Chapter 9 Angular Momentum

Quantum Mechanical Angular Momentum Operators

Classical angular momentum is a vector quantity denoted $\vec{L} = \vec{r} \text{ X } \vec{p}$. A common mnemonic to calculate the components is

$$\vec{L} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \big(yp_z - zp_y\big)\hat{\imath} + \big(zp_x - xp_z\big)\hat{\jmath} + \big(xp_y - yp_x\big)\hat{\jmath}$$
$$= L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{\jmath}.$$

Let's focus on one component of angular momentum, say $L_x = yp_z - zp_y$. On the right side of the equation are two components of position and two components of linear momentum. Quantum mechanically, all four quantities are operators. Since the product of two operators is an operator, and the difference of operators is another operator, we expect the components of angular momentum to be operators. In other words, quantum mechanically

$$\mathcal{L}_x = \mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y, \qquad \mathcal{L}_y = \mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z, \qquad \mathcal{L}_z = \mathcal{X}\mathcal{P}_y - \mathcal{Y}\mathcal{P}_x.$$

These are the components. Angular momentum is the vector sum of the components. The sum of operators is another operator, so angular momentum is an operator. We have not encountered an operator like this one, however, this operator is comparable to a vector sum of operators; it is essentially a ket with operator components. We might write

$$\big|\,\mathcal{L} > = \begin{pmatrix} \mathcal{L}_x \\ \mathcal{L}_y \\ \mathcal{L}_z \end{pmatrix} = \begin{pmatrix} \mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y \\ \mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z \\ \mathcal{X}\mathcal{P}_y - \mathcal{Y}\mathcal{P}_x \end{pmatrix}. \qquad (9-1)$$

A word of caution concerning common notation—this is usually written just $\mathcal{L}$, and the ket/vector nature of quantum mechanical angular momentum is not explicitly written but implied.

Equation (9-1) is in abstract Hilbert space and is completely devoid of a representation. We will want to pick a basis to perform a calculation. In position space, for instance

$$\mathcal{X} \to x, \qquad \mathcal{Y} \to y, \qquad \text{and} \qquad \mathcal{Z} \to z,$$

and

$$\mathcal{P}_x \to -i\hbar\frac{\partial}{\partial x}, \qquad \mathcal{P}_y \to -i\hbar\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathcal{P}_z \to -i\hbar\frac{\partial}{\partial z}.$$

Equation (9–1) in position space would then be written

$$\big|\,\mathcal{L} > = \begin{pmatrix} -i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y} \\ -i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z} \\ -i\hbar x\frac{\partial}{\partial y} + i\hbar y\frac{\partial}{\partial x} \end{pmatrix}. \qquad (9-2)$$

The operator nature of the components promise difficulty, because unlike their classical analogs which are scalars, the angular momentum operators do not commute.

Example 9–1: Show the components of angular momentum in position space do not commute.

Let the commutator of any two components, say $\big[\mathcal{L}_x,\, \mathcal{L}_y\big]$, act on the function x. This means

$$\big[\mathcal{L}_x,\, \mathcal{L}_y\big]x = \big(\mathcal{L}_x\,\mathcal{L}_y - \mathcal{L}_y\,\mathcal{L}_x\big)x$$

$$\rightarrow \left(-i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}\right)\left(-i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z}\right)x - \left(-i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z}\right)\left(-i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}\right)x$$

$$= \left(-i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}\right)\big(-i\hbar z\big) - \left(-i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z}\right)(0)$$

$$= \Big(\big(-i\hbar\big)^2 y\Big) = -\hbar^2 y \neq 0,$$

therefore $\mathcal{L}_x$ and $\mathcal{L}_y$ do not commute. Using functions which are simply appropriate position space components, other components of angular momentum can be shown not to commute similarly.

Example 9–2: What is equation (9–1) in the momentum basis?

In momentum space, the operators are

$$\mathcal{X} \rightarrow i\hbar\frac{\partial}{\partial p_x}, \qquad \mathcal{Y} \rightarrow i\hbar\frac{\partial}{\partial p_y}, \qquad \text{and} \qquad \mathcal{Z} \rightarrow i\hbar\frac{\partial}{\partial p_z},$$

and

$$\mathcal{P}_x \rightarrow p_x, \qquad \mathcal{P}_y \rightarrow p_y, \qquad \text{and} \qquad \mathcal{P}_z \rightarrow p_z.$$

Equation (9–1) in momentum space would be written

$$\big|\,\mathcal{L} > = \begin{pmatrix} i\hbar\frac{\partial}{\partial p_y}\,p_z - i\hbar\frac{\partial}{\partial p_z}\,p_y \\ i\hbar\frac{\partial}{\partial p_z}\,p_x - i\hbar\frac{\partial}{\partial p_x}\,p_z \\ i\hbar\frac{\partial}{\partial p_x}\,p_y - i\hbar\frac{\partial}{\partial p_y}\,p_x \end{pmatrix}.$$

Canonical Commutation Relations in Three Dimensions

We indicated in equation (9–3) the fundamental canonical commutator is

$$\big[\,\mathcal{X},\, \mathcal{P}\,\big] = i\hbar.$$

This is fine when working in one dimension, however, descriptions of angular momentum are generally three dimensional. The generalization to three dimensions[2,3] is

$$\big[\,\mathcal{X}_i,\, \mathcal{X}_j\,\big] = 0, \tag{9 – 3}$$

[2] Cohen-Tannoudji, *Quantum Mechanics* (John Wiley & Sons, New York, 1977), pp 149 – 151.

