Homework Assignment 5

Consider a system with angular momentum $l = 1$. In the L_z basis, the three L_i operators are given by:

$$L_x = (\hbar/\sqrt{2}) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = (\hbar/\sqrt{2}) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(a) Calculate L_x^2, L_y^2 and L_z^2 and then add them to find L^2.

(b) Calculate the angular momentum ladder operators L_+ and L_-.

(c) Show by explicit matrix multiplication for the $m = +1, 0$ and -1 states that:

$$L_- \mid l = 1,\ m> = \sqrt{l(l+1) - m(m-1)}\, \hbar \mid l = 1,\ m - 1>$$

$$L_+ \mid l = 1,\ m> = \sqrt{l(l+1) - m(m+1)}\, \hbar \mid l = 1,\ m + 1>$$

$$L^2 \mid l = 1\ , m> = l(l+1)\, \hbar^2 \mid l = 1,\ m>$$

$$L_z \mid l = 1,\ m> = m\, \hbar \mid l = 1,\ m>$$

Remember that

$$\mid l = 1,\ m\ = +1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mid l = 1,\ m\ = 0> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mid l = 1,\ m\ = -1> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Next consider the $t = 0$ wavefunction:

$$\mid \psi(t = 0)> = A\ (\ 3 \mid l = 1,\ m = 1> + 2 \mid l = 1,\ m = 0> + 4 \mid l = 1,\ m = -1>\ )$$

so

$$\mid \psi(t = 0)> = A \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}.$$

(d) Calculate the normalization constant A.

(e) If you measure L^2, what results can you obtain and with what probabilities will you obtain them?

(f) If you measure L_z, what results can you obtain and with what probabilities will you obtain them?

(g) Calculate $< L^2 >$, ΔL^2, $< L_z >$ and ΔL_z.

(h) Show that your results to parts (e), (f) and (g) are consistent by calculating the mean and the variance directly from the probabilities, *i.e.,* calculate:

$< \Omega >= \sum_i P(\omega_i)\, \omega_i$

and

$\Delta\Omega = \sqrt{\sum_i P(\omega_i)\, (\omega_i - < \Omega >)^2}$

for $\Omega = L^2$ and for $\Omega = L_z$.

(i) Plot $P(L^2)$ versus L^2 and $P(L_z)$ versus L_z, and indicate $< L^2 >$, ΔL^2, $< L_z >$ and ΔL_z on your plot.

(j) Show that $< L_x > = < L_y > = 0$ and that $< L^2 - L_z^2 > = < L_x^2 + L_y^2 >$ for any eigenstate of L_z.

Hint: $< L^2 - L_z^2 > = < L^2 > - < L_z^2 >$ and $< L_x^2 + L_y^2 > = < L_x^2 > + < L_y^2 >$.

(k) Sketch the semiclassical vector model for angular momentum states with $l = 1$ and $m = +1, 0, -1$. Explain how this geometry agrees with part j.

Now consider the time evolution of the wavefunction. You'll need the Hamiltonian. Suppose your system can only rotate about the z axis. Then it is described by the Hamiltonian $H = L_z^2/2I$ where I is the moment of inertia about the z axis.

(l) Write down the Hamiltonian matrix (in the L_z basis) and then find its eigenvectors and eigenvalues (by inspection).

(m) Calculate the time-dependent wavefunction $\mid \psi(t) >$ using parts (d) and (l).

(n) For your time-dependent wavefunction $\mid \psi(t) >$, calculate the expectation values of the energy and the angular momentum: $< E >$, $< L^2 >$, $< L_z >$, $< L_x >$ and $< L_y >$. Why are L_x and L_y time dependent? Why are L^2 and L_z time independent?

(o) Express the energy eigenfunctions in position space (think Y_{lm}). Express the TISE and the TDSE as a differential equation in position space.

(p) Express the TISE and the TDSE as matrix equations.