separated by $\hbar$, because that is the amount of change indicated by the raising and lowering operators. The picture corresponds to figure 11–1. If there is other than minimum separation, say there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 11–1 we expect

$$2\beta_{\max} = n\hbar \quad \Rightarrow \quad \beta_{\max} = \frac{n\hbar}{2}.$$

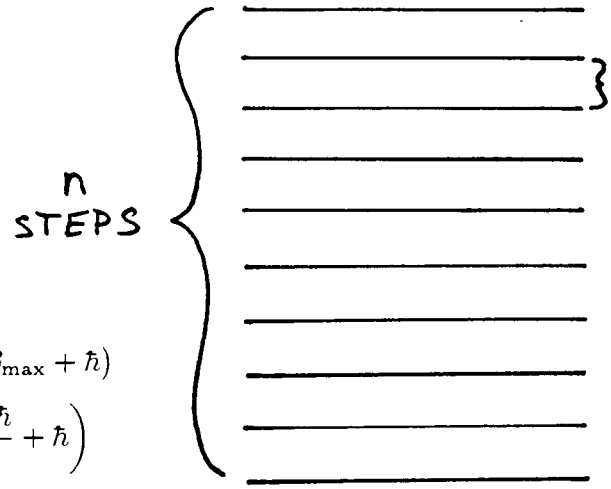

FIGURE 11-1. n STEPS WITH EACH STEP OF DISTANCE ħ.

Using this in equation (11–22),

$$\begin{aligned}\alpha &= \beta_{\max}(\beta_{\max} + \hbar) \\ &= \frac{n\hbar}{2}\left(\frac{n\hbar}{2} + \hbar\right) \\ &= \hbar^2\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right).\end{aligned}$$

We are going to re–label, letting $j = n/2$, so

$$\alpha = \hbar^2 j(j+1). \qquad (11-25)$$

Wait a minute.... The fact $j = \hbar/2$ vice just $\hbar$ does not appear consistent with the assumption that the rungs of the ladder are separated by $\hbar$...and it isn't. It appears the rungs of the ladder are separated by $\hbar/2$ vice $\hbar$.

What has occurred is that we have actually solved a more general problem than intended. Because of symmetry, the linear algebra arguments have given us the solution for <u>total angular momentum</u>. Total angular momentum is

$$\vec{J} = \vec{L} + \vec{S}, \qquad (11-26)$$

where $\vec{L}$ is <u>orbital angular momentum</u>, $\vec{S}$ is <u>spin angular momentum</u> or just <u>spin</u>. We posed the problem for orbital angular momentum, but because total angular momentum and spin obey analogous commutation relations to orbital angular momentum, we arrive at the solution for total angular momentum. Equations (11–7) indicated components of orbital angular momentum do not commute,

$$\left[\mathcal{L}_x, \mathcal{L}_y\right] = i\hbar\mathcal{L}_z, \qquad \left[\mathcal{L}_y, \mathcal{L}_z\right] = i\hbar\mathcal{L}_x, \qquad \text{and} \qquad \left[\mathcal{L}_z, \mathcal{L}_x\right] = i\hbar\mathcal{L}_y,$$

and for the ladder operator solution, we formed $\mathcal{L}_\pm = \mathcal{L}_x \pm i\mathcal{L}_y$. The commutation relations among the components of total angular momentum and spin angular momentum are exactly the same, *i.e.*,

$$\left[\mathcal{J}_x, \mathcal{J}_y\right] = i\hbar\mathcal{J}_z, \qquad \left[\mathcal{J}_y, \mathcal{J}_z\right] = i\hbar\mathcal{J}_x, \qquad \text{and} \qquad \left[\mathcal{J}_z, \mathcal{J}_x\right] = i\hbar\mathcal{J}_y,$$

and

$$\left[\mathcal{S}_x, \mathcal{S}_y\right] = i\hbar\mathcal{S}_z, \qquad \left[\mathcal{S}_y, \mathcal{S}_z\right] = i\hbar\mathcal{S}_x, \qquad \text{and} \qquad \left[\mathcal{S}_z, \mathcal{S}_x\right] = i\hbar\mathcal{S}_y.$$