[3] Sakurai, *Modern Quantum Mechanics* (Addison–Wesley Publishing Company, Reading, Massachusetts; 1994), pp 44 – 51.

which means any position component commutes with any other position component including itself,

$$\left[\,\mathcal{P}_i,\,\mathcal{P}_j\,\right]=0, \tag{9–4}$$

which means any linear momentum component commutes with any other linear momentum component including itself,

$$\left[\,\mathcal{X}_i,\,\mathcal{P}_j\,\right]=i\hbar\delta_{i,j}, \tag{9–5}$$

and the meaning of this equation requires some discussion. This means a position component will commute with an unlike component of linear momentum,

$$\left[\,\mathcal{X},\,\mathcal{P}_y\,\right]=\left[\,\mathcal{X},\,\mathcal{P}_z\,\right]=\left[\,\mathcal{Y},\mathcal{P}_x\,\right]=\left[\,\mathcal{Y},\,\mathcal{P}_z\,\right]=\left[\,\mathcal{Z},\,\mathcal{P}_x\,\right]=\left[\,\mathcal{Z},\,\mathcal{P}_y\,\right]=0,$$

but a position component and a like component of linear momentum are canonical commutators, *i.e.*,

$$\left[\,\mathcal{X}_x,\,\mathcal{P}_x\,\right]=\left[\,\mathcal{Y},\,\mathcal{P}_y\,\right]=\left[\,\mathcal{Z},\,\mathcal{P}_z\,\right]=i\hbar.$$

Commutator Algebra

In order to use the canonical commutators of equations (9–3) through (9–5), we need to develop some relations for commutators in excess of those discussed in chapter 3. For any operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, the relations below, some of which we have used previously, may be a useful list.

$$\begin{aligned}
\left[\,\mathcal{A},\,\mathcal{A}\,\right]&=0\\
\left[\,\mathcal{A},\,\mathcal{B}\,\right]&=-\left[\,\mathcal{B},\,\mathcal{A}\,\right]\\
\left[\,\mathcal{A},\,c\,\right]&=0,\qquad\text{for any scalar } c,\\
\left[\,\mathcal{A},\,c\,\mathcal{B}\,\right]&=c\left[\,\mathcal{A},\,\mathcal{B}\,\right],\qquad\text{for any scalar } c,\\
\left[\,\mathcal{A}+\mathcal{B},\,\mathcal{C}\,\right]&=\left[\,\mathcal{A},\,\mathcal{C}\,\right]+\left[\,\mathcal{B},\,\mathcal{C}\,\right]\\
\left[\,\mathcal{A},\,\mathcal{B}\mathcal{C}\,\right]&=\left[\,\mathcal{A},\,\mathcal{B}\,\right]\mathcal{C}+\mathcal{B}\left[\,\mathcal{A},\,\mathcal{C}\,\right]
\end{aligned} \tag{9–6}$$

$$\Big[\mathcal{A},\,\left[\mathcal{B},\,\mathcal{C}\,\right]\Big]+\Big[\mathcal{B},\,\left[\mathcal{C},\,\mathcal{A}\,\right]\Big]+\Big[\mathcal{C},\,\left[\mathcal{A},\,\mathcal{B}\,\right]\Big]=0.$$

You may have encountered relations similar to these in classical mechanics where the brackets are Poisson brackets. In particular, the last relation is known as the Jacobi identity. We are interested in quantum mechanical commutators and there are two important differences. Classical mechanics is concerned with quantities which are intrinsically real and are of finite dimension. Quantum mechanics is concerned with quantitites which are intrinsically complex and are generally of infinite dimension. Equation (9–6) is a relation we want to develop further.

Example 9–3: Prove equation (9–6).

$$\begin{aligned}
\left[\,\mathcal{A},\,\mathcal{B}\mathcal{C}\,\right]&=\mathcal{A}\mathcal{B}\mathcal{C}-\mathcal{B}\mathcal{C}\,\mathcal{A}\\
&=\mathcal{A}\mathcal{B}\mathcal{C}-\mathcal{B}\,\mathcal{A}\mathcal{C}+\mathcal{B}\,\mathcal{A}\mathcal{C}-\mathcal{B}\mathcal{C}\,\mathcal{A}\\
&=\left(\mathcal{A}\mathcal{B}-\mathcal{B}\,\mathcal{A}\right)\mathcal{C}+\mathcal{B}\left(\mathcal{A}\mathcal{C}-\mathcal{C}\,\mathcal{A}\right)\\
&=\left[\,\mathcal{A},\,\mathcal{B}\,\right]\mathcal{C}+\mathcal{B}\left[\,\mathcal{A},\,\mathcal{C}\,\right],
\end{aligned}$$

where we have added zero, in the form $-\mathcal{B}\,\mathcal{A}\mathcal{C}+\mathcal{B}\,\mathcal{A}\mathcal{C}$, in the second line.

Example 9–4: Develop a relation for $\left[\,\mathcal{A}\,\mathcal{B},\,\mathcal{C}\,\right]$ in terms of commutators of individual operators.

$$\begin{aligned}\left[\,\mathcal{A}\mathcal{B},\,\mathcal{C}\,\right] &= \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{C}\,\mathcal{A}\,\mathcal{B}\\ &= \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{A}\,\mathcal{C}\,\mathcal{B} + \mathcal{A}\,\mathcal{C}\,\mathcal{B} - \mathcal{C}\,\mathcal{A}\mathcal{B}\\ &= \mathcal{A}\left(\mathcal{B}\mathcal{C} - \mathcal{C}\,\mathcal{B}\right) + \left(\mathcal{A}\,\mathcal{C} - \mathcal{C}\,\mathcal{A}\right)\mathcal{B}\\ &= \mathcal{A}\left[\,\mathcal{B},\,\mathcal{C}\,\right] + \left[\,\mathcal{A},\,\mathcal{C}\,\right]\mathcal{B}.\end{aligned}$$

Example 9–5: Develop a relation for $\left[\,\mathcal{A}\mathcal{B},\,\mathcal{C}\,\mathcal{D}\,\right]$ in terms of commutators of individual operators.

