_R^2 + V(\vec{r}) \right] f \cdot g$$

$$= i\hbar \frac{d}{dt} (f \cdot g)$$

$$\frac{1}{\cancel{f} \cdot g} \cancel{f(\vec{R},t)} \left[-\frac{\hbar^2}{2\mu} \vec{\nabla}_r^2 + V(\vec{r}) - i\hbar \frac{d}{dt} \right] g(\vec{r},t)$$

depends only on $\vec{r}, t$

$$= \frac{1}{f \cdot \cancel{g}} \cancel{g(\vec{r},t)} \left[\frac{\hbar^2}{2M} + i\hbar \frac{d}{dt} \right] f(\vec{R},t)$$

depends only on $\vec{R}, t$

~~$f(\vec{R},t) =$ prob density~~

$f(\vec{R},t) =$ prob amplitude that cm is at $\vec{R}$ at time t

$g(\vec{r},t) =$ prob amplitude that separation is $\vec{r}$ at time t

SEPARATED EQUATIONS $7d \nearrow 3+1$ CM, $\searrow 3+1$ REL — 8d!

TDSE R: $$\left[-\frac{\hbar^2}{2M}\vec{\nabla}_R^2\right] f(\vec{R},t) = i\hbar\frac{d}{dt} f(\vec{R},t)$$

3d free particle

$$f(\vec{R},t) = e^{i(\vec{p}\cdot\vec{R} - (p^2/2M)t)/\hbar}$$

CM eigenstates

TDSE r: $$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2 + V(\vec{r})\right] g(\vec{r},t) = i\hbar\frac{d}{dt} g(\vec{r},t)$$

SEPARATE TIME DEPENDENCE

$$g(\vec{r},t) = \varphi(\vec{r})\, h(t)$$

$$\cancel{h(t)\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2 + V(\vec{r})\right]\varphi_m(\vec{r}) =}$$

$$\frac{1}{h\varphi_m} h(t)\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2 + V(\vec{r}) - i\hbar\frac{d}{dt}\right]\varphi_m(\vec{r})$$

$$= \frac{1}{h\cancel{\varphi_m}}\cancel{\varphi_m(\vec{r})}\left[i\hbar\frac{d}{dt}\right]h(t) \qquad \cancel{\frac{\epsilon}{h\varphi_m}}$$

SEPARATION CONSTANT E_m

energy eigenvalue = separation constant.

$$\frac{1}{\varphi_m}\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2 + V(\vec{r})\right]\varphi_m(\vec{r}) = E_m$$

$$E_m = \frac{1}{h(t)}\left[i\hbar\frac{dh}{dt}\right]$$

$$\frac{dh}{dt} = -i\frac{E_m}{\hbar}h$$

$$h(t) = e^{-iE_m t/\hbar}$$

$$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2 + V(\vec{r})\right]\varphi_m(\vec{r}) = E_m\,\varphi_m(\vec{r})$$

3d TISE for relative motion

SO, THE ENERGY EIGENSTATES ARE

$$\psi_m(\vec{r},\vec{R},t) = f(\vec{R},t)\;g(\vec{r},t)$$

$$= e^{i(\vec{P}\cdot\vec{R} - (P^2/2M))t/\hbar}\;\varphi_m(\vec{r})\;e^{-iE_m t/\hbar}$$

HARD JOB IS FINDING φ_m'S

TISE

$$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}^2+V(\vec{r})\right]\varphi_n(\vec{r}) = E_n\varphi_n(\vec{r})$$

SPECIAL CASE

central potential $\quad V(\vec{r}) = V(r) = -\frac{e^2}{r}$ HYDROGEN

$-\frac{Ze^2}{r}$ HYDROGEN-LIKE

$V(r)$ is spherically symmetric $\Rightarrow \vec{L}$ is conserved

SEPARATE AGAIN

$$\varphi_n(\vec{r}) = R(r)\,\Omega(\theta,\varphi)$$

$$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}^2+V(r)\right]R(r)\,\Omega(\theta,\varphi)$$

$$= E_n\,R(r)\,\Omega(\theta,\varphi)$$

$$-\frac{\hbar^2}{2\mu}\nabla^2 = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2}\left\{\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\right]$$

$$= -\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2}r+\frac{L^2}{2\mu r^2}$$

$$H = -\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2} r + \frac{\vec{L}^2}{2\mu r^2} + V(r)$$

coupled system of eigenequations

$$H\,\varphi_{nlm}(\vec{r}) = E_n\,\varphi_{nlm}(\vec{r})$$

$$L^2\,\varphi_{nlm}(\vec{r}) = l(l+1)\hbar^2\,\varphi_{nlm}(\vec{r})$$

$$L_z\,\varphi_{nlm}(\vec{r}) = m\hbar\,\varphi_{nlm}(\vec{r})$$

we already solved the L^2, L_z problem

$$\varphi_{nlm}(\vec{r}) = R_n(r)\,Y_{lm}(\theta,\varphi)$$

$$\left[-\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2} r - \frac{e^2}{r}\right] R_n Y_{lm}$$

$$+ \left[\frac{L^2}{2\mu r^2}\right] R_n Y_{lm} = E_n R_n Y_{lm}$$

$$l(l+1)\hbar^2$$

NEW TISE

$$\left[-\frac{\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2} r - \frac{e^2}{r} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} \right] R_n = E_n R_n$$

1d PROBLEM

WITH EFFECTIVE POTENTIAL

$$V_{eff}(r) = -\frac{e^2}{r} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

COULOMB ATTRACTION

CENTRIFUGAL ~~POTENTIAL~~ BARRIER

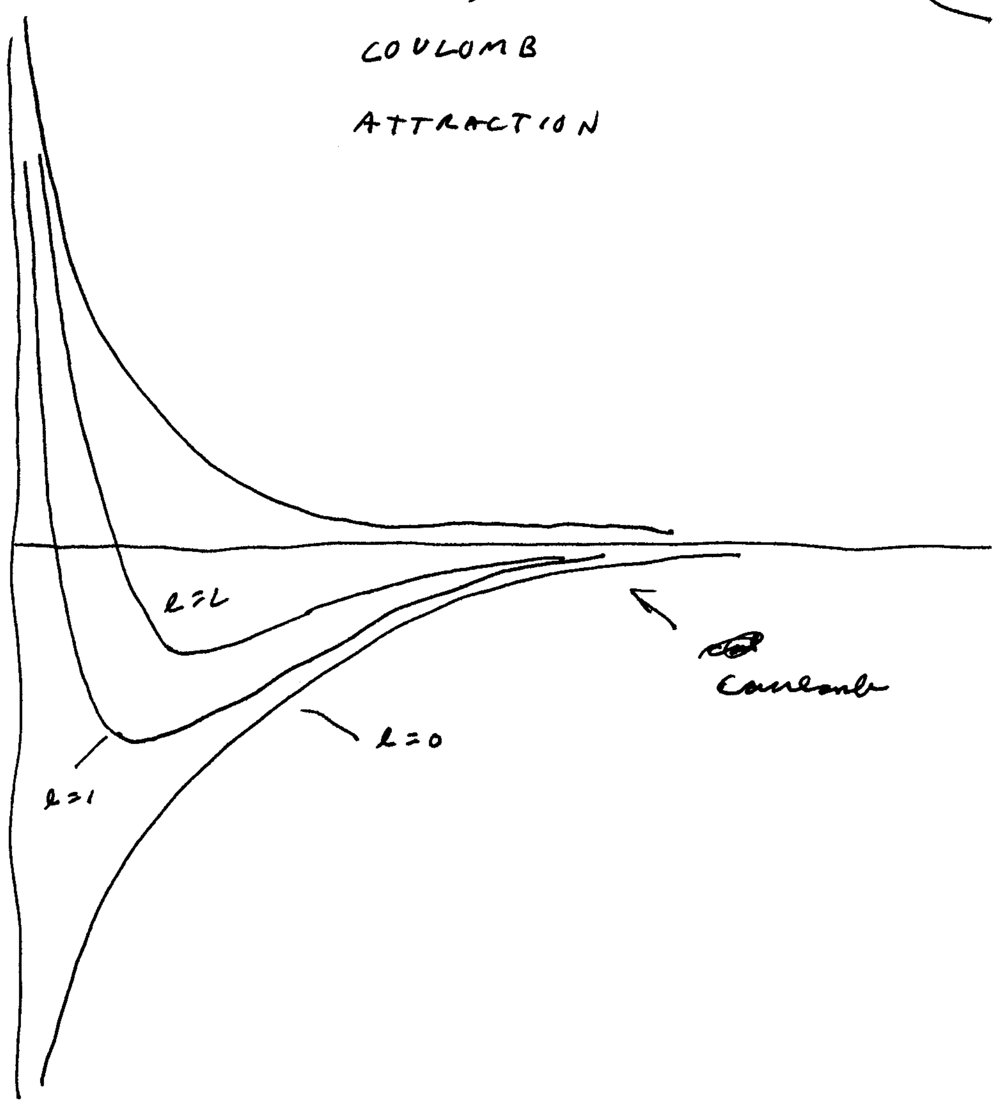
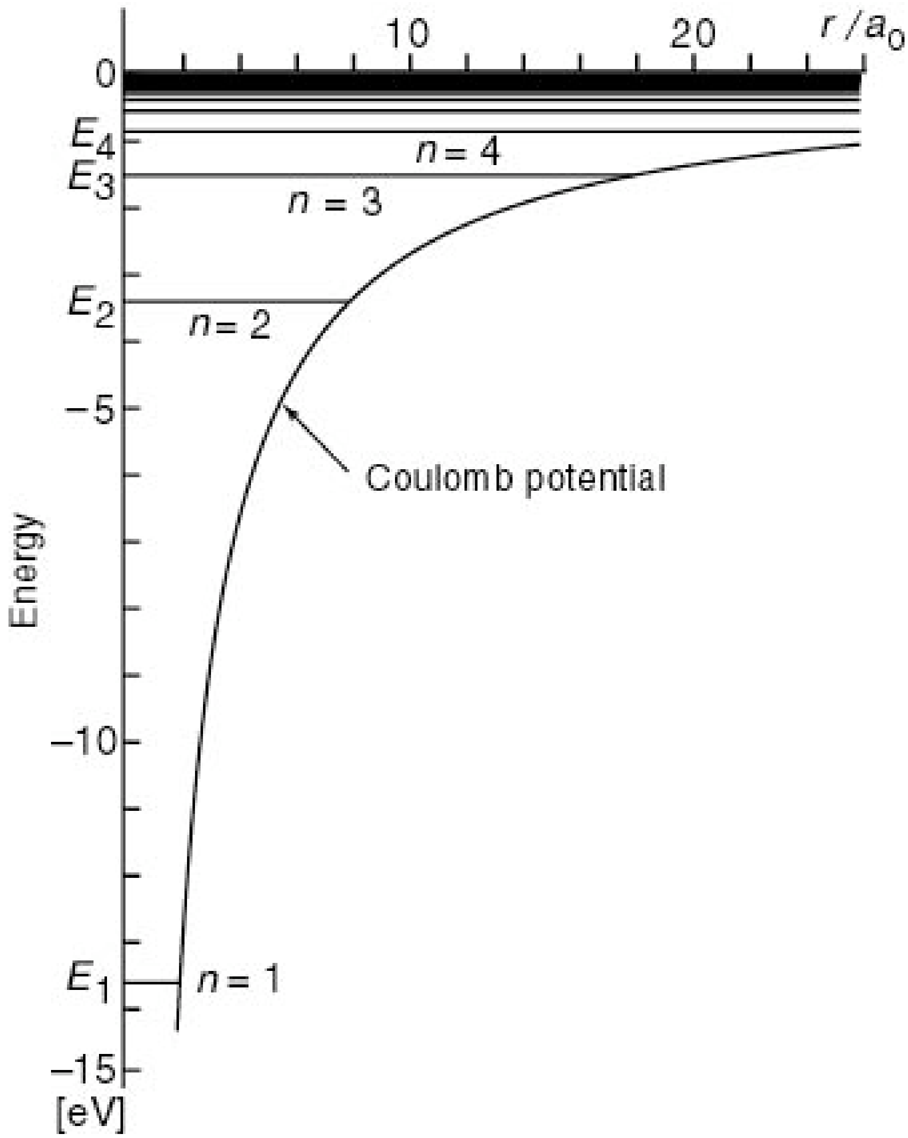


