LECTURE 12: ANGULAR MOMENTUM

TODAY: MOTIVATION

BACKGROUND

LADDER OP

NEXT TIME: DIFF EQ SOLN

SPHERICAL HARMONICS

LEGENDRE POLYNOMIALS

MOTIVATION: We want to solve hydrogen problem

1d $\rightarrow$ 3d

classical analogue: planetary motion

$\vec{L}$ conserved

SHO $\quad H = P^2 + X^2 \qquad\qquad H = P_x^2 + P_y^2 + P_z^2 + V(\vec{x})$

$a = x - iP \qquad\qquad = \left[\frac{P_r^2}{2m} + V(\vec{r})\right] + \frac{L^2}{2I}$

$a^\dagger = x + ip$

$L^2 \qquad L_z$

$L_\pm = L_x \pm i L_y$

CLASSICALLY $\qquad \vec{L} = \vec{r} \times \vec{p}$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$= (y p_z - z p_y)\hat{i} - (x p_z - z p_x)\hat{j} + (x p_y - y p_x)\hat{k}$$

QM

$$L_{x\,op} = y_{op} p_{z\,op} - z_{op} p_{y\,op}$$

$$L_{y\,op} = -(x_{op} p_{z\,op} - z_{op} p_{x\,op})$$

$$L_{z\,op} = x_{op} p_{y\,op} - y_{op} p_{x\,op}$$

POSITION SPACE

$$x_{op} \to x \qquad p_{x\,op} \to -i\hbar \frac{\partial}{\partial x}$$

$$y_{op} \to y \qquad p_{y\,op} \to -i\hbar \frac{\partial}{\partial y}$$

$$z_{op} \to z \qquad p_{z\,op} \to -i\hbar \frac{\partial}{\partial z}$$

MOMENTUM SPACE

$$x_{op} \to i\hbar \frac{\partial}{\partial p_x} \qquad p_{x\,op} \to p_x$$

$$y_{op} \to i\hbar \frac{\partial}{\partial p_y} \qquad p_{y\,op} \to p_y$$

$$z_{op} \to i\hbar \frac{\partial}{\partial p_z} \qquad p_{z\,op} \to p_z$$

DEEP IDEA HERE

SYMMETRY $\longrightarrow$ CONSERVATION LAW

TRANSLATIONAL INVARIANCE $\longrightarrow$ LINEAR MOMENTUM CONSERVATION

ROTATIONAL INVARIANCE $\longrightarrow$ ANGULAR MOMENTUM CONSERVATION

TIME TRANSLATION INVARIANCE $\longrightarrow$ ENERGY CONSERVATION

UNITARY OPERATORS

TRANSLATION BY $\vec{a} = (a_x, a_y, a_z)$

$$T(\vec{a}) = e^{-i\vec{p}\cdot\vec{a}/\hbar}$$

$$T(\vec{b})\, T(\vec{a}) = T(\vec{a}+\vec{b})$$

$$[p_i, p_j] = \delta_{ij}$$

ROTATIONS

$$R(\vec{\theta}) = e^{-i\vec{L}\cdot\vec{\theta}/\hbar}$$

$$R(\theta_z) = e^{-iL_z\cdot\theta_z/\hbar}$$

in coordinates

$$L_z = x_{op}\, p_{y\,op} - y_{op}\, p_{x\,op}$$

$$= -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

very messy to calculate ...

TIME TRANSLATION

$$U(t) = e^{-iHt/\hbar}$$

ROTATIONAL SYMMETRY $\Rightarrow$ WORK IN SPHERICAL COORDINATES

$$(x, y, z) \longrightarrow (r, \theta, \varphi)$$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$L_x = i\hbar \left[\sin\varphi \frac{\partial}{\partial \theta} + \cos\varphi \cot\theta \frac{\partial}{\partial \varphi} \right]$$

$$L_y = i\hbar \left[-\cos\varphi \frac{\partial}{\partial \theta} + \sin\varphi \cot\theta \frac{\partial}{\partial \varphi} \right]$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right]$$

$$L^2 = -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right]$$

LADDER OPERATORS $L_\pm = L_x \pm i L_y$

$$L_\pm = \pm\hbar e^{\pm i\varphi} \left[\frac{\partial}{\partial \theta} \pm i \cot\theta \frac{\partial}{\partial \varphi} \right]$$