Using the result of example 9–3,

$$\left[\,\mathcal{A}\mathcal{B},\,\mathcal{C}\,\mathcal{D}\,\right] = \left[\,\mathcal{A}\mathcal{B},\,\mathcal{C}\,\right]\mathcal{D} + \mathcal{C}\left[\,\mathcal{A}\mathcal{B},\,\mathcal{D}\,\right],$$

and using the result of example 9–4 on both of the commutators on the right,

$$\begin{aligned}\left[\,\mathcal{A}\mathcal{B},\,\mathcal{C}\,\mathcal{D}\,\right] &= \Big(\mathcal{A}\left[\,\mathcal{B},\,\mathcal{C}\,\right] + \left[\,\mathcal{A},\,\mathcal{C}\,\right]\mathcal{B}\Big)\mathcal{D} + \mathcal{C}\Big(\mathcal{A}\left[\,\mathcal{B},\,\mathcal{D}\,\right] + \left[\,\mathcal{A},\,\mathcal{D}\,\right]\mathcal{B}\Big)\\ &= \mathcal{A}\left[\,\mathcal{B},\,\mathcal{C}\,\right]\mathcal{D} + \left[\,\mathcal{A},\,\mathcal{C}\,\right]\mathcal{B}\mathcal{D} + \mathcal{C}\,\mathcal{A}\left[\,\mathcal{B},\,\mathcal{D}\,\right] + \mathcal{C}\left[\,\mathcal{A},\,\mathcal{D}\,\right]\mathcal{B},\end{aligned}$$

which is the desired result.

Angular Momentum Commutation Relations

Given the relations of equations (9–3) through (9–5), it follows that

$$\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] = i\hbar\,\mathcal{L}_z, \qquad \left[\,\mathcal{L}_y,\,\mathcal{L}_z\,\right] = i\hbar\,\mathcal{L}_x, \qquad \text{and} \qquad \left[\,\mathcal{L}_z,\,\mathcal{L}_x\,\right] = i\hbar\,\mathcal{L}_y. \tag{9 – 7}$$

Example 9–6: Show $\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] = i\hbar\,\mathcal{L}_z$.

$$\begin{aligned}\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] &= \left[\,\mathcal{Y}\,\mathcal{P}_z - \mathcal{Z}\,\mathcal{P}_y,\ \mathcal{Z}\,\mathcal{P}_x - \mathcal{X}\,\mathcal{P}_z\,\right]\\ &= \Big(\mathcal{Y}\,\mathcal{P}_z - \mathcal{Z}\,\mathcal{P}_y\Big)\Big(\mathcal{Z}\,\mathcal{P}_x - \mathcal{X}\,\mathcal{P}_z\Big) - \Big(\mathcal{Z}\,\mathcal{P}_x - \mathcal{X}\,\mathcal{P}_z\Big)\Big(\mathcal{Y}\,\mathcal{P}_z - \mathcal{Z}\,\mathcal{P}_y\Big)\end{aligned}$$

$$= \mathcal{Y}\,\mathcal{P}_z\mathcal{Z}\,\mathcal{P}_x - \mathcal{Y}\,\mathcal{P}_z\mathcal{X}\,\mathcal{P}_z - \mathcal{Z}\,\mathcal{P}_y\mathcal{Z}\,\mathcal{P}_x + \mathcal{Z}\,\mathcal{P}_y\mathcal{X}\,\mathcal{P}_z - \mathcal{Z}\,\mathcal{P}_x\mathcal{Y}\,\mathcal{P}_z + \mathcal{Z}\,\mathcal{P}_x\mathcal{Z}\,\mathcal{P}_y + \mathcal{X}\,\mathcal{P}_z\mathcal{Y}\,\mathcal{P}_z - \mathcal{X}\,\mathcal{P}_z\mathcal{Z}\,\mathcal{P}_y$$

$$= \Big(\mathcal{Y}\,\mathcal{P}_z\mathcal{Z}\,\mathcal{P}_x - \mathcal{Z}\,\mathcal{P}_x\mathcal{Y}\,\mathcal{P}_z\Big) + \Big(\mathcal{Z}\,\mathcal{P}_y\mathcal{X}\,\mathcal{P}_z - \mathcal{X}\,\mathcal{P}_z\mathcal{Z}\,\mathcal{P}_y\Big)$$

$$+ \Big(\mathcal{Z}\,\mathcal{P}_x\mathcal{Z}\,\mathcal{P}_y - \mathcal{Z}\,\mathcal{P}_y\mathcal{Z}\,\mathcal{P}_x\Big) + \Big(\mathcal{X}\,\mathcal{P}_z\mathcal{Y}\,\mathcal{P}_z - \mathcal{Y}\,\mathcal{P}_z\mathcal{X}\,\mathcal{P}_z\Big)$$

$$= \left[\,\mathcal{Y}\,\mathcal{P}_z,\ \mathcal{Z}\,\mathcal{P}_x\,\right] + \left[\,\mathcal{Z}\,\mathcal{P}_y,\ \mathcal{X}\,\mathcal{P}_z\,\right] + \left[\,\mathcal{Z}\,\mathcal{P}_x,\ \mathcal{Z}\,\mathcal{P}_y\,\right] + \left[\,\mathcal{X}\,\mathcal{P}_z,\ \mathcal{Y}\,\mathcal{P}_z\,\right].$$

Using the result of example 9–5, the plan is to express these commutators in terms of individual operators, and then evaluate those using the commutation relations of equations (9–3) through (9–5). In example 9–5, one commutator of the products of two operators turns into four commutators. Since we start with four commutators of the products of two operators, we are going to get 16 commutators in terms of individual operators. The good news is 14 of them are zero from equations (9–3), (9–4), and (9–5), so will be struck.