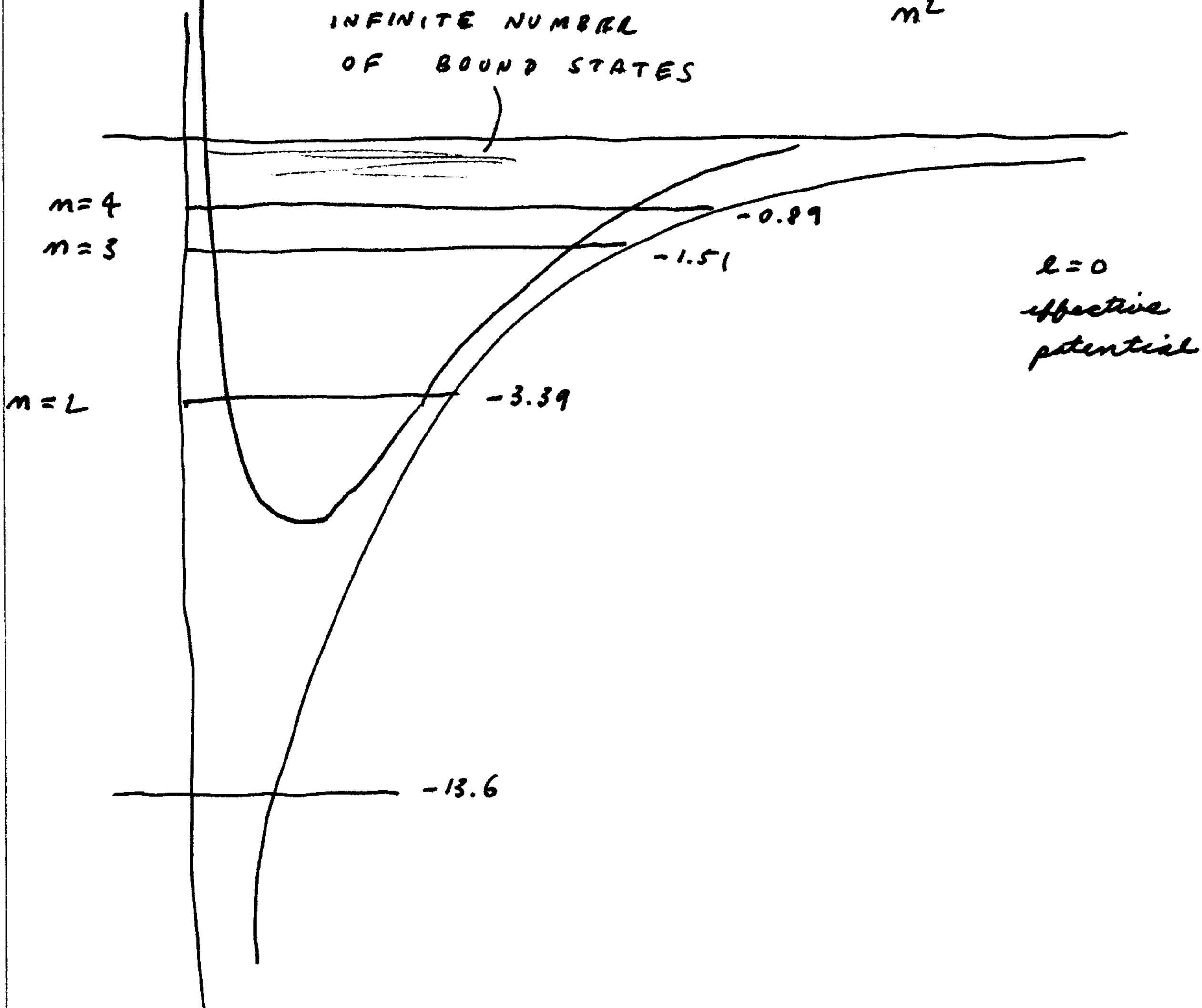
Fig. (B)



10
20
r / a0
0
E4
n = 4
E3
n = 3
E2
n = 2
−5
Coulomb potential
Energy
−10
E1
n = 1
−15
[eV]

