ANGULAR MOMENTUM 2 JULY 30, 2007

DISCUSSED THE EV, E$\vec{V}$ PROBLEM FOR $\vec{L}$

$$L^2 \,|\ell, m\rangle = \ell(\ell+1)\hbar^2 \,|\ell, m\rangle$$

$$L_z \,|\ell, m\rangle = m\hbar \,|\ell, m\rangle$$

$$L_\pm \,|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\;\hbar \,|\ell, m\pm 1\rangle$$

EXPRESS $|\ell, m\rangle$ IN POSITION SPACE

EXPLAIN THE PHYSICS OF $|\ell, m\rangle$

OUTLINE DIFF EQ VERSION

NEXT STEP ... go into real space

$$|\, l, m \rangle$$

$$\langle \theta, \varphi \,|\, l, m \rangle = Y_{lm}(\theta, \varphi)$$

$$\langle l, m \,|\, \theta, \varphi \rangle = Y^*_{lm}(\theta, \varphi)$$

$$\langle l', m' \,|\, l, m \rangle = \delta_{ll'}\, \delta_{mm'}$$

$$\int Y^*_{l'm'}(\theta, \varphi)\, Y_{lm}(\theta, \varphi)\, d(\cos\theta)\, d\varphi = \delta_{ll'}\delta_{mm'}$$

Q: HOW TO FIND ALL THE Y_{lm}'s?

A: Solve the differential equation ... OR

JUST LIKE SHO, WE HAVE TWO CHOICES

$$L^2 \, |l,m\rangle = l(l+1)\hbar^2 \, |l,m\rangle$$

$$L_z \, |l,m\rangle = m\hbar \, |l,m\rangle$$

$$-\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{lm}(\theta,\varphi) = l(l+1)\hbar^2 \, Y_{lm}(\theta,\varphi)$$

$$-i\hbar \frac{\partial}{\partial \varphi} Y_{lm}(\theta,\varphi) = m\hbar \, Y_{lm}(\theta,\varphi)$$

OR

$$L_+ Y_{ll}(\theta,\varphi) = 0 \quad \Rightarrow \quad \begin{array}{l} Y_{ll} \\ \downarrow L_- \\ Y_{l,l-1} \\ \downarrow L_- \end{array}$$

$$\text{or} \quad L_- Y_{l,-l}(\theta,\varphi) = 0 \quad \Rightarrow \quad L_{l,-l} \uparrow L_+$$

IMPORTANT TECHNIQUE # 137

SEPARATION OF VARIABLES

$$Y_{\ell m}(\theta, \varphi) = f_{\ell m}(\theta)\; g_{\ell m}(\varphi)$$

$$L_z | \ell, m\rangle = m\hbar\, | \ell, m\rangle$$

$$-i\hbar \frac{\partial}{\partial \varphi}\left(f_{\ell m}(\theta)\, g_{\ell m}(\varphi)\right) = m\hbar \left(f_{\ell m}(\theta)\, g_{\ell m}(\varphi)\right)$$

$$-i\hbar\; \cancel{f_{\ell m}(\theta)}\, \frac{\partial g_{\ell m}(\varphi)}{\partial \varphi} = m\hbar\; \cancel{f_{\ell m}(\theta)}\; g_{\ell m}(\varphi)$$

$$\frac{\partial g}{\partial \varphi} = i m g \quad \Rightarrow \quad g = e^{i m \varphi}$$

So it separates ~~and~~

and it does not depend on ℓ

next step find $F_{\ell m}(\theta)$'s

$$L_+ \, Y_{\ell\ell}(\theta, \varphi) = 0$$

$$\cancel{\hbar e^{i\varphi}} \left[\frac{\partial}{\partial \theta} + i \cot\theta \frac{d}{d\varphi} \right] \left[F_{\ell\ell}(\theta) \, e^{i\ell\varphi} \right] = 0$$

$$\frac{dF_{\ell\ell}}{d\theta} \cancel{e^{i\ell\varphi}} + i \cot\theta \, F_{\ell\ell} \, i\ell \, \cancel{e^{i\ell\varphi}} = 0$$