$$
\begin{aligned}
\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] &= \mathcal{Y}\left[\,\mathcal{P}_z,\,\mathcal{Z}\,\right]\mathcal{P}_x + \cancel{\left[\,\mathcal{Y},\,\mathcal{Z}\,\right]}\,\mathcal{P}_z\,\mathcal{P}_x + \mathcal{Z}\,\mathcal{Y}\,\cancel{\left[\,\mathcal{P}_z,\,\mathcal{P}_x\,\right]} + \mathcal{Z}\,\cancel{\left[\,\mathcal{Y},\,\mathcal{P}_x\,\right]}\,\mathcal{P}_z \\
&\quad + \mathcal{Z}\,\cancel{\left[\,\mathcal{P}_y,\,\mathcal{X}\,\right]}\,\mathcal{P}_z + \cancel{\left[\,\mathcal{Z},\,\mathcal{X}\,\right]}\,\mathcal{P}_y\,\mathcal{P}_z + \mathcal{X}\,\mathcal{Z}\,\cancel{\left[\,\mathcal{P}_y,\,\mathcal{P}_z\,\right]} + \mathcal{X}\left[\,\mathcal{Z},\,\mathcal{P}_z\,\right]\mathcal{P}_y \\
&\quad + \mathcal{Z}\,\cancel{\left[\,\mathcal{P}_x,\,\mathcal{Z}\,\right]}\,\mathcal{P}_y + \cancel{\left[\,\mathcal{Z},\,\mathcal{Z}\,\right]}\,\mathcal{P}_x\,\mathcal{P}_y + \mathcal{Z}\,\mathcal{Z}\,\cancel{\left[\,\mathcal{P}_x,\,\mathcal{P}_y\,\right]} + \mathcal{Z}\,\cancel{\left[\,\mathcal{Z},\,\mathcal{P}_y\,\right]}\,\mathcal{P}_x \\
&\quad + \mathcal{X}\,\cancel{\left[\,\mathcal{P}_z,\,\mathcal{Y}\,\right]}\,\mathcal{P}_z + \cancel{\left[\,\mathcal{X},\,\mathcal{Y}\,\right]}\,\mathcal{P}_z\,\mathcal{P}_z + \mathcal{Y}\,\mathcal{X}\,\cancel{\left[\,\mathcal{P}_z,\,\mathcal{P}_z\,\right]} + \mathcal{Y}\,\cancel{\left[\,\mathcal{X},\,\mathcal{P}_z\,\right]}\,\mathcal{P}_z \\
&= \mathcal{Y}\left[\,\mathcal{P}_z,\,\mathcal{Z}\,\right]\mathcal{P}_x + \mathcal{X}\left[\,\mathcal{Z},\,\mathcal{P}_z\,\right]\mathcal{P}_y \\
&= \mathcal{Y}\big(-i\hbar\big)\mathcal{P}_x + \mathcal{X}\big(i\hbar\big)\mathcal{P}_y \\
&= i\hbar\big(\mathcal{X}\,\mathcal{P}_y - \mathcal{Y}\,\mathcal{P}_x\big) \\
&= i\hbar\mathcal{L}_z.
\end{aligned}
$$

The other two relations, $\left[\,\mathcal{L}_y,\,\mathcal{L}_z\,\right] = i\hbar\,\mathcal{L}_x$ and $\left[\,\mathcal{L}_z,\,\mathcal{L}_x\,\right] = i\hbar\,\mathcal{L}_y$ can be calculated using similar procedures.

A Representation of Angular Momentum Operators

We would like to have matrix operators for the angular momentum operators $\mathcal{L}_x$, $\mathcal{L}_y$, and $\mathcal{L}_z$. In the form $\mathcal{L}_x$, $\mathcal{L}_y$, and $\mathcal{L}_z$, these are abstract operators in an infinite dimensional Hilbert space. Remember from chapter 2 that a subspace is a specific subset of a general complex linear vector space. In this case, we are going to find relations in a subspace C^3 of an infinite dimensional Hilbert space. The idea is to find three 3 X 3 matrix operators that satisfy relations (9–7), which are

$$
\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] = i\hbar\,\mathcal{L}_z, \qquad \left[\,\mathcal{L}_y,\,\mathcal{L}_z\,\right] = i\hbar\,\mathcal{L}_x, \qquad \text{and} \qquad \left[\,\mathcal{L}_z,\,\mathcal{L}_x\,\right] = i\hbar\,\mathcal{L}_y.
$$

One such group of objects is

$$
\mathcal{L}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar, \qquad \mathcal{L}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar, \qquad \mathcal{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar. \qquad (9-8)
$$

You have seen these matrices in chapters 2 and 3. In addition to illustrating some of the mathematical operations of those chapters, they were used when appropriate there, so you may have a degree of familiarity with them here. There are other ways to express these matrices in C^3. Relations (9–8) are dominantly the most popular. Since the three operators do not commute, we arbitrarily have selected a basis for one of them, and then expressed the other two in that basis. Notice $\mathcal{L}_z$ is diagonal. That means the basis selected is natural for $\mathcal{L}_z$. The terminology usually used is the operators in equations (9–8) are **in the $\underline{\mathcal{L}_z}$ basis**.

We could have selected a basis which makes $\mathcal{L}_x$ or $\mathcal{L}_y$, and expressed the other two in terms of the natural basis for $\mathcal{L}_x$ or $\mathcal{L}_y$. If we had done that, the operators are different than those seen in relations (9–8). The mathematics of this is not important at the moment, but it is important that you understand there are other self consistent ways to express these operators as 3 X 3 matrices.