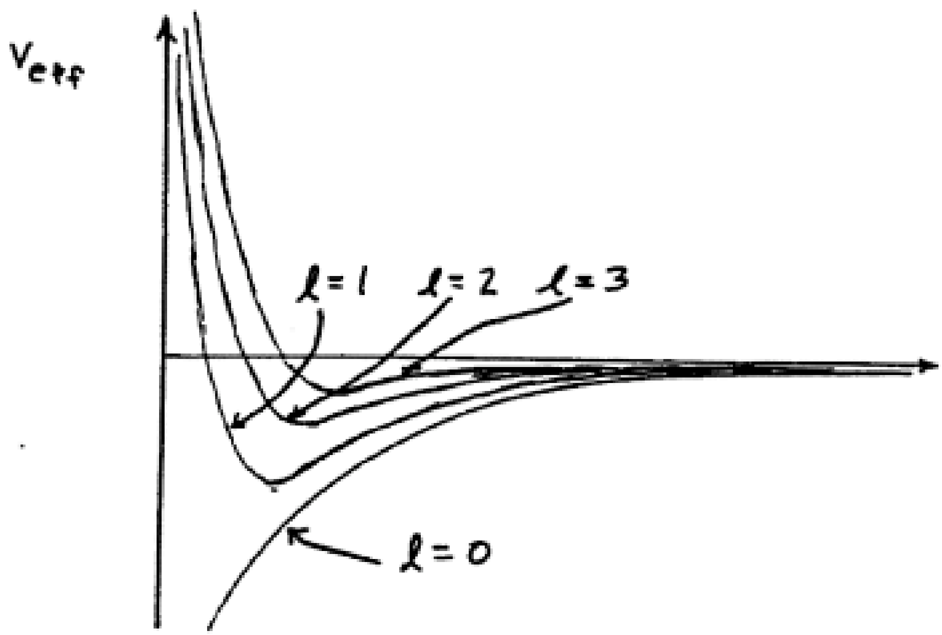
ENERGY DEGENERACY only n

$$E_n = \frac{E_0}{n^2} \qquad E_0 \approx -13.$$



The Effective Potential Depends on the Angular Momentum

=> Series of Nested Wells



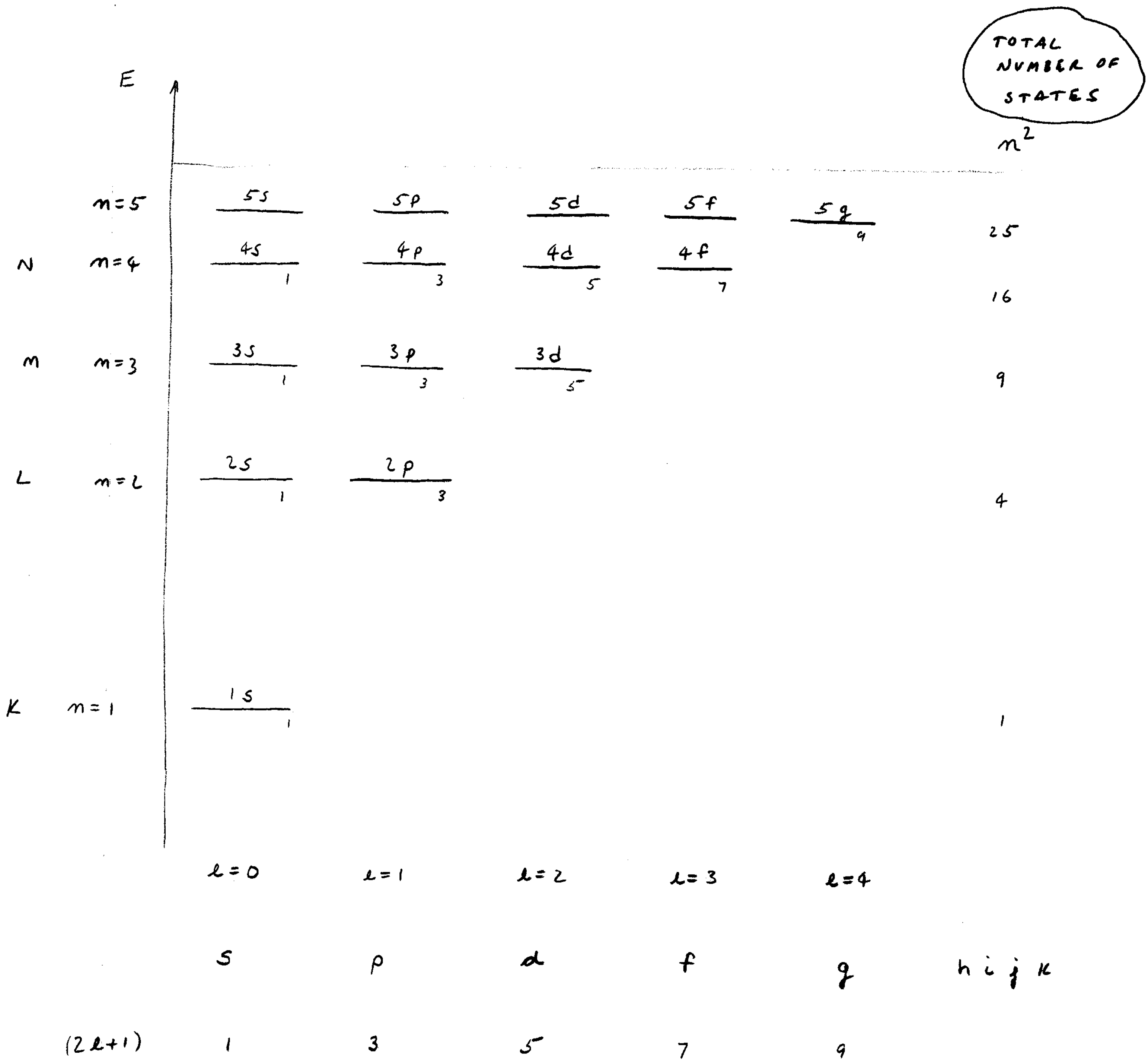
Series of States in each Well
Ground, 1st, 2nd, 3rd, ... excited

Energy
Degeneracy

$$E_n = - Z^2 \frac{E_0}{n^2}$$

for each n: $\ell = 0, 1, \dots, n$

for each ℓ: $m = -\ell, \dots, +\ell$



FIRST FEW RADIAL WAVEFUNCTIONS $\quad \mathfrak{z} = Z/a_0$

$n=1$
$$R_{10}(r) = 2\,\mathfrak{z}^{3/2}\, e^{-\mathfrak{z} r}$$

$n=2$
$$R_{20}(r) = \frac{1}{\sqrt{2}}\,\mathfrak{z}^{3/2}\left(1 - \tfrac{1}{2}\mathfrak{z} r\right) e^{-\mathfrak{z} r/2}$$

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\,\mathfrak{z}^{5/2}\,(r)\, e^{-\mathfrak{z} r/2}$$

$n=3$
$$R_{30}(r) = \frac{2}{3\sqrt{3}}\,\mathfrak{z}^{3/2}\left(1 - \tfrac{2}{3}\mathfrak{z} r + \tfrac{2}{27}\mathfrak{z}^2 r^2\right) e^{-\mathfrak{z} r/3}$$

$$R_{31}(r) = \frac{8}{27\sqrt{6}}\,\mathfrak{z}^{5/2}\left(\mathfrak{z} r - \tfrac{1}{6}\mathfrak{z}^2 r^2\right) e^{-\mathfrak{z} r/3}$$

$$R_{32}(r) = \frac{4}{81\sqrt{30}}\,\mathfrak{z}^{7/2}\,(r^2)\, e^{-\mathfrak{z} r/3}$$

general form
$$(\text{NORM})\,(\text{POLYNOMIAL})\, e^{-Zr/na_0}$$

WHAT DO THE RADIAL WAVEFCNS LOOK LIKE?

" " " RADIAL PROB DISTS " " ?

" " " 3d " " " " ?

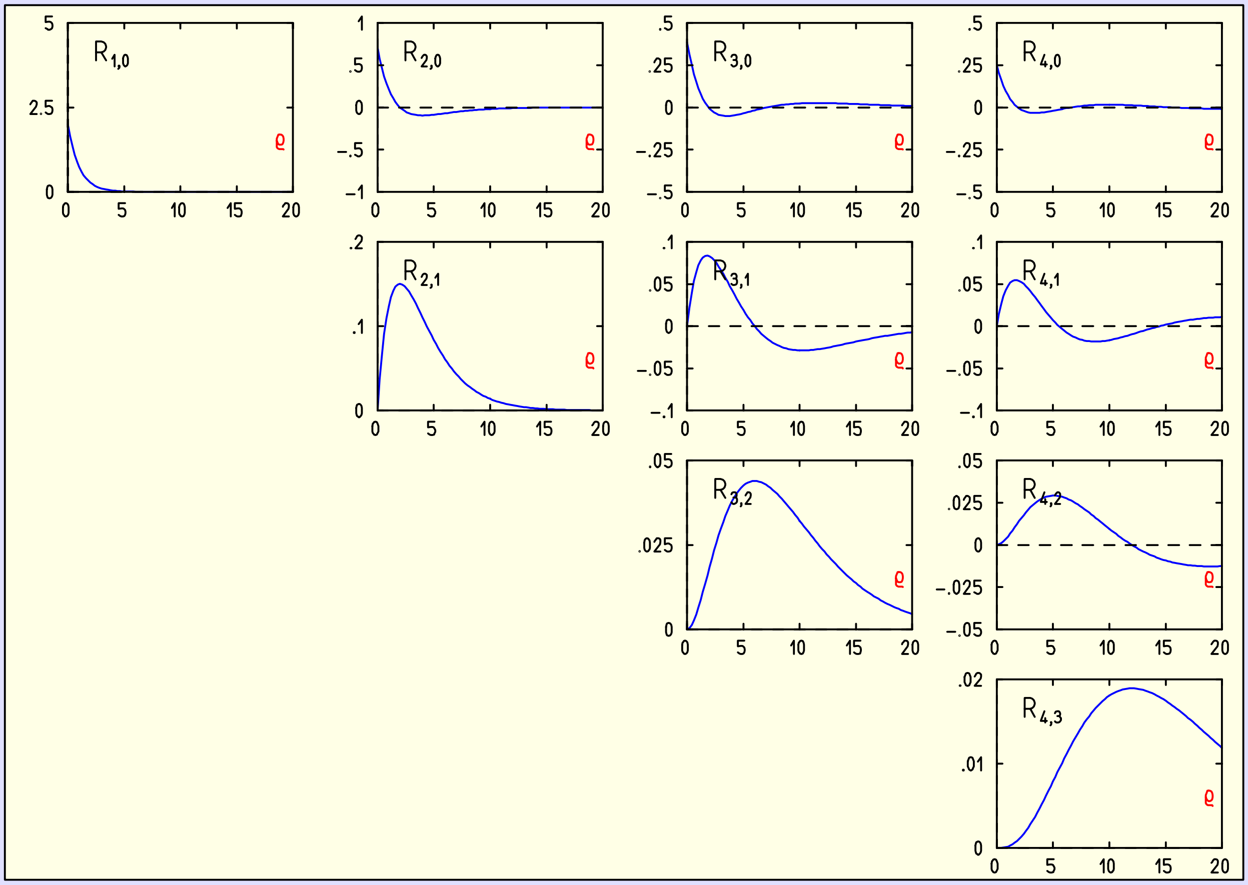
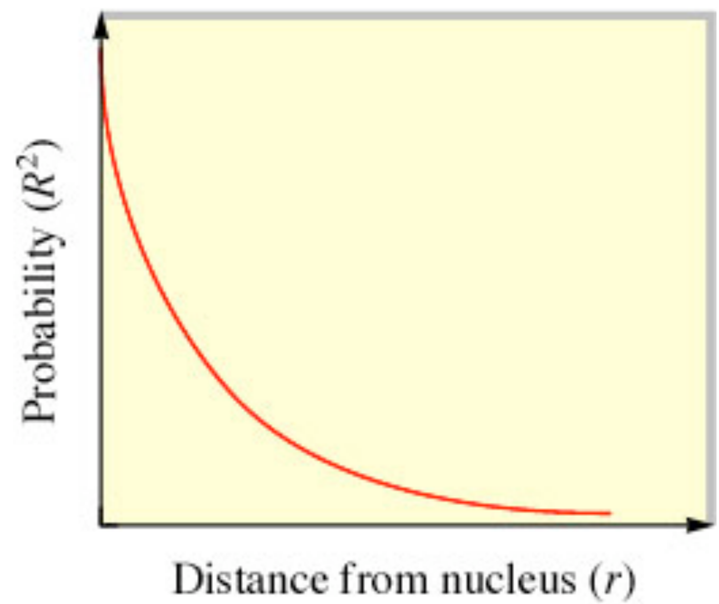