$$\left[\frac{d}{d\theta} - \ell \cot\theta \right] F_{\ell\ell}(\theta) = 0$$

TRY $\quad F_{\ell\ell} = A \, (\sin\theta)^{\ell}$

$$\frac{d}{d\theta} A \, (\sin\theta)^{\ell} = A \, (\sin\theta)^{\ell-1} \cos\theta$$

$$A \, (\sin\theta)^{\ell-1} \cos\theta - \ell \frac{\cos\theta}{\sin\theta} A \, (\sin\theta)^{\ell} \stackrel{?}{=} 0$$

yep!

$$\Rightarrow \quad Y_{\ell\ell}(\theta, \varphi) = A_{\ell} \, (\sin\theta)^{\ell} \, e^{i\ell\varphi}$$

TO GET THE REST

$$L_- Y_{\ell \ell} = a Y_{\ell\, \ell-1}$$

then normalise using

$$\int Y_{\ell m}^* Y_{\ell m} \, d\Omega = 1$$

$$L_- = \hbar e^{-i\varphi} \left(\frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right)$$

SPHERICAL HARMONICS

$\ell = 0$

$$Y_{00}(\theta, \varphi) = \langle \theta, \varphi | 0, 0 \rangle = \frac{1}{\sqrt{4\pi}}$$

$$\int \left(\frac{1}{\sqrt{4\pi}} \right)^* \frac{1}{\sqrt{4\pi}} \, d\Omega = 1 \quad \checkmark$$

$l = 1$

$$Y_{11}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{i\varphi}$$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

note symmetry

$$Y_{1-1}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{-i\varphi}$$

$l = 2$

$$Y_{2\pm2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} (\sin\theta)^2 \; e^{\pm 2i\varphi}$$

$$Y_{1\pm1}(\theta, \varphi) = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \; e^{\pm i\varphi}$$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

August 4, 2006

Conclude Angular Momentum

$\vec{L}$ mechanical gyroscope moving mass

$\vec{\mu}$ magnetic moment moving charge

Q: so what is moving?

A: The probability is moving!

PROBABILITY CURRENT (FLUX)

$$|\psi(\vec{r})|^2 = \psi^*(\vec{r})\,\psi(\vec{r}) = \text{PROB DENSITY}$$

1d	L^{-1}	$\rho(x)\,dx$
2d	L^{-2}	$\rho(x,y)\,dA$
3d	L^{-3}	$\rho(\vec{r})\,dV$

$$\rho(\vec{r},t) = \psi^*(\vec{r},t)\,\psi(\vec{r},t)$$

WANT A CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

$$\vec{j} = \left(\frac{-i\hbar}{2m}\right)\left[\psi^*\nabla\psi - \nabla\psi^*\,\psi\right]$$

$$= \frac{\hbar}{m}\,\mathrm{Im}\left[\psi^*\nabla\psi\right] = \mathrm{Re}\left[\psi^*\,\frac{\hbar}{im}\nabla\psi\right]$$

$\Rightarrow$ IF ψ is real, $\vec{j} = 0$

$\vec{j} = 0$ unless there is a gradient in the phase

PROB CONSERVATION $\Leftrightarrow$ $H = H^{+}$

NOW, TURN THIS AROUND

$$\psi(\vec{r}, t) = \sqrt{\rho(\vec{r}, t)} \; \exp\left(i \, \varphi(\vec{r}, t) \right)$$

$\rho > 0$

φ real

WHAT DOES φ MEAN PHYSICALLY?

$$\psi^{*} \nabla \psi = \sqrt{\rho} \; \nabla\left(\sqrt{\rho}\right) + \frac{i}{\hbar} \rho \nabla \varphi$$

$$\Rightarrow \quad \vec{j}(\vec{r}, t) = \frac{\rho \nabla \varphi}{m}$$

SPATIAL VARIATION OF PHASE $\Rightarrow$ PROB CURRENT

PROB FLUX

APPLY TO HYDROGEN

$$\vec{j}_{nlm}(\vec{r}) = \frac{\hbar}{2mi} \Phi^*_{nlm}(\vec{r}) \nabla \Phi_{nlm}(\vec{r}) + cc$$