Example 9–7: Show $\left[\mathcal{L}_x, \mathcal{L}_y\right] = i\hbar\,\mathcal{L}_z$ using relations (9–8).

$$\begin{aligned}
\left[\mathcal{L}_x, \mathcal{L}_y\right] &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar \\
&= \frac{\hbar^2}{2}\begin{pmatrix} i & 0 & -i \\ 0 & -i+i & 0 \\ i & 0 & -i \end{pmatrix} - \frac{\hbar^2}{2}\begin{pmatrix} -i & 0 & -i \\ 0 & i-i & 0 \\ i & 0 & i \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} i+i & 0 & -i+i \\ 0 & 0 & 0 \\ i-i & 0 & -i-i \end{pmatrix} \\
&= \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = i\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \\
&= i\hbar\,\mathcal{L}_z .
\end{aligned}$$

Again, the other two relations can be calculated using similar procedures. In fact, the arithmetic for the other two relations is simpler. Why would this be so? ...Because $\mathcal{L}_z$ is a diagonal operator.

Remember $\mathcal{L}$ is comparable to a vector sum of the three component operators, so in vector/matrix notation would look like

$$\big|\mathcal{L}> = \begin{pmatrix} \mathcal{L}_x \\ \mathcal{L}_y \\ \mathcal{L}_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \end{pmatrix} .$$

Again, this operator will normally be denoted just $\mathcal{L}$. The $\mathcal{L}$ operator is a different sort of object than the component operators. It is a different object in a different space. Yet, we would like a way to address angular momentum with a 3 X 3 matrix which is in the same subspace as the components. We can do this if we use $\mathcal{L}^2$. This operator is

$$\mathcal{L}^2 = 2\hbar^2\mathcal{I} = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (9-9)$$

Example 9–8: Show $\mathcal{L}^2 = 2\hbar^2\mathcal{I}$.

$$\mathcal{L}^2 = <\mathcal{L}\,\big|\,\mathcal{L}>$$

$$\rightarrow \langle \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar,\ \ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar,\ \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \,\Bigg|\, \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \end{pmatrix}\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1+1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\hbar^2 + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1+1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\hbar^2$$

$$= 2\hbar^2\mathcal{I}.$$

Complete Set of Commuting Observables ...A Discussion about Operators which do not Commute....

The intent of this section is to appreciate non–commutivity from a new perspective, and explain "what can be done about it" if the non–commuting operators represent physical quantities we want to measure. The following toy example is adapted from *Quantum Mechanics and Experience*[4].

We want two operators which do not commute. We are deliberately using simple operators in an effort to focus on principles. In a two dimensional linear vector space, the property of "hardness" is modelled

$$\mathrm{H}_{\mathrm{ard}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[4] Albert, *Quantum Mechanics and Experience* (Harvard University Press, Cambridge, Massachusetts, 1992), pp 30–33.

and has eigenvalues of ± 1 and eigenvectors

$$|1\rangle_{\text{hard}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-1\rangle_{\text{hard}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let's also consider the "color" operator,

$$C_{\text{olor}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues of ± 1 and eigenvectors

$$|1\rangle_{\text{color}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-1\rangle_{\text{color}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that a "hardness" or "color" eigenvector is a superposition of the eigenvectors of the other property, i.e.,

$$|1\rangle_{\text{hard}} = \frac{1}{\sqrt{2}}|1\rangle_{\text{color}} + \frac{1}{\sqrt{2}}|-1\rangle_{\text{color}}$$

$$|-1\rangle_{\text{hard}} = \frac{1}{\sqrt{2}}|1\rangle_{\text{color}} - \frac{1}{\sqrt{2}}|-1\rangle_{\text{color}}$$

$$|1\rangle_{\text{color}} = \frac{1}{\sqrt{2}}|1\rangle_{\text{hard}} + \frac{1}{\sqrt{2}}|-1\rangle_{\text{hard}}$$

$$|-1\rangle_{\text{color}} = \frac{1}{\sqrt{2}}|1\rangle_{\text{hard}} - \frac{1}{\sqrt{2}}|-1\,\rangle_{\text{hard}}$$

Hardness is a superposition of color states and color is a superposition of hardness states. That is the foundation of incompatibility, or non–commutivity. Each measurable state is a linear combination or superposition of the measurable states of the other property. To disturb one property is to disturb both properties.

Also in chapter 3, we indicated if two Hermitian operators commute, there exists a basis of common eigenvectors. Conversely, if they do not commute, there is no basis of common eigenvectors. We conclude there is no common eigenbasis for the "hardness" and "color" operators.

This is exactly the status of the three angular momentum component operators, except there are three vice two operators which do not commute with one another. None of the component operators commutes with any other. There is no common basis of eigenvectors between any two, so can be no common eigenbasis between all three.

Back to the hardness and color operators. If we can find an operator with which both commute, say the two dimensional identity operator $\mathcal{I}$, we can ascertain the eigenstate of the system. If we measure an eigenvalue of 1 for color, the eigenstate is proportional to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, were we to operate on this with the identity operator, the eigenstate of system is either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If we then measure with the hardness operator, the eigenvalue will be 1 if the state was $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or -1 if the state was $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have effectively removed the indeterminacy of the system by including

$\mathcal{I}$. If we measure either "hardness" or "color," and then operate with the identity, we attain a distinct, unique unit vector. There are two **complete sets of commuting operators** possible, $\mathcal{I}$ and $\mathrm{H}_{\mathrm{ard}}$, or $\mathcal{I}$ and $\mathrm{C}_{\mathrm{olor}}$.

The eigenvalues, indicated in the ket, and eigenvectors for the three angular momentum component operators are

$$|-\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \qquad |0> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad |\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},$$

for $\mathcal{L}_x$,

$$|-\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}, \qquad |0> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad |\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix},$$

for $\mathcal{L}_y$, and

$$|-1> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |0> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

for $\mathcal{L}_z$. Notice like the nonsense operators hardness and color, none of the angular momentum component operators commute and none of the eigenvectors correspond. Also comparable, $\mathcal{L}^2$ is proportional to the identity operator, except in three dimensions. We can do something similar to the "hardness, color" case to remove the indeterminacy. It must be similar and not the same...because we need a fourth operator with which the three non–commuting component angular momentum operators all commute, and any one of the angular momentum components to form a complete set of commuting observables. We choose $\mathcal{L}^2$, which commutes with all three component operators, and $\mathcal{L}_z$, which is the conventional choice of components.

