

ORBITAL ANGULAR MOMENTUM 3

IN DIRAC NOTATION

(IN THE HILBERT SPACE)

$$\mathbf{L}^2 |\ell, m\rangle = \ell(\ell+1)\hbar^2 |\ell, m\rangle$$

$$\mathbf{L}_z |\ell, m\rangle = m\hbar |\ell, m\rangle$$

$$\mathbf{L}_{\pm} |\ell, m\rangle = \sqrt{\ell(\ell+1)-m(m\pm 1)} \hbar |\ell, m\pm 1\rangle$$

TWO REPRESENTATIONS: MATRIX

FUNCTION $Y_{\ell m}(\theta, \varphi)$

IN MATRIX FORM

A STATE WITH

ANGULAR MOMENTUM J IS REPRESENTED

BY A VECTOR WITH $(2J+1)$ COMPONENTS

$$S = \frac{1}{2}$$

$$2S + 1 = 2$$

$$S^2 = \frac{1}{2}(\frac{1}{2}+1)\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad e\vec{n} = \text{ANY VECTOR } \begin{pmatrix} a \\ b \end{pmatrix}$$

$$S_z = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \sigma_x \quad e\vec{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_y \quad e\vec{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$S_y = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_z \quad e\vec{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$S_+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \uparrow \quad S_- = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

PAULI MATRICES

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

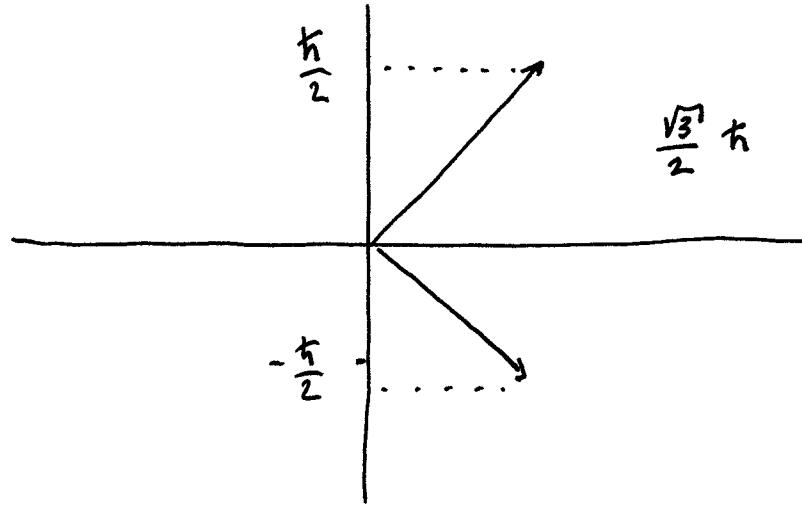
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\frac{1}{2}, \frac{1}{2}\rangle$$

$$\underline{\underline{\hspace{1cm}}} \quad |\frac{1}{2}, -\frac{1}{2}\rangle$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

REP IN SE BASIS



$$\langle s^2 \rangle = s(s+1)\hbar^2$$

LENGTH
VECTOR = $\frac{\sqrt{3}}{2}\hbar$

$$\langle s_z \rangle = m\hbar$$

$$\langle s_x \rangle = 0$$

$$\langle s_y \rangle = 0$$

For spin $\frac{1}{2}$

$$\underline{\Psi}(\chi) = \begin{pmatrix} \psi_+(\chi) \\ \psi_-(\chi) \end{pmatrix}$$

PROB AMP SPIN UP

PROB AMP SPIN DOWN

2 COMPONENT OBJECT
IS CALLED A SPINOR

for spin

For spin 0

$$\underline{\Psi}(\chi) = (\psi(\chi))$$

THIS WEEK: SPIN

UP UNTIL NOW, SCHRODINGER EQUATION

$$\cdot H |4\rangle = i\hbar \frac{d}{dt} |4\rangle$$

$$H \Psi(x) = i\hbar \frac{d}{dt} \Psi(x)$$

DOES NOT INCLUDE SPIN!