FOR ANY WAVEFCN $\psi(\vec{r})$

$$\psi(\vec{r}) = A(\vec{r})\, e^{i\phi(\vec{r})}$$

$$A(\vec{r}) \geq 0$$

$$0 \leq \varphi(\vec{r}) \leq 2\pi$$

$$\rho(\vec{r})\, dV = A^2(\vec{r})\, dV$$

PROB DENSITY DEPENDS ONLY ON AMPLITUDE

$$\vec{j} = \frac{\hbar}{m} A^2(\vec{r})\, \vec{\nabla} \varphi(\vec{r})$$

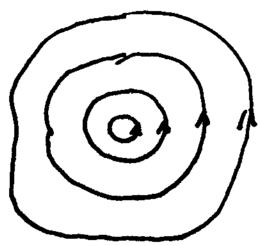
PROB CURRENT DEPENDS ON AMP AND PHASE

FOR HYDROGEN

$$A_{n\ell m}(\vec{r}) = \left| R_{n\ell}(r) \right| \left| Y_{\ell m}(\theta, \varphi) \right|$$

$$\varphi_{n\ell m}(\vec{r}) = m \hbar$$

$$\vec{j}(\vec{r}) = \frac{\hbar}{m} \frac{m}{r \sin\theta} \, \rho_{n\ell m}(\vec{r}) \, \hat{e}_\varphi$$



$m > 0$ CCW

$m < 0$ CW

$m = 0$ NO PROB CURRENT

HOW MUCH ANGULAR MOMENTUM?

$$dL_z = (\mu \vec{r} \times \vec{j}) \, d^3r$$

μ = reduced mass
also to avoid
confusion with m

$$L_z = \mu \int (\vec{r} \times \vec{j}) \cdot \hat{e}_z \, d^3r$$

$$= \mu \int |\vec{j}| \; r \sin\theta \; d^3r$$

$$= m\hbar \int \rho_{n\ell m}(\vec{r}) \, d^3r$$

$$\boxed{L_z = m\hbar}$$

TIME-DEPENDENCE

$$\Phi_{n\ell m}(\vec{r})$$

$$\Phi_{n\ell m}(\vec{r}, t) = \Phi_{n\ell m}(\vec{r})\, e^{-iE_n t/\hbar}$$

$$= R_{n\ell}(r)\, P_{\ell m}(\theta)\, e^{im\varphi}\, e^{-iE_n t/\hbar}$$

$$\propto e^{i(m\varphi - \omega_n t)}$$

$$\omega_n = E_n/\hbar$$

$\Rightarrow$ BEAUTIFUL SPINNING BALLS!

SO WHAT IS SPINNING?

OHANIAN AJP **54** 500 (1986)

KITA AJP **68** 259 (2000)

$\vec{L}$ THE PROB CURRENT IS FLOWING

$\vec{\mu}_L$

$\vec{S}$ THE FIELD IS SPINNING

$\vec{S}$ CIRCULATING FLOW OF ENERGY BELINFANTE 1939

$\vec{\mu}_S$ CIRCULATING FLOW OF CHARGE GORDON 1928

THIS IS A CLASSICAL PICTURE!

QM: no $\vec{E}$

no $\vec{B}$

only real and virtual photons!

WHAT IS SPIN?