The requirement for a complete set of commuting observables is equivalent to **removing or lifting a degeneracy**. The idea is closely related to the discussion at the end of example 3–33. If you comprehend the idea behind that discussion, you have the basic principle of this discussion.

Also, "complete" here means all possiblities are clear, *i.e.*, that any degeneracy is removed. This is the same word but a different context than "span the space" as the word was used in chapter 2. Both uses are conventional and meaning is ascertained only by usage, so do not be confused by its use in both contexts.

Precurser to the Hydrogen Atom

The Hamiltonian for a spherically symmetric potential commutes with $\mathcal{L}^2$ and the three component angular momentum operators. So $\mathcal{H}$, $\mathcal{L}^2$, and one of the three component angular momentum operators, conventially $\mathcal{L}_z$, is a complete set of commuting observables for a spherically symmetric potential.

We will use a Hamiltonian with a Coulomb potential for the hydrogen atom. The Coulomb potential is rotationally invariant, or spherically symmetric. We have indicated $\mathcal{H}$, $\mathcal{L}^2$, and $\mathcal{L}_z$ form a complete set of commuting observables for such a system. You may be familiar with the principal quantum number n, the angular momentum quantum number l, and the magnetic quantum number m. We will find there is a correspondence between these two sets of three quantities, which is n comes from application of $\mathcal{H}$, l comes from application of $\mathcal{L}^2$, and m

comes from application of $\mathcal{L}_z$. A significant portion of the reason to address angular momentum and explain the concept of a complete set of commuting observables now is for use in the next chapter on the hydrogen atom.

Ladder Operators for Angular Momentum

We are going to address angular momentum, like the SHO, from both a linear algebra and a differential equation perspective. We are going to assume rotational invariance, or spherical symmetry, so we have $\mathcal{H}$, $\mathcal{L}^2$, and $\mathcal{L}_z$ as a complete set of commuting observables. We will address linear algebra arguments first. And we will work only with the components and $\mathcal{L}^2$, saving the Hamiltonian for the next chapter.

The four angular momentum operators are related as

$$\mathcal{L}^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2 \quad \Rightarrow \quad \mathcal{L}^2 - \mathcal{L}_z^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2.$$

The sum of the two components $\mathcal{L}_x^2 + \mathcal{L}_y^2$ would appear to factor

$$\big(\mathcal{L}_x + i\mathcal{L}_y\big)\big(\mathcal{L}_x - i\mathcal{L}_y\big),$$

and they would if the factors were scalars, but they are operators which do not commute, so this is not factoring. Just like the SHO, it is a good mnemonic, nevertheless.

Example 9–9: Show $\mathcal{L}_x^2 + \mathcal{L}_y^2 \neq \big(\mathcal{L}_x + i\mathcal{L}_y\big)\big(\mathcal{L}_x - i\mathcal{L}_y\big)$.

$$\begin{aligned}\big(\mathcal{L}_x + i\mathcal{L}_y\big)\big(\mathcal{L}_x - i\mathcal{L}_y\big) &= \mathcal{L}_x^2 - i\mathcal{L}_x\mathcal{L}_y + i\mathcal{L}_y\mathcal{L}_x + \mathcal{L}_y^2\\ &= \mathcal{L}_x^2 + \mathcal{L}_y^2 - i\big(\mathcal{L}_x\mathcal{L}_y - \mathcal{L}_y\mathcal{L}_x\big)\\ &= \mathcal{L}_x^2 + \mathcal{L}_y^2 - i\big[\mathcal{L}_x,\, \mathcal{L}_y\big]\\ &= \mathcal{L}_x^2 + \mathcal{L}_y^2 - i\big(i\hbar\mathcal{L}_z\big)\\ &= \mathcal{L}_x^2 + \mathcal{L}_y^2 + \hbar\mathcal{L}_z\\ &\neq \mathcal{L}_x^2 + \mathcal{L}_y^2,\end{aligned}$$

where the expression in the next to last line is a significant intermediate result, and we will have reason to refer to it.

Like the SHO, the idea is to take advantage of the commutation relations of equations (9–7). We will use the notation

$$\mathcal{L}_+ = \mathcal{L}_x + i\mathcal{L}_y, \qquad \text{and} \qquad \mathcal{L}_- = \mathcal{L}_x - i\mathcal{L}_y, \tag{9 – 12}$$

which together are often denoted $\mathcal{L}_\pm$. We need commutators for $\mathcal{L}_\pm$, which are

$$\big[\mathcal{L}^2,\, \mathcal{L}_\pm\big] = 0, \tag{9 – 13}$$

$$\big[\mathcal{L}_z,\, \mathcal{L}_\pm\big] = \pm\hbar\, \mathcal{L}_\pm. \tag{9 – 14}$$

Example 9–10: Show $\big[\mathcal{L}^2,\, \mathcal{L}_+\big] = 0$.

$$\big[\mathcal{L}^2,\, \mathcal{L}_+\big] = \big[\mathcal{L}^2,\, \mathcal{L}_x + i\mathcal{L}_y\big] = \big[\mathcal{L}^2,\, \mathcal{L}_x\big] + i\big[\mathcal{L}^2,\, \mathcal{L}_y\big] = 0 + i(0) = 0.$$

Example 9–11: Show $\big[\mathcal{L}_z,\, \mathcal{L}_+\big] = \hbar\, \mathcal{L}_+$.

$$\big[\mathcal{L}_z,\, \mathcal{L}_+\big] = \big[\mathcal{L}_z,\, \mathcal{L}_x + i\mathcal{L}_y\big] = \big[\mathcal{L}_z,\, \mathcal{L}_x\big] + i\big[\mathcal{L}_z,\, \mathcal{L}_y\big] = i\hbar\mathcal{L}_y + i\big(-i\hbar\mathcal{L}_x\big) = \hbar\big(\mathcal{L}_x + i\mathcal{L}_y\big) = \hbar\mathcal{L}_+.$$

We will proceed essentially as we did the the raising and lowering operators of the SHO. Since $\mathcal{L}^2$ and $\mathcal{L}_z$ commute, they share a common eigenbasis.