Fig. 13.14. **Radial eigenfunctions $R_{n\ell}(\rho)$ for the electron in the hydrogen atom. Their zeros are the $n-\ell-1$ zeros of the Laguerre polynomials $L_{n-\ell-1}^{2\ell+1}(2\rho/n)$. Here the argument of the Laguerre polynomial is $2\rho/n$ with n being the principal quantum number and $\rho = r/a$ the distance between electron and nucleus divided by the Bohr radius a.**

A hydrogen atom lost its electron and went to the police station to file a missing electron report. He was questioned by the police: "Haven't you just misplaced it somewhere? Are you sure that your electron is really lost?" *"I'm positive."* replied the atom.

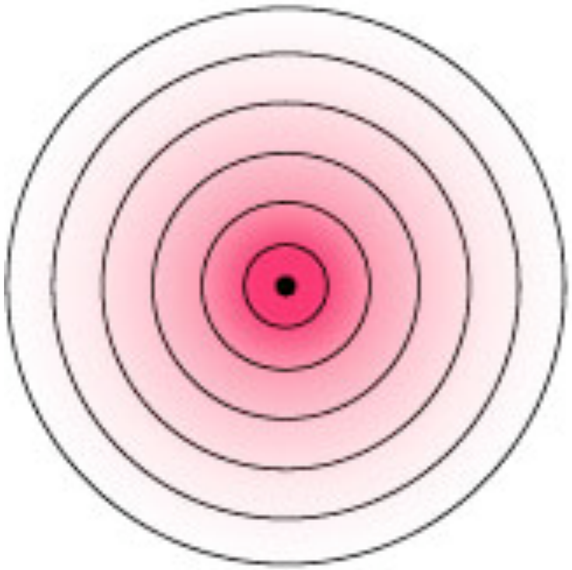
Probability Distribution for the 1s Wave Function

(a)

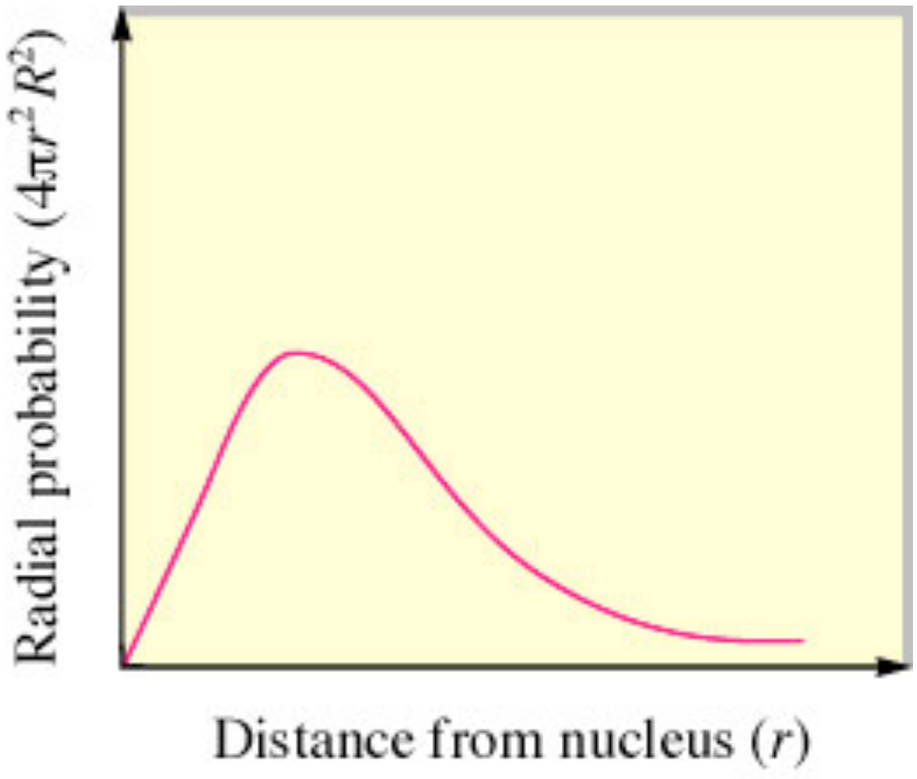


(b)

Radial Probability Distribution



(a)