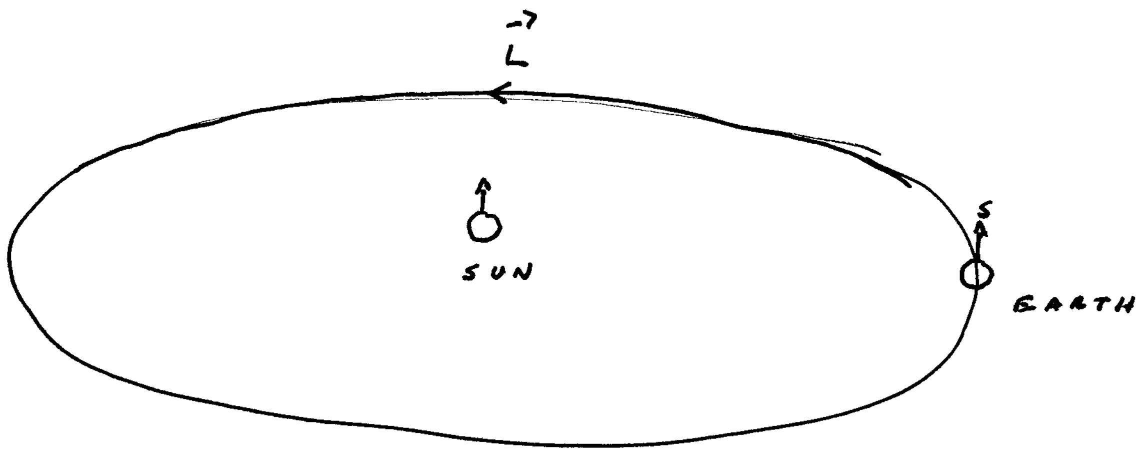
Real particles have intrinsic angular momentum.

The associated degree of freedom is called spin.

Fixed property of the particle; you cannot change it.

You can change $\vec{L}$.

CLASSICAL PICTURE



Okay, maybe the electron is like a little ~~spinning~~ spinning ball. How fast does it spin?

HOW BIG IS AN ELECTRON?

TWO CLASSICAL ANSWERS

(1) The classical radius r_0

$$\begin{pmatrix}\text{COULOMB} \\ \text{ENERGY}\end{pmatrix} = \begin{pmatrix}\text{REST MASS} \\ \text{ENERGY}\end{pmatrix}$$

$$\frac{e^2}{r_0} = mc^2$$

$$r_0 = \frac{e^2}{mc^2} = 2.8179 \times 10^{-15}\ m$$

$$r_0 \sim 3 \times 10^{-5}\ \text{Å} = \frac{a_0}{(137)^2}$$

MUCH SMALLER THAN AN ATOM

PHYSICAL MEANING: E & M cross section
elastic photon scattering cross section

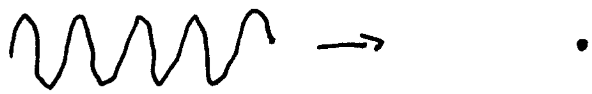
(2) The Compton radius $r_c = 3.8616 \times 10^{-13}\ m$

$$r_c = \frac{\hbar}{mc} \sim 4 \times 10^{-3}\ \text{Å}$$

$$r_c = \frac{a_0}{137} = 137\, r_0$$

PHYSICAL MEANING: inelastic electron scattering cross section

THOMPSON SCATTERING



PHOTON

FREE ELECTRON

$$\vec{E} = E_0 \hat{n} \cos(kz - \omega t)$$

$$\vec{F} = m\vec{a}$$

$$-eE_0 = m \frac{d^2x}{dt^2}$$

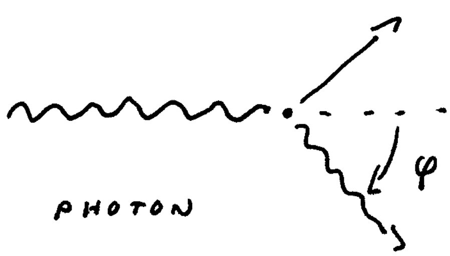
$$a = \frac{-e}{m} E_0$$

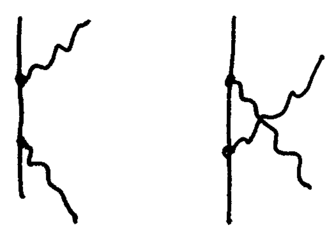
$$\frac{d\sigma}{d\Omega} = r_0^2 \left| \hat{\epsilon}_i \cdot \hat{\epsilon}_s \right|^2$$

$$= r_0^2 \sin^2\theta$$

SEAGULL DIAGRAM

COMPTON SCATTERING





$$\Delta\lambda = \frac{h}{mc}(1 - \cos\varphi) = 0.02426\ \text{Å}\ (1 - \cos\varphi)$$

$$\frac{d\sigma}{d\Omega} = r_c^2 = (137)^2 r_0^2$$

HOW FAST WOULD THE ELECTRON SPIN?

