Hints for Homework Set 5

(a) To find the squared operators, just square the three individual matrices. To find the total L^2 , just add up the three components

$$L^2 = L_x^2 + L_y^2 + L_z^2.$$

(b) To find the raising and lowering operators, add L_x and $\pm iL_y$. Do not let the complex components of L_y confuse you. When you multiply L_y by *i* you will find that all the components of $\pm iL_y$ are real numbers.

(c) All you are supposed to do here is to multiply the four matrices L_+ , L_- , L^2 , and L_z , by the three vectors |l = 1, m = +1 >, |l = 1, m = 0 >, and |l = 1, m = -1 > and verify that the general rules work for these twelve particular cases. . . .

(d) Of course, you can do the normalization calculation using Dirac notation, or you can use your old friend by now, the vector representation method, by realizing that

$$|\psi(t=0)\rangle = A \begin{pmatrix} 3\\ 2\\ 4 \end{pmatrix}.$$

(e) By now you can probably do these possibilities and probabilities calculations in your sleep!!! You already know the eigenvalues and the eigenvectors, so you're all set!!! Hint: the L^2 operator is the identity matrix, and you can easily calculate

$$P(l = 1, m = i) = | < l = 1, m = i | \psi(t = 0) > |^{2}$$

for i = 1, 2, 3.

(f) Without even waking up from the dream-like state that you entered by working part e, you should be able to knock off part f in REM sleep.

(g) You've got the state vector and you've got the matrix representations of the two operators, so the pedogogical intention here is for you to calculate the expectation value and the standard deviation using the matrix methods:

$$<\Omega>=<\psi \mid \Omega \mid \psi>,$$

and

$$\Delta \Omega = \left[<\psi \,|\, \Omega^2 \,|\, \psi > - <\psi \,|\, \Omega \,|\, \psi >^2 \right]^{1/2},$$

respectively. As many of you discovered when working the last exam (under less than ideal circumstances, and much to my chagrin!!!), it's usually much harder to use the alternate matrix form for the standard deviation

$$\Delta\Omega = \langle (\Omega - \langle \Omega \rangle I)^2 \rangle^{1/2} .$$

(h) For this part, you are supposed to calculate the expectation value and the standard deviation using the two probability based expressions

$$<\Omega> = \sum_{i} P(\omega_i) \; \omega_i \qquad \text{and} \qquad \Delta \Omega = \sqrt{\sum_{i} P(\omega_i) \; (\omega_i - <\Omega>)^2}$$

for $\Omega = L^2$ and for $\Omega = L_z$. If you keep all of your numerics to five digits when you calculate the standard deviation of L_z , you'll find everything works out nicely—unless you have made an error.

(i) You've already made several of these plots! I guess the idea is just to really hammer home the message that when you measure something for a given state vector that, in general, you can get many possible results and that a statistical representation of them is a lot more compact and hence often more useful than a list of every possible outcome. However, for this problem, there are only three possible outcomes, so this isn't as dramatic as it would be if there were say 999 possible outcomes. Remember that the standard deviation is a measure of the spread of a series of values around their mean, so it is customary to plot plus and minus the standard deviation centered on the mean.... You will find a very small spread in the standard deviation for L^2 —*i.e.*, since there is only one possible value for L^2 , the standard deviation is zero!

(j) These are very straightforward calculations since you have all the operators, and you know the three eigenvectors of the L_z operator. The hint is just to remind you that the expectation value is linear, so you can add the L^2 matrix to the L_z^2 matrix and then calculate the expectation value of the sum matrix, or you can calculate the expectation value of L^2 and add it to the expectation value of L_z^2 . Just like standard deviation calculations, sometimes one is easier and sometimes the other is easier. Here, they are both about the same. The hard part of this problem is understanding what it means....however, that's the point of the next problem!

(k) There is a discussion on the semiclassical vector model in many modern physics books. For example, both Bieser's, Perspectives of Modern Physics, and Krane's, Modern Physics have reasonable discussions in their discussion of the hydrogen atom. The big take home message from this problem is that when the quantization axis is along \hat{z} , the total angular momentum represented by L^2 and the z component of the angular momentum represented by L_z are quantized, but the x and y components of the angular momentum are not quantized! The semiclassical vector model sketch is the best visual representation we know how to make of this situation....

(l) You know the matrix form of the L_z^2 operator, and this problem is intended to be a simple "just write it down" problem! The matrix form of the L_z^2 operator is so simple, you should be able to find it's eigenvalues and eigenvectors by inspection!