EIGENVALUE PROBLEM

$$L^2 \, | \, l, m \rangle = l(l+1)\hbar^2 \, | \, l, m \rangle$$

$$L_z \, | \, l, m \rangle = m\hbar \, | \, l, m \rangle$$

IF H IS ROTATIONALLY INVARIANT

$$[H, L_i] = 0$$

$$[H, L^2] = 0$$

$$[H, L_z] = 0$$

(H eigenfcns) = (radial eigenfcn)(spherical harmonic)

LADDER OPERATOR SOLUTION

$$L^2 = \vec{L} \cdot \vec{L} = L_x^2 + L_y^2 + L_z^2$$

$$[L_i, L_j] = i\hbar \, \epsilon_{ijk} L_k$$

EINSTEIN SUMMATION OVER K.

ϵ_{ijk} CALLED TOTALLY ANTISYMMETRIC TENSOR

1 2 3	2 1 3	any two equal
2 3 1	1 3 2	
3 1 2	3 2 1	
$\epsilon_{ijk} = +1$	$\epsilon_{ijk} = -1$	$\epsilon_{ijk} = 0$

1 2 3

x y z

ANOTHER VIEW: $\epsilon_{123} = +1$

changes sign when interchange two indices

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

$$\vec{L} \times \vec{L} = i\hbar \vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix}$$

CROSS PRODUCTS

$$\vec{c} = \vec{a} \times \vec{b}$$

$$c_i = \epsilon_{ijk} \, a_j \, b_k$$

EINSTEIN SUM OVER j, k.

SINCE L_x, L_y, L_z DO NOT COMMUTE...

CHOOSE L^2, L_z

RAISING AND LOWERING OPERATORS

$$L_+ = L_x + i L_y$$

$$L_- = L_x - i L_y$$

WORK OUT COMMUTATORS

$$[L^2, L_i] = 0$$

$$[L^2, L_\pm] = 0$$

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

PROCEED AS BEFORE

$$L^2 \,|\, \alpha, \beta \rangle = \alpha \,|\, \alpha, \beta \rangle$$

$$L_z \,|\, \alpha, \beta \rangle = \beta \,|\, \alpha, \beta \rangle$$

$$[\, L_z ,\ L_+] = +\hbar L_+$$

$$L_z L_+ - L_+ L_z = \hbar L_+$$

$$L_z \left[L_+ \,|\, \alpha, \beta \rangle \right] = (\hbar L_+ + L_+ L_z) \,|\, \alpha, \beta \rangle$$

$$= (\beta + \hbar) \left[L_z \,|\, \alpha, \beta \rangle \right]$$

CONCLUDE

$L_+ \,|\, \alpha, \beta \rangle$ is $e\vec{v}$ of L_z with ev $\beta + \hbar$

$L_- \,|\, \alpha, \beta \rangle$ $\qquad\qquad$ $\beta - \hbar$

STEP SIZE $= \hbar$

HOW ABOUT L^2

$$[L^2, L_+] = 0$$

$$L^2 L_+ - L_+ L^2 = 0$$

$$L^2 [L_+ |\alpha, \beta\rangle] = L_+ [L^2 |\alpha, \beta\rangle]$$

$$= \alpha L_+ |\alpha, \beta\rangle$$

$L_+ |\alpha, \beta\rangle$ is e.v. of L^2 with e.v. α

$L_- |\alpha, \beta\rangle$ $\qquad\qquad\qquad\qquad\qquad\qquad$ α

NEW FEATURE!