$$L = \frac{1}{2}\hbar$$

$$L = I\omega = \left(\frac{2}{5} m R^2\right)\left(\frac{v}{R}\right) = \frac{2}{5} m v R$$

$$\Rightarrow v = \frac{5}{4}\frac{\hbar}{mR}$$

$R = r_0 \Rightarrow v_0 = 171\,c$ SUPERLUMINAL!

$R = r_c \Rightarrow v_0 = 1.25\,c$ STILL SUPERLUMINAL...

PRESENT UNDERSTANDING:

The electron is a point particle.

Nothing inside to spin!

no mass => no angular momentum

no charge => no magnetic moment

The proton has internal structure 3 quarks

intrinsic angular momentum of the quarks

orbital angular momentum of the quarks

THIS WEEK: SPIN

UP UNTIL NOW, SCHRODINGER EQN

$$H\,|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$$

$$H\,\psi(x) = i\hbar \frac{d}{dt}\psi(x)$$

DOES NOT INCLUDE SPIN!

TWO WAYS TO INCLUDE SPIN:

(1) PAULI- SCHRODINGER EQN

PAULI
FRAUENFELDER
SORENSEN
YOU

$$H\,\Psi(x) = i\hbar \frac{d}{dt}\Psi(x)$$

NON-RELATIVISTIC, ANY SPIN
works for

FOR SPIN 1/2

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}$$

2 COMPONENT OBJECT IS CALLED A SPINOR

FOR SPIN 1

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_0(x) \\ \psi_-(x) \end{pmatrix}$$

FOR SPIN 0 $\Psi(x) = (\psi(x))$

THE ALGEBRA OF SPIN

$$\vec{S} \times \vec{S} = i\hbar \vec{S}$$

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

FOR SPIN 1/2

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \Rightarrow \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

PAULI

MATRICES

σ_x, σ_y and σ_z

ARE THE

large number of such repetitions into our teaching. It may bore a few poor students, but almost all benefit.

IX. THE IMPORTANCE OF CALCULATING WITH NUMBERS

The world has changed quite a bit in the past 30 or 40 years. When I was an undergraduate we learned that there are only four angles in this world, namely, 30°, 45°, 60°, and 90°. Furthermore, all measurements are divisible by 2, often by 3 and 4, and, curiously, not infrequently by 49. It came as something of a surprise, when I embarked on experimental research, to find that most measurements are embarrassingly inelegant numbers, and that angles, as often as not, wander somewhere between those canonical values we learned in class.

I understand why my student problems had such remarkably simple numbers. It was just that nobody liked long division, and the alternatives were few.

Of course, we did have pocket calculators, or, more accurately, hip calculators. But they were hard to use, required a fair amount of manual dexterity to get results accurate to three figures. They were slow, and very expensive. My present shirt pocket calculator, whose batteries have already lasted two years, not only gives me nine figures and hyperbolic functions, but even does arithmetic in hexadecimal. It cost $14.29. When students grumble about the expense, I delight to tell them that my 1945 log log duplex trig calculator, required on every test, cost me $176 (in 1985 dollars, using an average inflation rate of 5% per annum).

My point is this. Calculating power today is dirt cheap. It costs far less than textbooks and it lasts from one course to another. It gives us the opportunity to teach the physics of the real world rather than the physics of the textbook. Our students, furthermore, at least our technically inclined students, will spend their lives making use of these calculators.

This needs to be recognized in what we do in our calculus-based physics. Thirty, 60, and 90 ought to be reduced to their proper place. In my classes, tests, if not textbook problems, have angles like 27.6°. Automobiles have speeds of 37 km/h. Electrons move in orbits of radius 0.26 centimeters. The only difficulties students have with this is that too frequently their calculations seem to be accurate to one part in ten to the ninth.