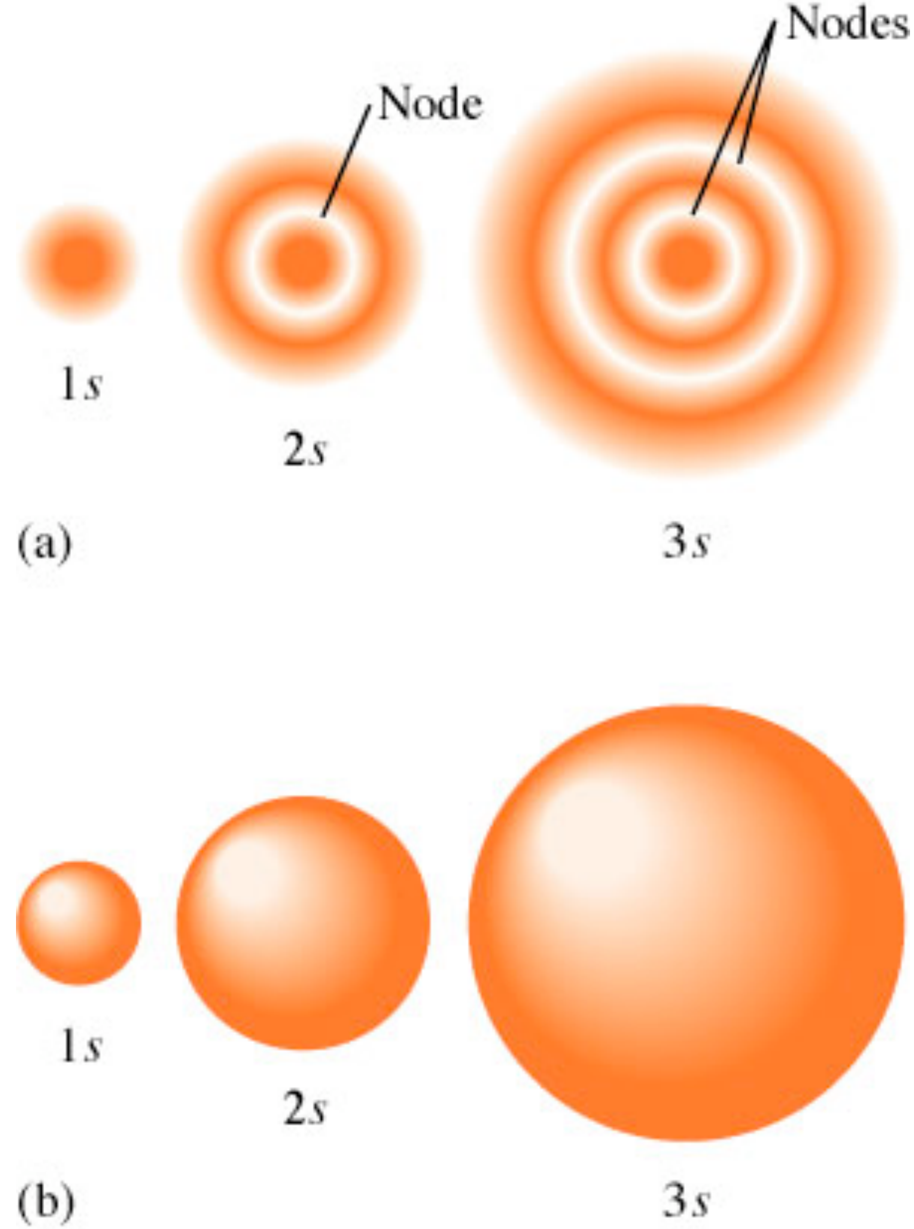


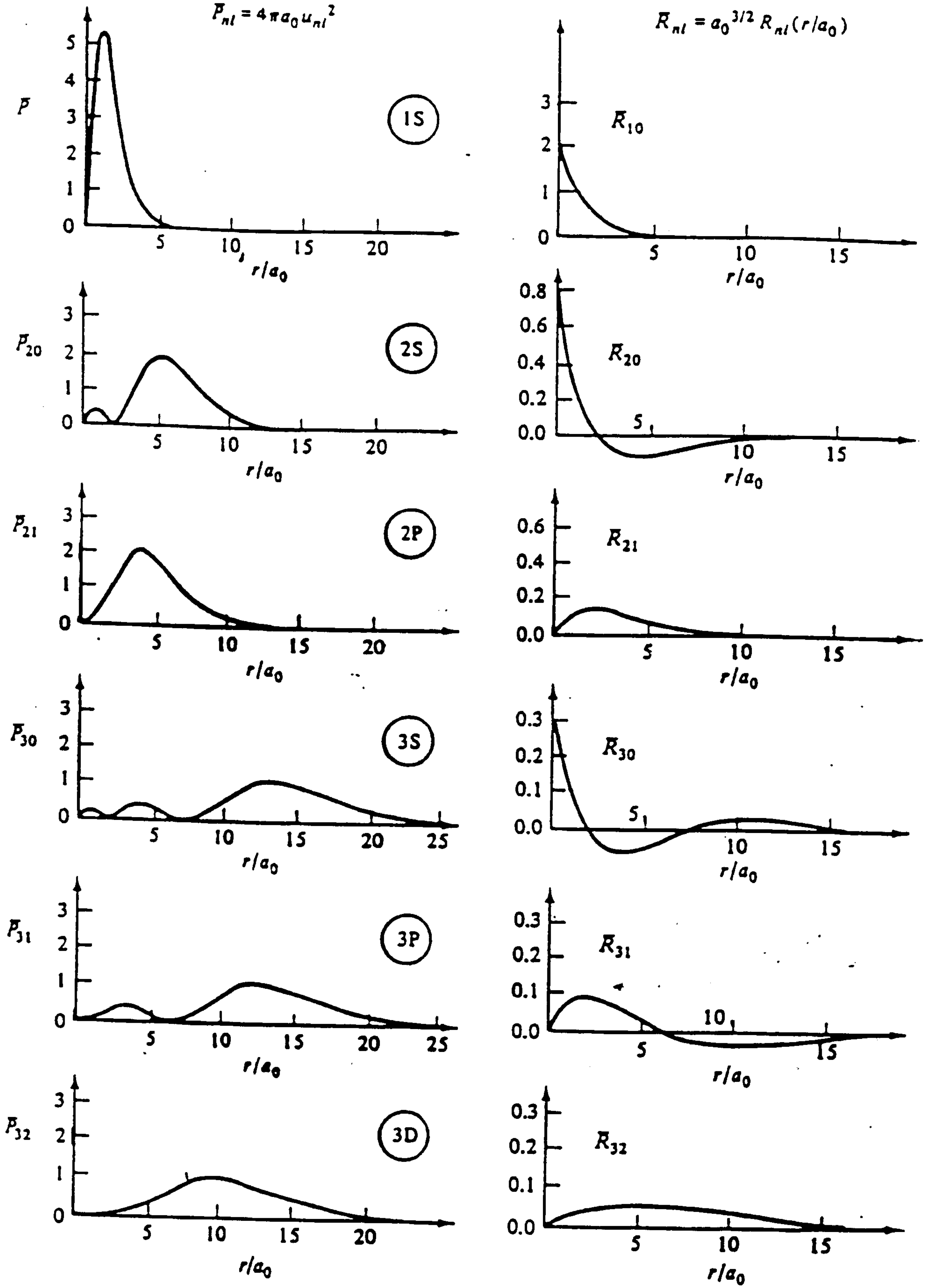
(b)

Probability Distribution

- ✓ square of the wave function
- ✓ probability of finding an electron at a given position
- ✓ Radial probability distribution is the probability distribution in each spherical shell.

Two Representations of the Hydrogen 1s, 2s, and 3s Orbitals





$\bar{P}_{nl} = 4\pi a_0 u_{nl}^2$