This ladder has a top and a bottom

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 - L_z^2 = L_x^2 + L_y^2$$

$$\langle \alpha, \beta | L^2 - L_z^2 | \alpha, \beta\rangle = \langle \alpha, \beta | L_x^2 + L_y^2 | \alpha, \beta\rangle$$

$$\alpha - \beta^2$$

$$(\alpha - \beta^2) \langle \alpha, \beta | \alpha, \beta \rangle = \langle \alpha, \beta | L_x^2 + L_y^2 | \alpha, \beta \rangle \geq 0$$

$$\beta^2 \leq \alpha$$

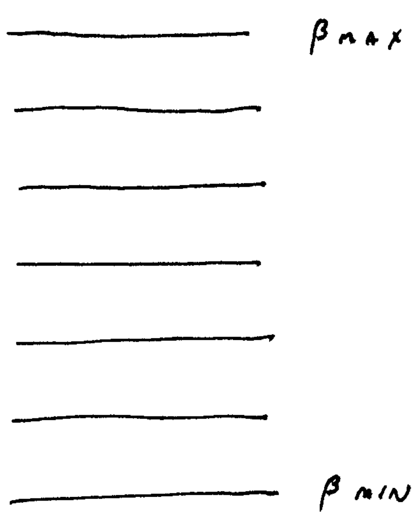
physically, given a total angular momentum, there is a maximum z projection

TOP AND BOTTOM TO LADDER

$$L_+ | \alpha, \beta_{MAX} \rangle = 0$$

$$L_- | \alpha, \beta_{MIN} \rangle = 0$$

next show $\beta_{MAX} = -\beta_{MIN}$



$$\beta_{MAX} = \hbar j$$

$$\alpha = j(j+1)\hbar^2$$

OH, MY GOSH! WE ACCIDENTALLY SOLVED A MORE GENERAL PROBLEM!

ORBITAL ANGULAR MOMENTUM L^2, L_z CAN ONLY HAVE INTEGRAL L!

TOTAL ANGULAR MOMENTUM J^2, J_z CAN HAVE INTEGER OR HALF INTEGER J...

SPIN ANGULAR MOMENTUM S^2, S_z CAN HAVE INTEGER OR HALF INTEGER S

$$\vec{J} = \vec{L} + \vec{S}$$

NEXT QUARTER $\vec{J}$, and $\vec{S}$: adding angular momentum

CLEBSCH-GORDON COEFF'S

GRIFFITH'S QUOTE...

LET'S CONSIDER SOME EXAMPLES

If you think this is starting to sound like mystical numerology, I don't blame you. We will not be using the Clebsch-Gordon tables much in the rest of this book, but I wanted you to know where they fit into the scheme of things, in case you encounter them later on. In a mathematical sense this is all applied group theory---what we are talking about is the decomposition of the direct product of two irreducible representations of the rotation group into a direct sum of irreducible representations (you can quote that, to impress your friends).

ORBITAL ANGULAR MOMENTUM

$$\ell = 0, 1, 2, 3, \ldots$$

$$m = -\ell, \ldots, 0, \ldots \ell$$

$$\boxed{\begin{aligned} L^2 \,|\ell, m\rangle &= \ell(\ell+1)\hbar^2 \,|\ell, m\rangle \\ L_z \,|\ell, m\rangle &= m\hbar \,|\ell, m\rangle \\ L_\pm \,|\ell, m\rangle &= \sqrt{\ell(\ell+1) - m(m\pm1)}\,\hbar \,|\ell, m\pm1\rangle \end{aligned}}$$

SPECIAL CASES:

$$\ell = 0 \quad \Rightarrow m = 0$$

$$|0,0\rangle$$

$$L^2 \,|0,0\rangle = 0\,\hbar^2 \,|0,0\rangle$$

$$L_z \,|0,0\rangle = 0\,\hbar \,|0,0\rangle$$

$\ell = 1 \qquad m = -1, 0, +1$

$|1,1\rangle$

$|1,0\rangle$

$|1,-1\rangle$

$$L^2 |1, m\rangle = 1(1+1)\hbar^2 |1, m\rangle = 2\hbar^2 |1, m\rangle$$

$$L_z |1, m\rangle = m\hbar |1, m\rangle$$

$\ell = 2 \qquad m = -2, -1, 0, +1, +2$

$|2,2\rangle$

$|2,1\rangle$

$|2,0\rangle$

$|2,-1\rangle$

$|2,-2\rangle$

$$L^2 |2, m\rangle = 6\hbar^2 |2, m\rangle$$

$$L_z |2, m\rangle = m\hbar |2, m\rangle$$

SEMICLASSICAL VECTOR MODEL

length of ang momentum vector $\sqrt{\ell(\ell+1)\hbar^2}$

z projection of $\vec{L}$ $m\hbar$

$\ell = 1$ $\sqrt{\ell(\ell+1)\hbar^2} = \sqrt{2}\,\hbar$

$m\hbar = -\hbar,\ 0,\ +\hbar$

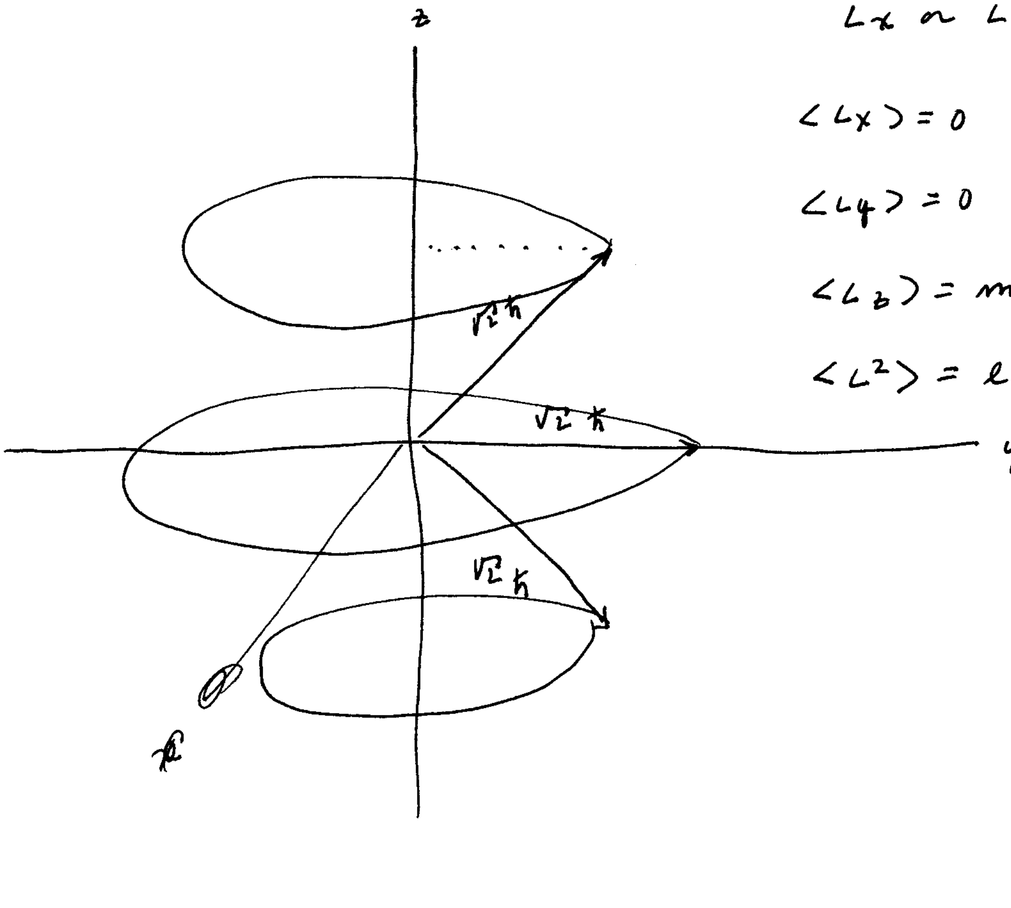
L_z does not commute with L_x or L_y $\Rightarrow$