Example 9–12: Show $\mathcal{L}^2$ and $\mathcal{L}_z$ commute.

$$\begin{aligned}
\big[\mathcal{L}^2,\, \mathcal{L}_z\big] &= \big[\mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2,\, \mathcal{L}_z\big] \\
&= \big[\mathcal{L}_x^2,\, \mathcal{L}_z\big] + \big[\mathcal{L}_y^2,\, \mathcal{L}_z\big] + \big[\mathcal{L}_z^2 \cancel{,}\, \mathcal{L}_z\big] \\
&= \big[\mathcal{L}_x\, \mathcal{L}_x,\, \mathcal{L}_z\big] + \big[\mathcal{L}_y\, \mathcal{L}_y,\, \mathcal{L}_z\big] \\
&= \mathcal{L}_x\, \big[\mathcal{L}_x,\, \mathcal{L}_z\big] + \big[\mathcal{L}_x,\, \mathcal{L}_z\big]\, \mathcal{L}_x + \mathcal{L}_y\, \big[\mathcal{L}_y,\, \mathcal{L}_z\big] + \big[\mathcal{L}_y,\, \mathcal{L}_z\big]\, \mathcal{L}_y \\
&= \mathcal{L}_x\big(-i\hbar\mathcal{L}_y\big) + \big(-i\hbar\mathcal{L}_y\big)\mathcal{L}_x + \mathcal{L}_y\big(i\hbar\mathcal{L}_x\big) + \big(i\hbar\mathcal{L}_x\big)\mathcal{L}_y \\
&= \big(-i\hbar\mathcal{L}_x\, \mathcal{L}_y + i\hbar\mathcal{L}_x\, \mathcal{L}_y\big) + \big(-i\hbar\mathcal{L}_y\, \mathcal{L}_x + i\hbar\mathcal{L}_y\, \mathcal{L}_x\big) \\
&= 0,
\end{aligned}$$

where we have used the results of example 9–4 and two of equations (9–7) in the reduction.

We assume $\mathcal{L}^2$ and $\mathcal{L}_z$ will have different eigenvalues when they operate on the same basis vector, so we need two indices for each basis vector. The first index is the eigenvalue for $\mathcal{L}^2$, we will use α for the eigenvalue, and the second index is the eigenvalue for $\mathcal{L}_z$, denoted by β. If we had a third commuting operator, for instance $\mathcal{H}$ which we will add in the next chapter, we would need three eigenvalues to uniquely identify each ket. Here we are considering two commuting operators, so we need two indices representing the eigenvalues of the two commuting operators.

Considering just $\mathcal{L}^2$ and $\mathcal{L}_z$ here, the form of the eigenvalue equations must be

$$\mathcal{L}^2\big|\alpha,\, \beta\big> = \alpha\big|\alpha,\, \beta\big>, \tag{9 – 15}$$

$$\mathcal{L}_z\big|\alpha,\, \beta\big> = \beta\big|\alpha,\, \beta\big>, \tag{9 – 16}$$

where $\big|\alpha,\, \beta\big>$ is the eigenstate, α is the eigenvalue of $\mathcal{L}^2$, and β is the eigenvalue of $\mathcal{L}_z$. Equation (9–14)/example 9–11 give us

$$\big[\mathcal{L}_z,\, \mathcal{L}_+\big] = \mathcal{L}_z\, \mathcal{L}_+ - \mathcal{L}_+\, \mathcal{L}_z = \hbar\, \mathcal{L}_+$$

$$\Rightarrow \quad \mathcal{L}_z\, \mathcal{L}_+ = \mathcal{L}_+\, \mathcal{L}_z + \hbar\, \mathcal{L}_+.$$

Using this in equation (9–16),

$$\begin{aligned}
\mathcal{L}_z\, \mathcal{L}_+\big|\alpha,\, \beta\big> &= \big(\mathcal{L}_+\, \mathcal{L}_z + \hbar\, \mathcal{L}_+\big)\big|\alpha,\, \beta\big> \\
&= \mathcal{L}_+\, \mathcal{L}_z\big|\alpha,\, \beta\big> + \hbar\, \mathcal{L}_+\big|\alpha,\, \beta\big> \\
&= \mathcal{L}_+\, \beta\big|\alpha,\, \beta\big> + \hbar\, \mathcal{L}_+\big|\alpha,\, \beta\big> \\
&= \big(\beta + \hbar\big)\mathcal{L}_+\big|\alpha,\, \beta\big> \,.
\end{aligned} \tag{9 – 17}$$

Summarizing,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\big) = \big(\beta + \hbar\big)\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\big),$$

which means $\mathcal{L}_+\big|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}_z$ with eigenvalue $\big(\beta+\hbar\big)$. The effect of $\mathcal{L}_+$ is to increase the eigenvalue of $\mathcal{L}_z$ by the amount $\hbar$, so it is called the **raising operator**. Note that it raises only the eigenvalue of $\mathcal{L}_z$. A better name would be the raising operator for $\mathcal{L}_z$, but the convention is when angular momentum is being discussed is to refer simply to the raising operator, and you need to know it applies only to $\mathcal{L}_z$.