(m) This is another "just write it down right" problem. You know the normalized state vector from part d, and you know that the eigenvectors of the Hamiltonian are the eigenvectors of the L_z operator (since the Hamiltonian is proportional to L_z^2). So you have enough information to construct the propagator

$$U(t) = \sum_{n} |n\rangle \langle n| e^{-iE_{n}t/\hbar}$$

and apply it to the initial state vector, or to just use our equivalent shortcut method of expanding the state vector in terms of the energy eigenvectors and inserting the time dependence. You can express your final answer in terms of the abstract kets, or in terms of the vector representation.

(n) To see how to approach this, let's consider the example state vector

$$|\psi(t)\rangle = \frac{1}{\sqrt{14}} \left[2 \begin{pmatrix} 1\\0\\0 \end{pmatrix} e^{-i\hbar t/2I} + 3 \begin{pmatrix} 0\\1\\0 \end{pmatrix} + \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i\hbar t/2I} \right].$$

Note that we can write this in the vector representation as

$$|\psi(t)\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 2e^{-i\alpha} \\ 3\\ e^{-i\alpha} \end{pmatrix}.$$

Here we've defined $\alpha = \hbar t/2I$ to keep the notation compact.

The expectation value of the energy for this state is given by

$$\langle \psi \mid H \mid \psi \rangle = \frac{1}{\sqrt{14}} \left(2e^{+i\alpha}, 3, e^{+i\alpha} \right) \frac{\hbar^2}{2I} \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 2e^{-i\alpha}\\ 3\\ e^{-i\alpha} \end{pmatrix}$$

$$=\frac{\hbar^2}{28I}\left(2e^{+i\alpha},\ 3,\ e^{+i\alpha}\right)\begin{pmatrix}2e^{-i\alpha}\\0\\e^{-i\alpha}\end{pmatrix}$$

$$= \frac{\hbar^2}{28I} (4+1)$$
$$= \frac{5\hbar^2}{28I}.$$

And the expectation value of L_x is given by

$$\langle \psi \,|\, L_x \,|\, \psi \rangle = \frac{1}{\sqrt{14}} \left(2e^{+i\alpha}, \ 3, \ e^{+i\alpha} \right) \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 2e^{-i\alpha}\\ 3\\ e^{-i\alpha} \end{pmatrix}$$

$$= \frac{\hbar}{14\sqrt{2}} \left(2e^{+i\alpha}, 3, e^{+i\alpha} \right) \begin{pmatrix} 3\\ 3e^{-i\alpha}\\ 3 \end{pmatrix}$$
$$= \frac{\hbar}{14\sqrt{2}} \left(6e^{+i\alpha} + 9e^{-i\alpha} + 3e^{+i\alpha} \right)$$
$$= \frac{9\hbar}{14\sqrt{2}} \left(e^{+i\alpha} + e^{-i\alpha} \right) = \frac{9\sqrt{2}\hbar}{14} \left(\frac{e^{+i\alpha} + e^{-i\alpha}}{2} \right)$$
$$= \frac{9\sqrt{2}\hbar}{14} \cos \alpha$$
$$= \frac{9\sqrt{2}\hbar}{14} \cos \left(\frac{\hbar t}{2I} \right).$$

There are two aspects involved in explaining the time dependence and the time independence: an abstract aspect, and a computational aspect. Abstract: Figure out which operators commute with the Hamiltonian and which operators do not, and then use what you know about the time dependence of operators that commute with the Hamiltonian. Computational: Note that for L_x we see state mixing because the exponential terms do not cancel!!! Can you generalize this???

(o) The idea behind this part and the next part is to help you to see the relationship between the discrete vector/matrix representation of angular momentum l = 1, and the corresponding continuous spherical harmonic and differential operator representation. This part will probably be pretty clear if you work through the steps in Shankar's Example 12.5.10 on pages 337-8.

The Hamiltonian is given in terms of the L_z operator as

$$H = \frac{L_z^2}{2I},$$

and the L_z operator is given in spherical coordinates by

$$L_z = -i\hbar \frac{\partial}{\partial \phi},$$

so your job is to put these together to get the differential operator form of the Hamiltonian, and then to write down the TDSE and the TISE.

(p) You already know the Hamiltonian matrix, you know the eigenvectors of the Hamiltonian, and you know the full time dependent state vector....so you only need to write down the abstract versions of the TDSE and the TISE and then transcribe all the terms into the corresponding matrices, vectors, and complex numbers. Warning: you don't have to do the matrix multiplication on the LHS of the equation and to evaluate the time derivatives on the RHS to show that the TDSE works, and you don't have to do the matrix multiplication for all three eigenvectors of H to show that the TISE works....but hopefully it will be obvious to you exactly what these two equations say and that they do in fact work!!!!!