$\langle L_x \rangle = 0$

$\langle L_y \rangle = 0$

$\langle L_z \rangle = m\hbar$

$\langle L^2 \rangle = \ell(\ell+1)\hbar^2$



GENERALIZED UNCERTAINTY RELATION

$$(\Delta A)^2 \quad (\Delta B)^2$$

$$\sigma_A^2 \quad \sigma_B^2 \geq \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2$$

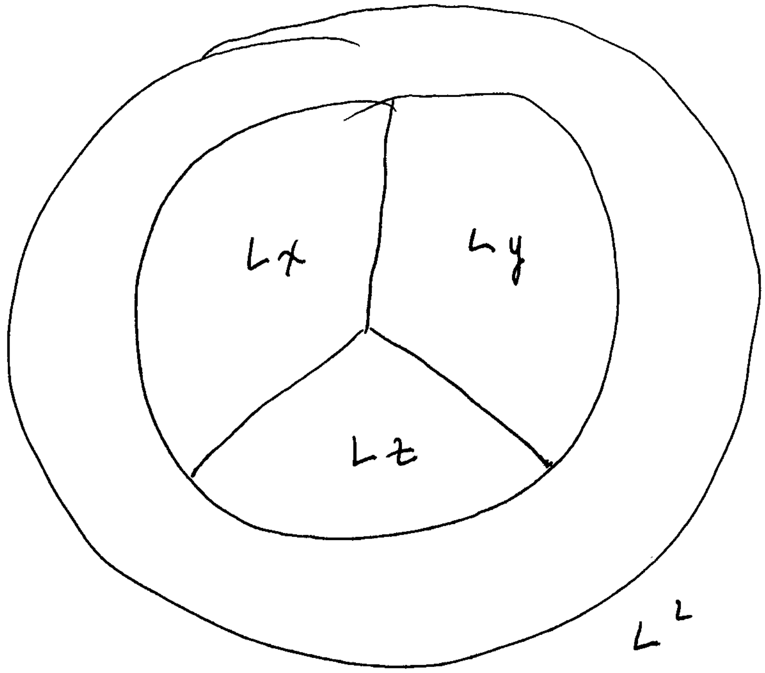
$$(\Delta L_x)^2 \quad (\Delta L_y)^2$$

$$\sigma_{L_x}^2 \quad \sigma_{L_y}^2 \geq \left(\frac{1}{2i} \langle i \hbar L_z \rangle \right)^2$$

$$\geq \frac{\hbar^2}{4} \langle L_z \rangle^2$$

$$\Delta L_x \quad \Delta L_y$$

$$\sigma_{L_x} \quad \sigma_{L_y} \geq \frac{\hbar}{2} \left| \langle L_z \rangle \right|$$


Lx
Ly
Lz
L2

NEXT STEP ... go into real space

$$| \ell, m \rangle$$

$$\langle \theta, \varphi | \ell, m \rangle = Y_{\ell m}(\theta, \varphi)$$

$$\langle \ell, m | \theta, \varphi \rangle = Y^*_{\ell m}(\theta, \varphi)$$

$$\langle \ell', m' | \ell, m \rangle = \delta_{\ell \ell'} \delta_{m m'}$$

$$\int Y^*_{\ell' m'}(\theta, \varphi) \; Y_{\ell m}(\theta, \varphi) \; d(\cos\theta) \, d\varphi = \delta_{\ell\ell'}\delta_{mm'}$$

Q: HOW TO FIND ALL THE $Y_{\ell m}$'s ?

A: Solve the differential equation ... OR

JUST LIKE SHO, WE HAVE TWO CHOICES

$$L^2 \, |l, m\rangle = l(l+1)\hbar^2 \, |l, m\rangle$$

$$L_z \, |l, m\rangle = m\hbar \, |l, m\rangle$$

$$-\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{lm}(\theta, \varphi) = l(l+1)\hbar^2 \, Y_{lm}(\theta, \varphi)$$

$$-i\hbar \frac{\partial}{\partial \varphi} Y_{lm}(\theta, \varphi) = m\hbar \, Y_{lm}(\theta, \varphi)$$