All this is fine for the science and engineering students. What about the liberal arts students? Years ago, I would not have dreamed of asking them to buy slide rules. I hesitate now to ask them to have calculators, yet I note that almost all do. I continue to give them problems with nice numbers, yet I find them using a calculator to divide 8 by 4. I'm beginning to think that they too should always deal with real-world numbers. If they have to use a calculator to divide 8 by 4, they might as well be dividing 8.63 by 4.79.

By now I have run the device of numbers into the ground. It has given me a handy framework to air my grievances about and my hopes for physics teaching. I hope I will hear more about these dirty problems of physics teaching in less than ideal circumstance from the rest of you. Let me thank the AAPT once again for giving me this award. Thank you all for hearing me out.

What is spin?

Hans C. Ohanian
Rensselaer Polytechnic Institute, Troy, New York 12180

(Received 5 February 1984; accepted for publication 1 May 1985)

According to the prevailing belief, the spin of the electron or of some other particle is a mysterious internal angular momentum for which no concrete physical picture is available, and for which there is no classical analog. However, on the basis of an old calculation by Belinfante [Physica **6**, 887 (1939)], it can be shown that the spin may be regarded as an angular momentum generated by a circulating flow of energy in the wave field of the electron. Likewise, the magnetic moment may be regarded as generated by a circulating flow of charge in the wave field. This provides an intuitively appealing picture and establishes that neither the spin nor the magnetic moment are "internal"—they are not associated with the internal structure of the electron, but rather with the structure of its wave field. Furthermore, a comparison between calculations of angular momentum in the Dirac and electromagnetic fields shows that the spin of the electron is entirely analogous to the angular momentum carried by a classical circularly polarized wave.

I. INTRODUCTION

When Goudsmit and Uhlenbeck proposed the hypothesis of the spin of the electron, they had in mind a mechanical picture of the electron as a small rigid body rotating about its axis. Such a picture had earlier been considered by Kronig and discarded on the advice of Pauli, Kramers, and Heisenberg, who deemed it a fatal flaw of this picture that the speed of rotation—calculated from the magnitude of the spin and a plausible estimate of the radius of the electron—was in excess of the speed of light. However, the great success of the spin hypothesis in explaining the Zeeman effect and the doublet structure of spectral lines quickly led to its acceptance.[1] Since the naive mechanical picture of spin proved untenable, physicists were left with the concept of spin minus its physical basis, like the grin of the Cheshire cat. Pauli pontificated that spin is "an essentially quantum-mechanical property,...a classically not describable two-valuedness"[2] and he insisted that the lack of a concrete picture was a satisfactory state of affairs:

> After a brief period of spiritual and human confusion, caused by a provisional restriction to 'Anschaulichkeit', a general agreement was reached following the substitution of abstract mathematical symbols, as for instance psi, for concrete pictures. Especially the concrete picture of rotation has been replaced by mathematical characteristics of the representations of rotations in three-dimensional space.[3]

Thus physicists gradually came to regard the spin as an abstruse quantum property of the electron, a property not amenable to physical explanation.

Judging from statements found in modern textbooks on atomic physics and quantum theory, one would think our understanding of spin (or the lack thereof) has not made any progress since the early years of quantum mechanics. The spin is usually said to be a nonorbital, "internal," "intrinsic," or "inherent" angular momentum (the words are often used interchangeably, although they should not be), and it is often treated as an irreducible entity that cannot be explained further. Sometimes the (unsubstantiated) suggestion is made that the spin is due to an (unspecified) internal structure of the electron.[4] And sometimes the consolation is offered that the spin arises in a natural way from Dirac's equation[5] or from the analysis of the representations of the Lorentz group. It is true that the Dirac equation contains a wealth of information about spin: The equation tells us that the spinor wavefunctions are indeed endowed with a spin angular momentum of $\hbar/2$, it supplies the mathematical description of the kinematics of a free-electron or other particle of spin one-half, and—in conjunction with the principle of minimal coupling—it supplies the equations governing the dynamics of a charged particle immersed in a electromagnetic field, equations which directly yield the correct value of the gyromagnetic ratio for the electron. It is also true that the analysis of the representations of the Lorentz group is very informative: The analysis tells us that the quantum-mechanical wavefunctions must be certain types of tensors or spinors characterized by a value of the mass and (if the mass is not negative) an integer or half-integer value of the spin. But in all of this the spin merely plays the role of an extra, nonorbital angular momentum of unknown etiology. Thus the mathematical formalism of the Dirac equation and of group theory demands the existence of the spin to achieve the conservation of angular momentum and to construct the generators of the rotation group, but fails to give us any understanding of the physical mechanism that produces the spin.