$\bar{R}_{nl} = a_0^{3/2} R_{nl}(r/a_0)$
$\bar{P}$
1S
$\bar{R}_{10}$
$\bar{P}_{20}$
2S
$\bar{R}_{20}$
$\bar{P}_{21}$
2P
$\bar{R}_{21}$
$\bar{P}_{30}$
3S
$\bar{R}_{30}$
$\bar{P}_{31}$
3P
$\bar{R}_{31}$
$\bar{P}_{32}$
3D
$\bar{R}_{32}$
r/a_0

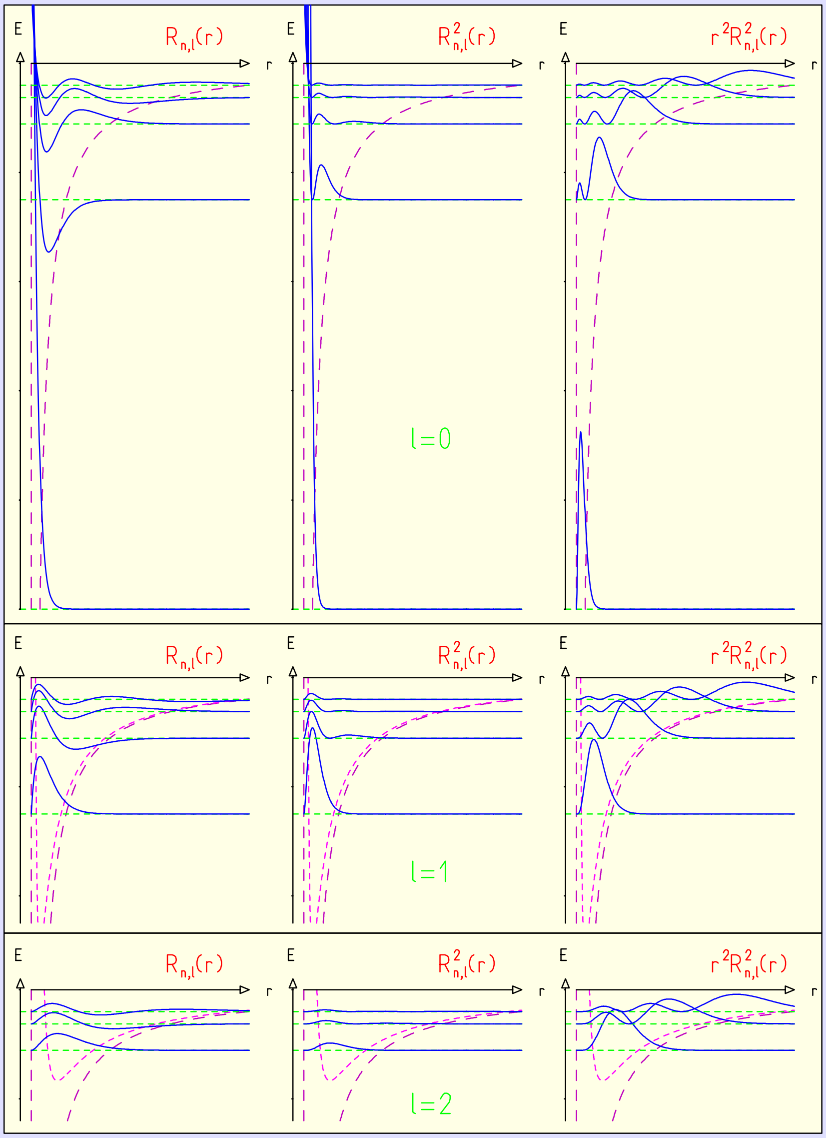
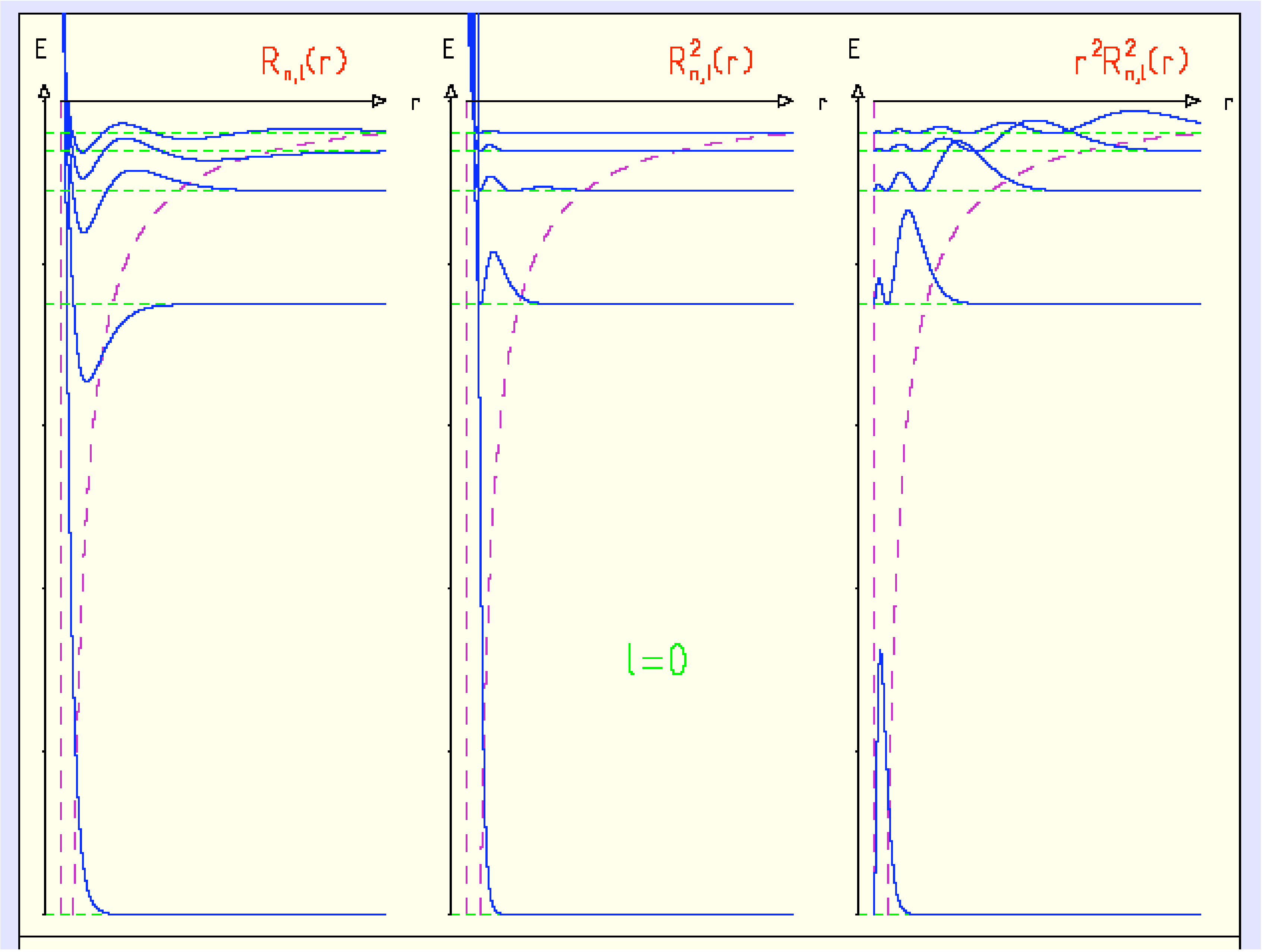
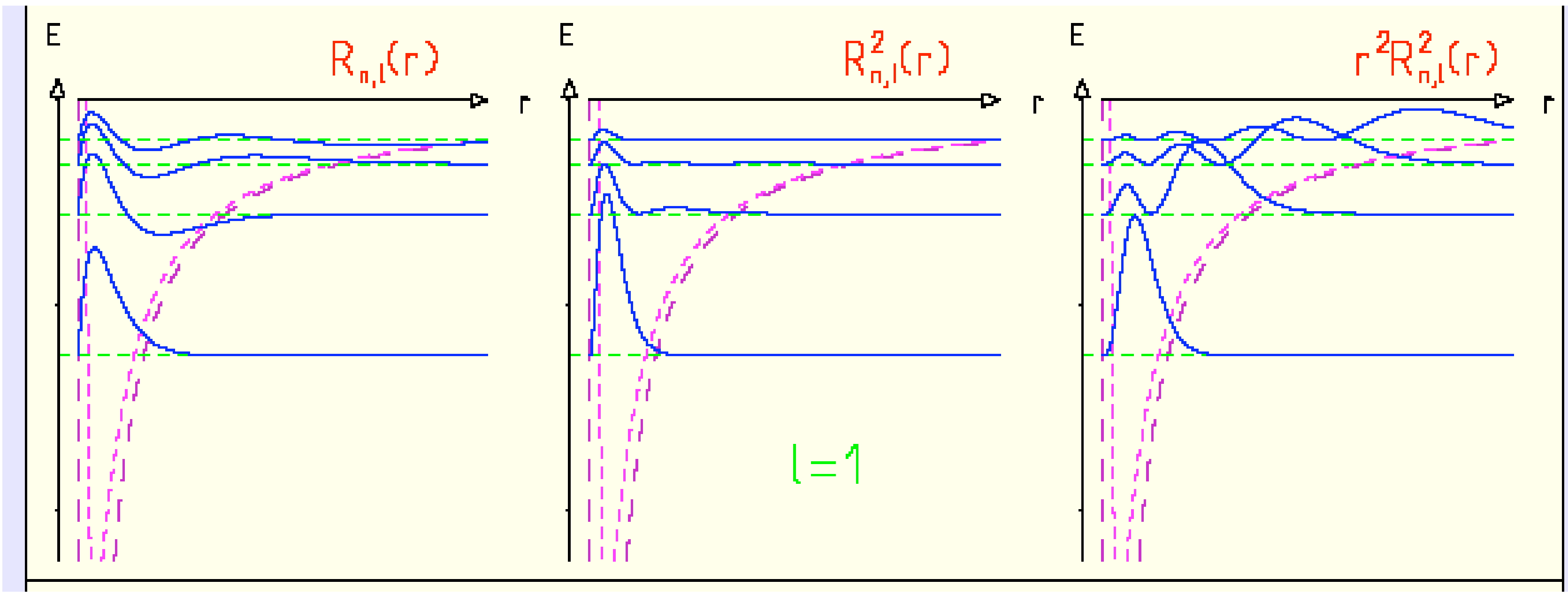