Angular Momentum Eigenvalue Picture for Eigenstates

What is $|l, m>$? It is an eigenstate of the commuting operators $\mathcal{L}^2$ and $\mathcal{L}_z$. The quantum numbers l and m are not eigenvalues. The corresponding eigenvalues are $\hbar^2 l(l+1)$ and $m\hbar$. Were we to use eigenvalues in the ket, the eigenstate would look like $|\hbar^2 l(l+1), m\hbar>$. But just l and m uniquely identify the state, and that is more economical, so only the quantum numbers are conventionally used. This is essentially the same sort of convenient shorthand used to denote an eigenstate of a SHO $|n>$, vice using the eigenvalue $|(n+\frac{1}{2})\hbar\omega>$.

Only one quantum number is needed to uniquely identify an eigenstate of a SHO, but two are needed to uniquely identify an eigenstate of angular momentum. Because the angular momentum component operators do not commute, a complete set of commuting observables are needed. Each of the component operators commutes with $\mathcal{L}^2$, so we use it and one other, which is $\mathcal{L}_z$ chosen by convention. One quantum number is needed for each operator in the complete set. Multiple quantum numbers used to identify a ket denote a complete set of commuting observables is needed.

Remember that a system is assumed to exist in a linear combination of all possible eigenstates until we measure. If we measure, what are the possible outcomes? Possible outcomes are the eigenvalues. For a given value of the orbital angular momentum quantum number, the magnetic quantum number can assume integer values ranging from $-l$ to l. The simplest case is $l = 0 \Rightarrow m = 0$ is the only possible value of the magnetic quantum number. The possible outcomes of a measurement of such a system are eigenvalues of $\hbar^2(0)(0+1) = 0\hbar^2$ or just 0 for $\mathcal{L}^2$, and $m\hbar = (0)\hbar$ or just 0 for $\mathcal{L}_z$ as well, corresponding to figure 11–2.a.

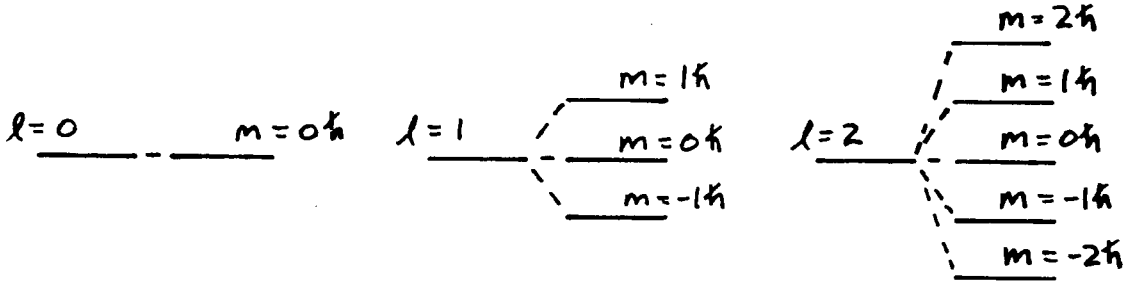

Figure 11 – 2.a. $l = 0$. Figure 11 – 2.b. $l = 1$. Figure 11 – 2.c. $l = 2$.

If we somehow knew $l = 1$, which could be ascertained by a measurement of $\hbar^2(1)(1+1) = 2\hbar^2$ for $\mathcal{L}^2$, the possible values of the magnetic quantum number are $m = -1$, 0, or 1, so the eigenvalues which could be measured are $-\hbar$, 0, or $\hbar$ for $\mathcal{L}_z$, per figure 11–2.b. If we measured $\hbar^2(2)(2+1) = 6\hbar^2$ for $\mathcal{L}^2$, we would know we had $l = 2$, and the possible values of the magnetic quantum number are $m = -2$, -1, 0, 1, or 2, so the eigenvalues which could be measured are $-2\hbar$, $-\hbar$, 0, $\hbar$, or $2\hbar$ for $\mathcal{L}_z$, per figure 11–2.c. Though the magnetic quantum number is bounded by the orbital angular momentum quantum number, the orbital angular momentum quantum number is not bounded, so we can continue indefinitely. Notice there are $2l+1$ possible values of m for every value of l.

