Quantum mechanics in a two-dimensional state space

Consider a system described by the Hamiltonian $\mathbf{H}$

$$\mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

and by a second observable operator $\mathbf{\Omega}$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

At $t = 0$, this system is initially in the state $\mid \psi(t = 0) >$ represented by $N \begin{pmatrix} i \\ 1 \end{pmatrix}$.

(a) Calculate the normalization constant N.

(b) Find the eigenvalues and the eigenvectors of the Hamiltonian operator. Whether you find them by inspection, or by full calculation, show that they work!

(c) Find the eigenvalues and the eigenvectors of the Omega operator. Whether you find them by inspection, or by full calculation, show that they work!

(d) Use outer products the eigenvectors of the Hamiltonian operator to calculate the two matrices that represent the two energy-subspace projection operators. Show that the eigenvectors of the Hamiltonian operator $\mathbf{H}$ form a basis for the two-dimensional state space by showing that the sum of the two energy-subspace projection operators is equal to the identity operator.

(e) Use outer products the eigenvectors of the Omega operator to calculate the two matrices that represent the two omega-subspace projection operators. Show that the eigenvectors of the $\mathbf{\Omega}$ operator form a basis for the two-dimensional state space by showing that the sum of the two omega-subspace projection operators is equal to the identity operator.

(f) Calculate the commutator $[\,\mathbf{H}, \mathbf{\Omega}\,]$. Do $\mathbf{H}$ and $\mathbf{\Omega}$ commute?

(g) If you were to measure the energy at time $t = 0$, what results could you obtain, and with what probabilities would you obtain them? What would the state vector be right after each energy measurement?

(h) If instead you were to measure the omega-ness at time $t = 0$, what results could you obtain, and with what probabilities would you obtain them? What would the state vector be right after each omega-ness measurement?

(i) Calculate the $t = 0$ expectation values of $\mathbf{H}$ and $\mathbf{\Omega}$ using

$< \mathbf{H} >=< \psi(0)|\mathbf{H}|\psi(0) >$ and

$< \mathbf{\Omega} >=< \psi(0)|\mathbf{\Omega}|\psi(0) >$.

Then show that your results agree with the values that you obtained above, *i.e.*, show that

$< \psi(0)|\mathbf{H}|\psi(0) >= \sum P(E_i)\ E_i$

and show that

$< \psi(0)|\mathbf{\Omega}|\psi(0) >= \sum P(\omega_i)\ \omega_i$.

(j) Calculate the $t = 0$ uncertainties $\Delta\mathbf{H}$ and $\Delta\boldsymbol{\Omega}$ using

$\Delta\mathbf{H} =< \psi(0)|\ (\mathbf{H}- < \mathbf{H} >)^2\ |\psi(0) >^{0.5}$ and

$\Delta\boldsymbol{\Omega} =< \psi(0)|\ (\boldsymbol{\Omega}- < \boldsymbol{\Omega} >)^2\ |\psi(0) >^{0.5}$

Then show that your results agree with the values that you obtained above, *i.e.*, show that

$< \psi(0)|\ (\mathbf{H}- < \mathbf{H} >)^2\ |\psi(0) >^{0.5} = (\ \sum P(E_i)\ (E_i- < \mathbf{H} >)^2\ )^{0.5}$

and show that

$< \psi(0)|\ (\boldsymbol{\Omega}- < \boldsymbol{\Omega} >)^2\ |\psi(0) >^{0.5} =\ (\ \sum P(\omega_i)\ (\omega_i- < \boldsymbol{\Omega} >^2)\ )^{0.5}$.

(k) For measurements at $t = 0$, sketch $P(E_i)$ versus E and sketch $P(\omega_i)$ versus ω. Indicate your calculated values of $< \mathbf{H} >$ and $\Delta\mathbf{H}$ and your calculated values of $< \boldsymbol{\Omega} >$ and $\Delta\boldsymbol{\Omega}$ on your respective sketches.

(l) Consider extremely fast alternating energy and omega-ness measurements—so fast that you can neglect the much slower time evolution produced by the Hamiltonian. Draw and explain the branches of the probability tree for the case where the first measurement is an energy measurement. You can stop at the fourth generation: H, Ω, H, Ω. Your probability tree should show the possibilities, the probabilities, and the resulting state vectors for each measurement.

(m) Consider extremely fast alternating omega-ness and energy measurements—so fast that you can neglect the much slower time evolution produced by the Hamiltonian. Draw and explain the branches of the probability tree for the case where the first measurement is an omega-ness measurement. You can stop at the fourth generation: Ω, H, $\Omega\ H$. Your probability tree should show the possibilities, the probabilities, and the resulting state vectors for each measurement.

Finally, consider the time-dependence of this system.

(n) Expand $|\ \psi(0) >$ in the energy eigenbasis and then write down the time evolution of $|\ \psi(t) >$.

(o) If you were to measure the energy at time t, what results could you obtain, and with what probabilities would you obtain them? What would the state vector be right after each energy measurement?

(p) If instead you were to measure the omega-ness at time t, what results could you obtain, and with what probabilities would you obtain them? What would the state vector be right after each omega-ness measurement?

(q) Explain why the energy measurements are time-independent, but the ω measurements are not.