Fig. 13.15. Radial eigenfunctions $R_{n\ell}(r)$, their squares $R^2_{n\ell}(r)$, and the functions $r^2R^2_{n\ell}(r)$ for the lowest eigenstates of the electron in the hydrogen atom and the lowest angular-momentum quantum numbers $\ell = 0, 1, 2$. Also shown are the energy eigenvalues as horizontal dashed lines, the form of the Coulomb potential $V(r)$, and, for $\ell \neq 0$, the forms of the effective potential $V_\ell^{\mathrm{eff}}(r)$. The eigenvalue spectra are degenerate for all ℓ values, except that the minimum value of the principal quantum number is $n = \ell + 1$.

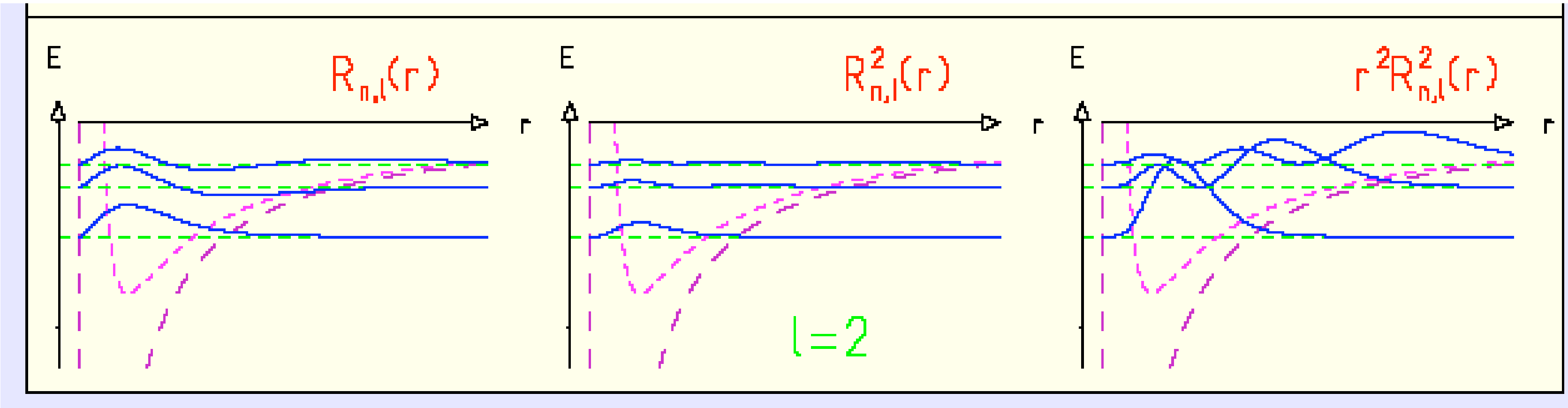
1s, 2s, 3s, 4s, 5s



2p, 3p, 4p, 5p



3d, 4d, 5d



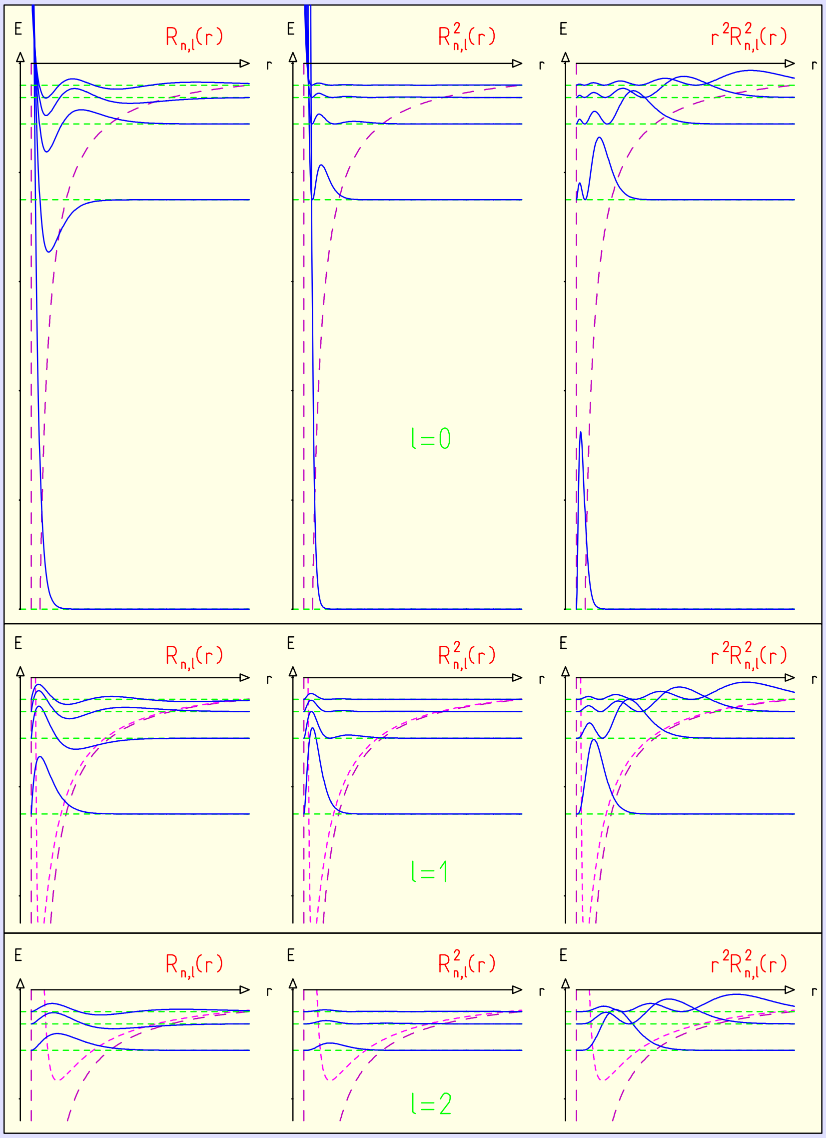


Fig. 13.15. Radial eigenfunctions $R_{n\ell}(r)$, their squares $R^2_{n\ell}(r)$, and the functions $r^2R^2_{n\ell}(r)$ for the lowest eigenstates of the electron in the hydrogen atom and the lowest angular-momentum quantum numbers $\ell = 0, 1, 2$. Also shown are the energy eigenvalues as horizontal dashed lines, the form of the Coulomb potential $V(r)$, and, for $\ell \neq 0$, the forms of the effective potential $V^{\text{eff}}_\ell(r)$. The eigenvalue spectra are degenerate for all ℓ values, except that the minimum value of the principal quantum number is $n = \ell + 1$.

http://itl.chem.ufl.edu/4412_aa/radwfct.html

START TO SOLVE

make equation dimensionless $r \to \rho$

length $a_0 = \frac{\hbar^2}{me^2}$ BOHR RADIUS $= 0.52$ Å

$$\rho = r/a_0$$

energy $E_0 = \frac{\hbar^2}{2ma_0^2} = 13.6$ ev RYDBERG

$E \to \lambda^2$

$$\lambda^2 = E_n/E_0$$

$$\frac{2ma_0^2}{\hbar^2}\left[-\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{e^2}{r}\right]R_{n\ell} = E_n R_{n\ell}$$

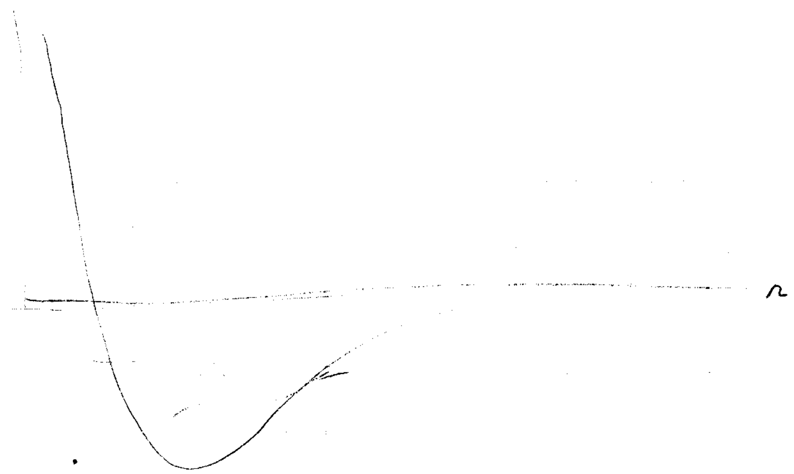
$$r\times \quad \left[-\frac{a_0^2}{r}\frac{\partial^2}{\partial r^2}r + \left(\frac{a_0^2}{r^2}\right)\ell(\ell+1) - \frac{2me^2}{\hbar^2}\frac{a_0^2}{r}\right]R_{n\ell} = \lambda^2 R_{n\ell}$$

$$-\frac{\partial^2}{\partial\left(\frac{r}{a_0}\right)^2} + \frac{\ell(\ell+1)}{\rho^2} - \underbrace{\left(\frac{2}{a_0}\right)\left(\frac{a_0^2}{r}\right)}_{\frac{2}{\rho}}$$

$S \equiv r R_{n\ell}$

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - \lambda^2\right]S(\rho) = 0$$

Bound states



$r \to 0 \qquad V \to \infty \qquad \Rightarrow \qquad \psi \to 0$

$r \to \infty \qquad V \to 0 \qquad \Rightarrow \qquad \psi \sim e^{-\alpha r}$.

$\rho \to \infty$

$$\frac{d^2 S}{d\rho^2} - \left[\underbrace{\cancel{\frac{\ell(\ell+1)}{\rho^2}}}_{\to 0} + \underbrace{\cancel{\frac{2}{\rho}}}_{\to 0} - \lambda^2 \right] S = 0$$

$$S = A e^{-\lambda\rho} + \cancel{B} e^{+\lambda\rho \to 0}$$

$$S(\rho) = e^{-\lambda\rho} \; Y(\rho)$$

separate again

$$\frac{d^2}{d\rho^2}\left[e^{-\lambda\rho} Y(\rho) \right] - \left[\frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - \lambda^2 \right] \left[e^{-\lambda\rho} Y(\rho) \right] = 0$$

$$\frac{d}{d\rho}\left[-\lambda e^{-\lambda\rho}\, y + e^{-\lambda\rho}\frac{dy}{d\rho}\right]$$

$$+\lambda^2 e^{-\lambda\rho}\, y \underbrace{- \lambda e^{-\lambda\rho}\frac{dy}{d\rho} - \lambda e^{-\lambda\rho}\frac{dy}{d\rho}}_{-2\lambda\,\ldots} + e^{-\lambda\rho}\frac{d^2y}{d\rho^2}$$

$$\cancel{e^{-\lambda\rho}}\left[\frac{d^2}{d\rho^2} - 2\lambda\frac{d}{d\rho} + \lambda^2 - \frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - \cancel{\lambda^2}\right] y(\rho) = 0$$

want to solve this equation such that $\psi(r)\to 0$ as $r\to 0$

$$y(\rho) = \rho^m \sum_{n=0}^{\infty} c_n \rho^n = \sum_{n=0}^{\infty} c_n \rho^{n+m}$$

insures

$\psi(r)\to 0$ as $r\to 0$.

$$\frac{dy}{d\rho} = \frac{d}{d\rho}\left[\sum c_n \rho^{n+m}\right] = \sum (n+m)\, c_n\, \rho^{(n+m-1)}$$

$$\frac{d^2y}{d\rho^2} = \frac{d}{d\rho}\left[\sum (n+m)\, c_n\, \rho^{(n+m-1)}\right] = \sum_{n=0}^{\infty} (n+m)(n+m-1)\, c_n\, \rho^{n+m-2}$$

lowest power of ρ is $(m-2)$ TH.

$$(\cancel{n}^{0}+m)(\cancel{n}^{0}+m-1)\, c_0\, \rho^{m-2} - \frac{\ell(\ell+1)}{\rho^2}\, c_0\, \rho^{m}$$

$$[m(m-1) - \ell(\ell+1)]\, c_0 = 0 \qquad \Rightarrow \quad m = \ell+1$$

$$\text{or } \; m = -\ell$$

coefficient of the $(n+m-2)$ TH TERM

$$\sum_{n=0}^{\infty} (n+m)(n+m-1)\, c_n\, \rho^{n+m-2} - 2\lambda (n+m)\, c_n\, \rho^{n+m-1}$$

$$+ \frac{2}{\rho}\, c_n\, \rho^{(n+m)-1} - \frac{\ell(\ell+1)}{\rho^2}\, c_n\, \rho^{(n+m)-2} = 0$$

$$[(n+m)(n+m-1) - \ell(\ell+1)]\, c_n + [-2\lambda(n-1+m) + 2]\, c_{n-1} = 0$$

$\ell = m-1$

$$[(n+\ell+1)(n+\ell) - \ell(\ell+1)]\, c_n = 2[(n+\ell)\lambda - 1]\, c_{n-1}$$

$$n[n+2\ell+1]\, c_n = 2[(n+\ell)\lambda - 1]\, c_{n-1}$$

$$\boxed{c_n = \frac{2[(n+\ell)\lambda - 1]}{n[n+2\ell+1]}\, c_{n-1}}$$

special values of λ terminate series

TWO CONDITIONS

1) POLYNOMIAL STARTS WITH $\rho^{\ell+1}$ — finite $\psi(0)$

2) POLYNOMIAL STOPS WITH $\rho^{n+\ell}$ — finite energy.