A semi–classical diagram is often used. A simple interpretation of $|l, m>$ is that it is a vector quantized in length of

$$|\mathcal{L}| \rightarrow |\mathcal{L}| = \sqrt{\mathcal{L}^2} = \hbar\sqrt{l(l+1)}.$$

This vector has values for which the z component is also quantized in units of $m\hbar$. These

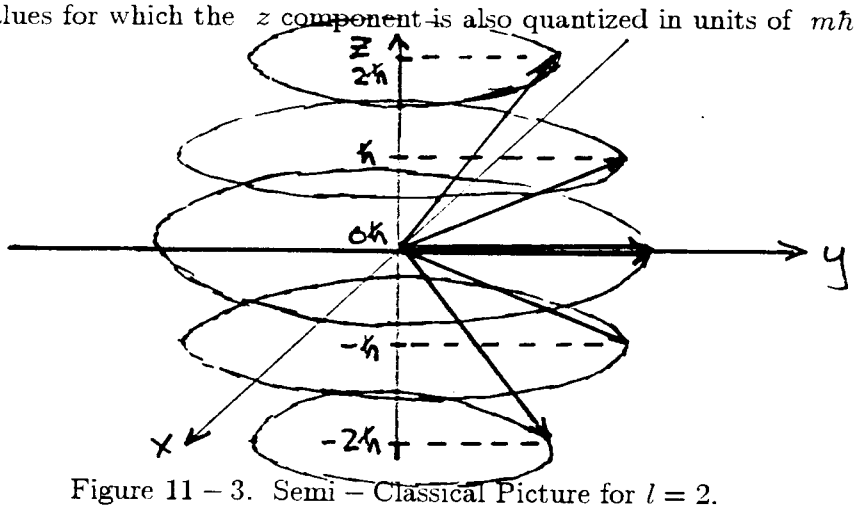

Figure 11 – 3. Semi – Classical Picture for $l = 2$.

features are illustrated in figure 11–3 for $l = 2$. The vectors are free to rotate around the z axis at any azimuthal angle ϕ, but are fixed at polar angles θ determined by the fact the projection on the z axis must be $-2\hbar$, $-\hbar$, 0, $\hbar$, or $2\hbar$. Notice there is no information concerning the x or y components other than the square of their sum is fixed. We could express this for $|\psi(t)> = |l, m>$ by stating the projection on the xy plane will be $\cos(\omega t)$ or $\sin(\omega t)$. In such a case we can determine x and y component expectation values from symmetry alone, *i.e.*,

$$<\mathcal{L}_x> = 0, \qquad <\mathcal{L}_y> = 0.$$

Finally, what fixes any axis in space? And how do we know which axis is the z axis? The answer is we must introduce some asymmetry. Without an asymmetry of some sort, the axes and their labels are arbitrary. The practical assymmetry to introduce is a magnetic field, and that will establish a component quantization axis which will be the z axis.

Eigenvalue/Eigenvector Equations for the Raising and Lowering Operators

Using quantum number notation, the fact $\mathcal{L}_+|\alpha, \beta>$ is and eigenstate of $\mathcal{L}_z$ would be written

$$\begin{aligned} \mathcal{L}_z\mathcal{L}_+|l, m> &= (m\hbar + \hbar)\mathcal{L}_+|l, m> \\ &= (m+1)\hbar\mathcal{L}_+|l, m> \\ &= \gamma\mathcal{L}_z|l, m+1>, \end{aligned}$$

where γ is a proportionality constant. Then

$$\begin{aligned} \mathcal{L}_z\mathcal{L}_+|l, m> &= \mathcal{L}_z\gamma|l, m+1> \\ \Rightarrow \quad \mathcal{L}_+|l, m> &= \gamma|l, m+1>, \end{aligned}$$

is the eigenvalue/eigenvector equation for the raising operator, where γ is evidently the eigenvalue, and the eigenvector is raised by one element of quantization in the z component. This means if the z component of the state on which the raising operator acts is $m\hbar$, the new state has a z component of $m\hbar + \hbar = (m+1)\hbar$, and thus the index $m+1$ is used in the new eigenket.