TWO WAYS TO INCLUDE SPIN:

(1) PAULI - SCHRODINGER EQUATION

PAULI
FRAUENFELDER
SORENSEN
YOU

$$H \underline{\Psi}(x) = i\hbar \frac{d}{dt} \underline{\Psi}(x)$$

NON-RELATIVISTIC, ANY SPIN
FOR SPIN $\frac{1}{2}$

$$\underline{\Psi}(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}$$

2 COMPONENT

OBJECT IS

CALLED A

FOR SPIN 1

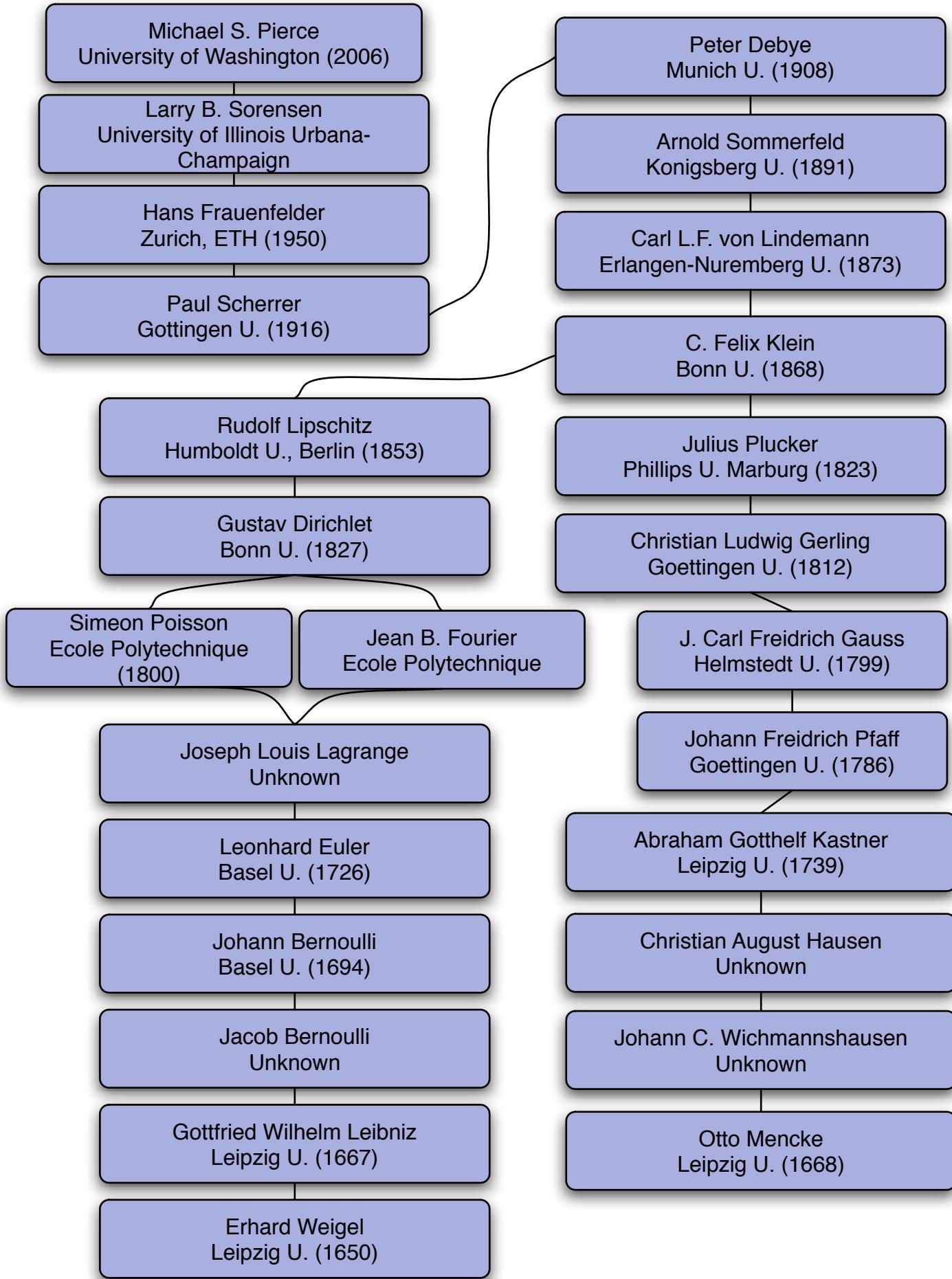
SPINOR

$$\underline{\Psi}(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_0(x) \\ \Psi_-(x) \end{pmatrix}$$

FOR SPIN 0

$$\underline{\Psi}(x) = (\Psi(x))$$

1
11



$$\ell = 1$$

$$L^2 = \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

every vector is an $e\vec{v}$

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \frac{i}{\sqrt{2}} \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$L_4 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$e^{\hat{L}_4 t} \Rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