The lack of a concrete picture of the spin leaves a grievous gap in our understanding of quantum mechanics. The prevailing acquiescence to this unsatisfactory situation becomes all the more puzzling when one realizes that the means for filling this gap have been at hand since 1939, when Belinfante[6] established that the spin could be regarded as due to a circulating flow of energy, or a momentum density, in the electron wave field. He established that this picture of the spin is valid not only for electrons, but also for photons, vector mesons, and gravitons—in all cases the spin angular momentum is due to a circulating energy flow in the fields. Thus contrary to the common prejudice, the spin of the electron has a close classical analog: It is an angular momentum of exactly the same kind as carried by the fields of a circularly polarized electromagnetic wave. Furthermore, according to a result established by Gordon[7] in 1928, the magnetic moment of the electron is due to the circulating flow of charge in the electron wave field. This means that neither the spin nor the magnetic moment are internal properties of the electron—they have nothing to do with the internal structure of the electron, but only with the structure of its wave field.

Unfortunately, this clear picture of the physical origin of the spin and of the magnetic moment has not received the wide recognition it deserves, perhaps because neither Belinfante nor Gordon loudly proclaimed that their calculations provided a new physical explanation of the spin and of the magnetic moment. These calculations are sometimes reproduced in texts on quantum field theory,[8] but usually without any commentary on their physical interpretation. In the present paper, it is my objective to revive these forgotten explanations of the spin and the magnetic moment in the hope that the intuitive picture of circulating energy and charge will become part of the lore learned by all students of physics. I want to emphasize that, in contrast to some other attempts at explaining the spin,[9] the present explanation is completely consistent with the standard interpretation of quantum mechanics.

A crucial ingredient in Belinfante's calculation of the spin angular momentum is the use of the symmetrized energy-momentum tensor. It is well known that in a field theory we can construct several energy-momentum tensors, all of which satisfy the conservation law $\partial_\nu T^{\mu\nu} = 0$, and all of which yield the same net energy ($\int T^{00}\, d^3x$) and momentum ($\int T^{k0}\, d^3x$) as the canonical energy-momentum tensor.[10] These diverse energy-momentum tensors differ by terms of the form $\partial_\alpha U^{\mu\nu\alpha}$, which are antisymmetric in the last two indices ($U^{\mu\nu\alpha} = -U^{\mu\alpha\nu}$), and therefore identically satisfy the conservation law $\partial_\nu \partial_\alpha U^{\mu\nu\alpha} = 0$. Belinfante showed that by a suitable choice of the term $\partial_\alpha U^{\mu\nu\alpha}$, it is always possible to construct a *symmetrized* energy-momentum tensor ($T^{\mu\nu} = T^{\nu\mu}$). The symmetrized energy-momentum tensor has the distinctive advantage that the angular momentum calculated directly from the momentum density T^{k0} is a conserved quantity (in the absence of external torques). This means that the momentum density gives rise to both orbital angular momentum and spin angular momentum. If instead of the symmetrized energy-momentum tensor, we were to use the unsymmetrized canonical energy-momentum tensor, then the momentum density would not give rise to the spin angular momentum. This does not mean that the spin would vanish from the theory—an examination of the conservation law for angular momentum shows that the spin emerges as a mysterious extra quantity that must be added to the orbital angular momentum to achieve conservation—but the simple and clear physical mechanism underlying spin would vanish. I will take it for granted that the symmetrized energy-momentum tensor is the correct energy-momentum