The name spherical harmonic comes from the geometry the functions naturally describe, spherical, and the fact any solution of Laplace's equation is known as harmonic. Picture a ball. The surface may be smooth, which is likely the first picture you form. Put a rubber band around the center, and you get a minima at the center and bulges, or maxima, in the top and bottom half. Put rubber bands on the circumference, like lines of longitude, and you get a different pattern of maxima and minima. We could imagine other, more complex patterns of maxima and minima. When these maxima and minima are symmetric with respect to an origin, the center of the ball, Legendre functions, associated Legendre functions, and spherical harmonics provide useful descriptions.

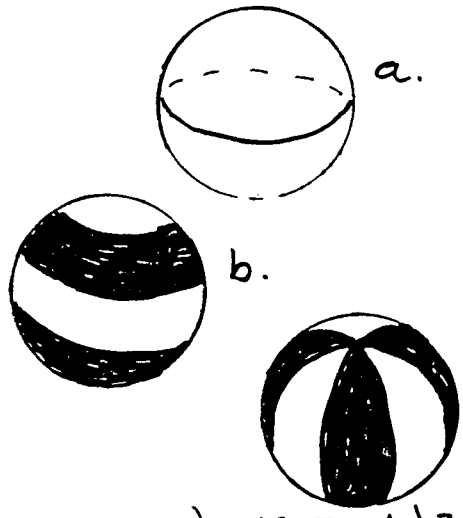

FIGURE 11-4. a) SPHERE, b) ZONAL HARMONIC, c) SECTORAL HARMONIC.

Properties that makes these special functions particularly useful is they are orthogonal and complete. Any set that is orthogonal can be made orthonormal. We have used orthonormality in a number of calculations, and the property of orthonormality continues to be a practical necessity. They are also complete in the sense any phenomenon can be described by an appropriate linear combination. Other complete sets of orthonormal functions we have encountered are sines and cosines for the square well, and Hermite polynomials for the SHO. A set of complete, orthonormal functions is equivalent to a linear vector space; these special functions are different manifestations of a complex linear vector space.

Spherical Harmonics

The ket $|l,\, m>$ is an eigenstate of the commuting operators $\mathcal{L}^2$ and $\mathcal{L}_z$, but it is an abstract eigenstate. That $|l,\, m>$ is abstract is irrelevant for the eigenvalues, since eigenvalues are properties of the operators. We would, however, like a representation useful for description for the eigenvectors. Per chapter 4, we can form an inner product with an abstract vector to attain a representation. Using a guided choice, the angles of spherical coordinate system will yield an appropriate representation. Just as $<x|g> = g(x)$, we will write

$$<\theta,\, \phi|l,\, m> = Y_{l,m}(\theta, \phi).$$

The functions of polar and azimuthal angles, $Y_{l,m}(\theta, \phi)$, are the spherical harmonics.

The spherical harmonics are related so strongly to the geometry of the current problem, they can be derived from the spherical coordinate form of the eigenvalue/eigenvector equation (11–29), $\mathcal{L}_z|l,\, m> = m\hbar|l,\, m>$, and use of the raising/lowering operator equation (11–31).

Using the spherical coordinate system form of the operator and the functional forms of the eigenstates, equation (11–29) is

$$-i\hbar\frac{\partial}{\partial\phi}Y_{l,m}(\theta, \phi) = m\hbar\, Y_{l,m}(\theta, \phi).$$

We are going to assume the spherical harmonics are separable, that they can be expressed as a product of a function of θ and a second function of ϕ, or

$$Y_{l,m}(\theta, \phi) = f_{l,m}(\theta)\, g_{l,m}(\phi).$$

illustrated. The radial coordinate will be examined in the next chapter. The size of any of the individual pictures in figure 11–2 is arbitrary; they could be very large or very small. We assume a radius of one unit to draw the sketches. In other words, you can look at the smooth sphere of $Y_{0,0}$ as having radius one unit, and the relative sizes of other spherical harmonic functions are comparable on the same radial scale.

Figure 11 – 2. Illustrations of the First Sixteen Spherical Harmonic Functions.