OR

$$L_+ Y_{ll}(\theta, \varphi) = 0 \quad \Rightarrow \quad Y_{ll} \xrightarrow{L_-} Y_{l,l-1} \xrightarrow{L_-}$$

$$\text{or} \quad L_- Y_{l,-l}(\theta, \varphi) = 0 \quad \Rightarrow \quad L_{l,-l} \xrightarrow{L_+}$$

IMPORTANT TECHNIQUE # 137

SEPARATION OF VARIABLES

$$Y_{\ell m}(\theta, \varphi) = f_{\ell m}(\theta)\, g_{\ell m}(\varphi)$$

$$L_z | \ell, m\rangle = m\hbar\, | \ell, m\rangle$$

$$-i\hbar \frac{\partial}{\partial \varphi}\left(f_{\ell m}(\theta)\, g_{\ell m}(\varphi)\right) = m\hbar \left(f_{\ell m}(\theta)\, g_{\ell m}(\varphi)\right)$$

$$-i\hbar\, \cancel{f_{\ell m}(\theta)}\, \frac{\partial g_{\ell m}(\varphi)}{\partial \varphi} = m\hbar\, \cancel{f_{\ell m}(\theta)}\, g_{\ell m}(\varphi)$$

$$\frac{\partial g}{\partial \varphi} = i m g \quad \Rightarrow \quad g = e^{i m \varphi}$$

So it separates ~~and~~

and it does not depend on ℓ

next step find $F_{\ell m}(\theta)$'s

$$L_+ \; Y_{\ell\ell}(\theta, \varphi) = 0$$

$$\cancel{\hbar e^{i\varphi}} \left[\frac{\partial}{\partial \theta} + i \cot\theta \frac{d}{d\varphi} \right] \left[F_{\ell\ell}(\theta) \, e^{i\ell\varphi} \right] = 0$$

$$\frac{dF_{\ell\ell}}{d\theta} \cancel{e^{i\ell\varphi}} + i \cot\theta \; F_{\ell\ell} \; i\ell \; \cancel{e^{i\ell\varphi}} = 0$$

$$\left[\frac{d}{d\theta} - \ell \cot\theta \right] F_{\ell\ell}(\theta) = 0$$

TRY $\quad F_{\ell\ell} = A \, (\sin\theta)^{\ell}$

$$\frac{d}{d\theta} A \, (\sin\theta)^{\ell} = A \, (\sin\theta)^{\ell-1} \cos\theta$$

$$A \, (\sin\theta)^{\ell-1} \cos\theta - \ell \frac{\cos\theta}{\sin\theta} A \, (\sin\theta)^{\ell} \stackrel{?}{=} 0$$

yep! $\quad \Rightarrow \quad Y_{\ell\ell}(\theta, \varphi) = A_\ell \, (\sin\theta)^{\ell} \, e^{i\ell\varphi}$

TO GET THE REST

$$L_- Y_{\ell\ell} = a\, Y_{\ell\,\ell-1}$$

then normalize using

$$\int Y^*_{\ell m} Y_{\ell m}\, d\Omega = 1$$

$$L_- = \hbar e^{-i\varphi} \left(\frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right)$$

SPHERICAL HARMONICS

$\ell = 0$

$$Y_{00}(\theta, \varphi) = \langle \theta, \varphi | 0, 0 \rangle = \frac{1}{\sqrt{4\pi}}$$

$$\int \left(\frac{1}{\sqrt{4\pi}} \right)^* \frac{1}{\sqrt{4\pi}}\, d\Omega = 1 \quad \checkmark$$

$\ell = 1$

$$Y_{11}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\ \sin\theta\ e^{i\varphi}$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\ \cos\theta$$

$$Y_{1-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\ \sin\theta\ e^{-i\varphi}$$

note symmetry

$\ell = 2$

$$Y_{2\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\ (\sin\theta)^2\ e^{\pm 2i\varphi}$$

$$Y_{1\pm1}(\theta,\varphi) = \sqrt{\frac{15}{8\pi}}\ \sin\theta\ \cos\theta\ e^{\pm i\varphi}$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\ (3\cos^2\theta - 1)$$