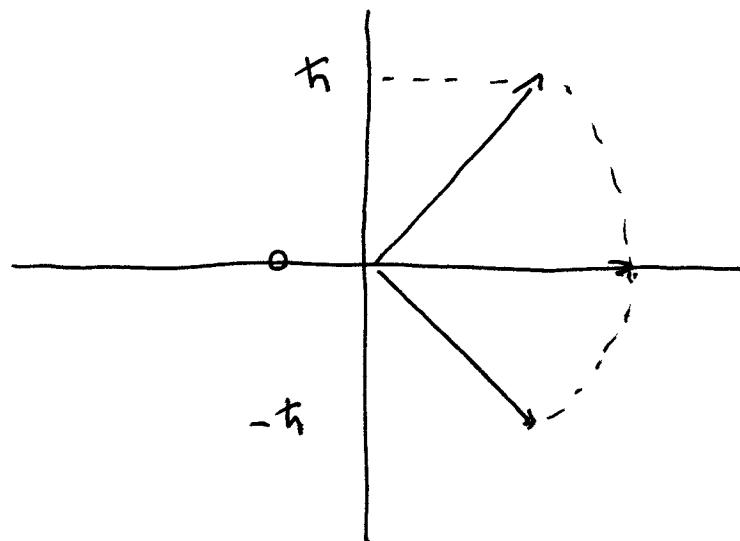
$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_0(x) \\ \psi_-(x) \end{pmatrix}$$

PROB AMP $S_2 = 1$
 PROB AMP $S_2 = 0$
 PROB AMP $S_2 = -1$

$|l, l\rangle$
 $|l, 0\rangle$
 $|l, -l\rangle$

$$\langle l^2 \rangle = l(l+1)\hbar^2 = 2\hbar^2$$

length = $\sqrt{2}\hbar$



$$\langle l^2 \rangle = l(l+1)\hbar^2 = 2\hbar^2$$

$$\text{LENGTH} = \sqrt{2}\hbar$$

$$\langle s_z \rangle = m\hbar$$

$$\langle s_x \rangle = 0$$

$$\langle s_y \rangle = 0$$

2. Consider a system initially in the state $|\psi(0)\rangle$ with the Hamiltonian H , where

$$|\psi(0)\rangle = N \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ in the } L_z \text{ basis, and where } H = (2\omega/\hbar)L^2 + (3\omega)L_z.$$

- (a) What angular momentum is described by the 3-component vector $|\psi(0)\rangle$? What is the length of this vector? What are the allowed z -projections?
- (b) Calculate the normalization constant N and the Hamiltonian matrix.

Hint: $L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$ and $L_z |l, m\rangle = m\hbar |l, m\rangle$.

- (c) Calculate the eigenvalues and the eigenvectors of the Hamiltonian.

Hint: H , L^2 and L_z all commute.

- (d) Calculate the time evolution of the state vector $|\psi(t)\rangle$ by expanding $|\psi(0)\rangle$ as a sum of energy eigenvectors and using the time evolution of the energy eigenvectors.
- (e) If the energy is measured at time t , what results can be found and with what probabilities will these results be found?
- (f) Calculate $\langle E \rangle$ and ΔE . Plot $P(E)$ vs. E and indicate $\langle E \rangle$ and ΔE on your plot.
- (g) If L^2 and L_z are measured at time t , what results can be found and with what probabilities will these results be found?

FOR $\ell = 1$

$$H = A L^2 + B L_z$$

$$L^2 \rightarrow \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_z \rightarrow \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$H \rightarrow \begin{pmatrix} 2A\hbar^2 + B\hbar & 0 & 0 \\ 0 & 2A\hbar^2 & 0 \\ 0 & 0 & 2A\hbar^2 - B\hbar \end{pmatrix}$$

GENERALIZED UNCERTAINTY RELATION

FOR ANY OBSERVABLES A and B

$$(\Delta A)^2 (\Delta B)^2 \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|^2$$

FOR L_x AND L_y

$$\begin{aligned} (\Delta L_x)^2 (\Delta L_y)^2 &\geq \left| \frac{1}{2i} \langle i\hbar L_z \rangle \right|^2 \\ &\geq \frac{\hbar^2}{4} |\langle L_z \rangle|^2 \end{aligned}$$

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle|$$