eigenenergies

$$c_n = \frac{2\left[(n+\ell)\lambda - 1 \right]}{n(n+2\ell+1)} \; c_{n-1}$$

$$(n+\ell)\lambda - 1 = 0$$

$$\lambda = \left(\frac{1}{n+\ell} \right) = \frac{1}{(j+\ell)}$$

energy $\quad \lambda^2 = \frac{E_n}{E_0}$

$$E_n = \frac{1}{(j+\ell)^2} \; E_0$$

$$\boxed{E_n = \frac{1}{n^2} \; E_0}$$

$$\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r) \; Y_{\ell m}(\theta, \varphi) = \frac{e^{-\lambda\rho}}{a_0 \rho} \; y_{n\ell}(\rho) \; Y_{\ell m}(\theta, \varphi)$$

$$= N \; e^{-\rho/n} \left[\frac{y_{n\ell}(\rho)}{\rho} \right] Y_{\ell m}(\theta, \varphi)$$

y goes from $\ell \ldots n$

$[\;]$ goes from $\ell-1 \ldots n-1$

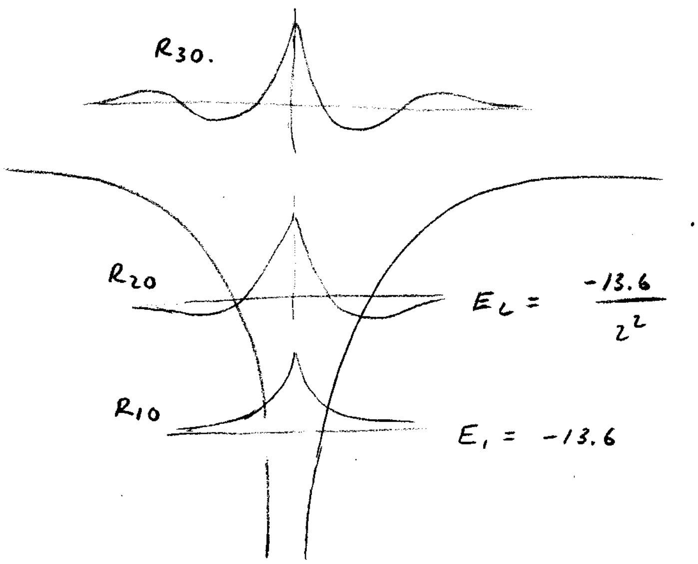
Probability distributions

$$P(\vec{r}) = |\psi_{n\ell m}(\vec{r})|^2 \, d^3r.$$

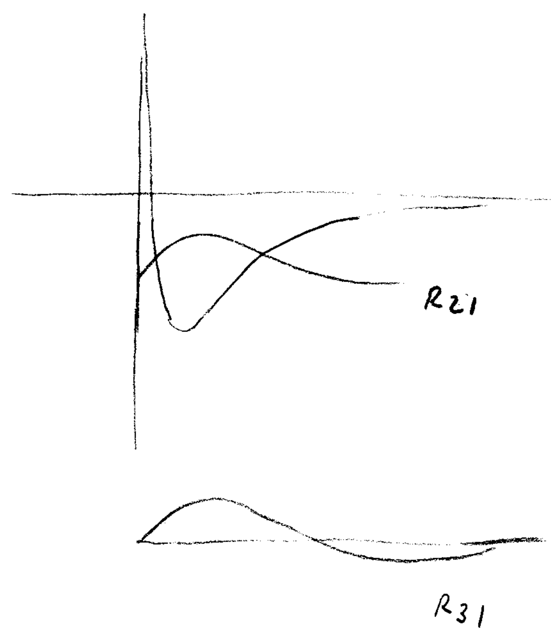
$$|R_{n\ell}(r)|^2 \, r^2 \, dr$$

$$|Y_{\ell m}(\theta, \varphi)|^2 \, d\Omega$$

$\ell = 0$



$\ell = 1$



p 142 Pauling

p 266 Eisberg.

Angular dependence

$$|Y_{\ell m}(\theta,\varphi)|^2 = \Theta(\theta)\, e^{im\varphi}\ \Theta^*(\theta)\, e^{-im\varphi}$$

phase changes as you go around z axis but the prob does not change

$$|\Theta(\theta)|^2$$

Polar plot

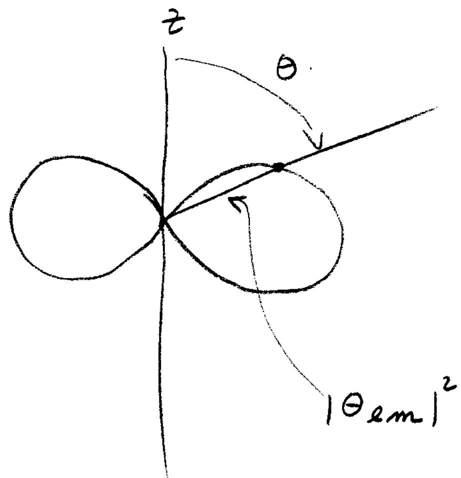
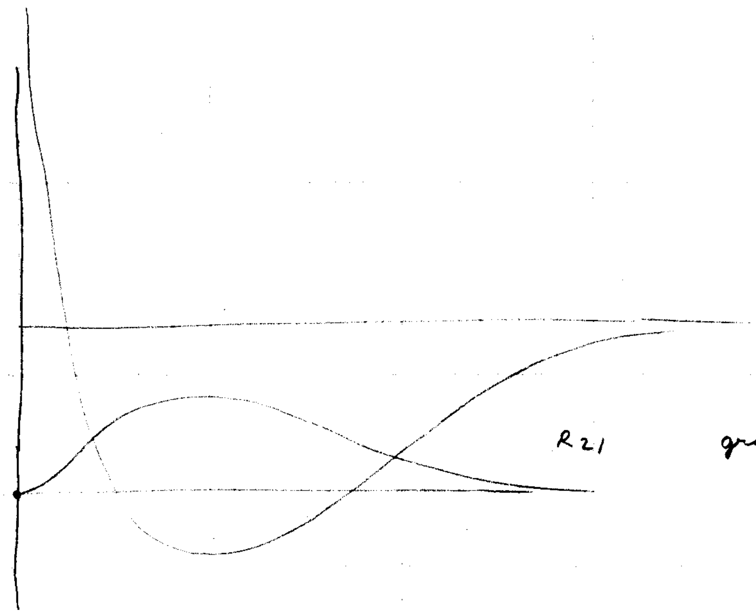


Figure of revolution around z axis

p 271 272 Eisberg

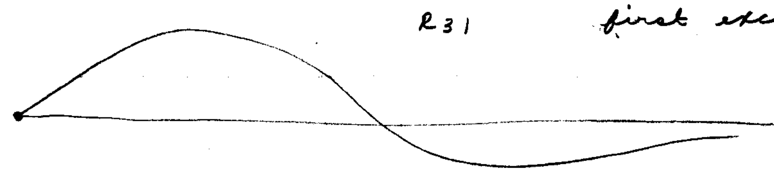
$l = 1$ well



ZERO AT ORIGIN

$V = \infty \Rightarrow$

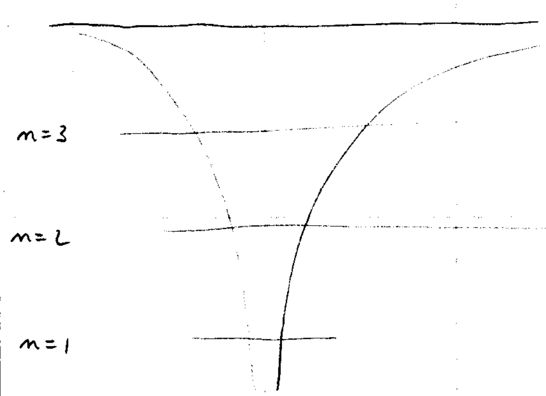
R_{21} ground state of $l=1$ well



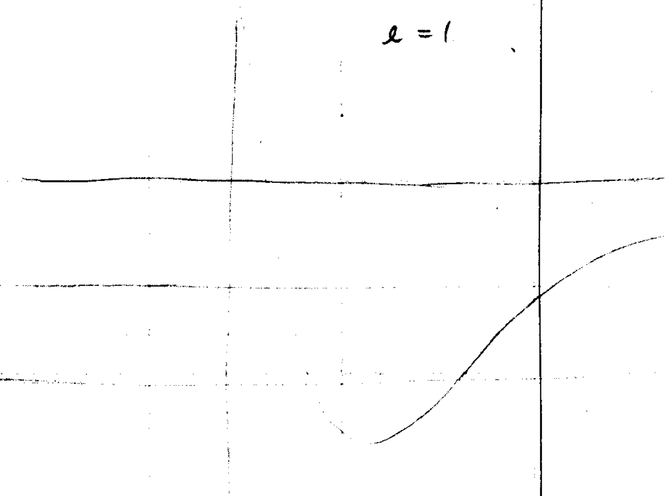
R_{31} first excited state of $l=1$ well.

degeneracy:

$l=0$



$l=1$



R_{20} has same energy as R_{21}

all R_{nl}'s have same energy.

$$R_{30} = \frac{2}{3\sqrt{3}} \, z^{3/2} \left(1 - \tfrac{2}{3} z r + \frac{2}{27} z^2 r^2\right) e^{-zr/3}$$

$$R_{31} = \frac{8}{27\sqrt{6}} \, z^{3/2} \left(z r - \frac{1}{6} z^2 r^2\right) e^{-zr/3}$$

$$R_{32} = \frac{4}{81\sqrt{30}} \, z^{7/2} \, r^2 \, e^{-zr/3}$$

$\vdots$

for each n: $\quad l = 0, \ldots, n-1$

$$e^{-Zr/na_0}$$

$l = 1$