Were we to calculate similarly, we would find $\mathcal{L}_-\big|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}_z$ with eigenvalue $\big(\beta-\hbar\big)$. The effect of $\mathcal{L}_-$ is to decrease the eigenvalue by the amount $\hbar$, so it is called the **lowering operator**. Again, the convention when angular momentum is being discussed is to refer to the lowering operator without reference to $\mathcal{L}_z$.

Example 9–13: Show $\big|\alpha,\,\beta\!>$ is an eigenvector of $\mathcal{L}^2$. Equation (9–13) yields

$$\big[\mathcal{L}^2,\,\mathcal{L}_+\big] = \mathcal{L}^2\,\mathcal{L}_+ - \mathcal{L}_+\,\mathcal{L}^2 = 0$$

$$\Rightarrow \quad \mathcal{L}^2\,\mathcal{L}_+ = \mathcal{L}_+\,\mathcal{L}^2.$$

Then

$$\mathcal{L}^2\,\mathcal{L}_+\big|\alpha,\,\beta\!> = \mathcal{L}_+\,\mathcal{L}^2\big|\alpha,\,\beta\!> = \mathcal{L}_+\,\alpha\big|\alpha,\,\beta\!> = \alpha\,\mathcal{L}_+\big|\alpha,\,\beta\!>,$$

or summarizing

$$\mathcal{L}^2\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\big) = \alpha\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\big),$$

so $\mathcal{L}_+\big|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}^2$ with eigenvalue α. Similarly, $\mathcal{L}_-\big|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}^2$ with eigenvalue α.

It is important that $\mathcal{L}_+\big|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}^2$, but be sure to notice that the raising/lowering operator has no effect on the eigenvalue of $\mathcal{L}^2$. The eigenvalue of $\mathcal{L}^2$ acting on an eigenstate is α. The eigenvalue of $\mathcal{L}^2$ acting on a combination of the raising/lowering operator and an eigenstate is still α.

Eigenvalue Solution for the Square of Orbital Angular Momentum

Recalling the relation between the four angular momentum operators,

$$\mathcal{L}^2 - \mathcal{L}_z^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2,$$

we are going use the eigenvalue equations and apply these operators to the generic eigenstate, *i.e.*,

$$\begin{aligned}
\big(\mathcal{L}^2 - \mathcal{L}_z^2\big)\big|\alpha,\,\beta\!> &= \mathcal{L}^2\big|\alpha,\,\beta\!> - \mathcal{L}_z^2\big|\alpha,\,\beta\!> \\
&= \alpha\big|\alpha,\,\beta\!> - \mathcal{L}_z\beta\big|\alpha,\,\beta\!> \\
&= \alpha\big|\alpha,\,\beta\!> - \beta^2\big|\alpha,\,\beta\!> \\
&= \big(\alpha-\beta^2\big)\big|\alpha,\,\beta\!> .
\end{aligned}$$

Forming an adjoint eigenstate and a braket,

$$\begin{aligned}
<\!\alpha,\beta\big|\mathcal{L}^2 - \mathcal{L}_z^2\big|\alpha,\,\beta\!> &= <\!\alpha,\beta\big|\mathcal{L}_x^2 + \mathcal{L}_y^2\big|\alpha,\,\beta\!> && (9-18) \\
&= <\!\alpha,\beta\big|\alpha-\beta^2\big|\alpha,\,\beta\!> \\
&= \big(\alpha-\beta^2\big) <\!\alpha,\beta\big|\alpha,\,\beta\!> && (9-19) \\
&= \alpha-\beta^2 \geq 0, && (9-20)
\end{aligned}$$

where we have assumed orthonormality of eigenstates in equation (9–19). The condition that the difference in equation (9–20) is non–negative is from the fact the braket is expressible in terms of a sum of $\mathcal{L}_x^2$ and $\mathcal{L}_y^2$, as seen in equation (9–18). Both $\mathcal{L}_x$ and $\mathcal{L}_y$ are Hermitian, so their eigenvalues are real. The sum of the squares of the eigenvalues, corresponding to operations by $\mathcal{L}_x^2$ and $\mathcal{L}_y^2$ in equation (9–18), must be non–negative. In mathematical vernacular, $\mathcal{L}_x^2$ and $\mathcal{L}_y^2$ are positive definite.

Equation (9–20) is equivalent to $\alpha \geq \beta^2$, which means β is bounded for a given value of α. Therefore there is an eigenstate $|\alpha, \beta_{\text{max}}>$ which cannot be raised, and another eigenstate $|\alpha, \beta_{\text{min}}>$ which cannot be lowered. In other words, we have a ladder which has a bottom, like the SHO, and a top, unlike the SHO. In a calculation similar to example 9–9,

$$\mathcal{L}_-\mathcal{L}_+ = \mathcal{L}_x^2 + \mathcal{L}_y^2 - \hbar = \mathcal{L}^2 - \mathcal{L}_z^2 - \hbar\mathcal{L}_z,$$

so

$$\begin{aligned}
&\mathcal{L}_-\mathcal{L}_+|\alpha, \beta_{\text{max}}> = \vec{0} \\
\Rightarrow\quad &\big(\mathcal{L}^2 - \mathcal{L}_z^2 - \hbar\mathcal{L}_z\big)|\alpha, \beta_{\text{max}}> = 0 && (9-21)\\
\Rightarrow\quad &\mathcal{L}^2|\alpha, \beta_{\text{max}}> - \mathcal{L}_z^2|\alpha, \beta_{\text{max}}> - \hbar\mathcal{L}_z|\alpha, \beta_{\text{max}}> = 0 \\
\Rightarrow\quad &\alpha|\alpha, \beta_{\text{max}}> - \beta_{\text{max}}^2|\alpha, \beta_{\text{max}}> - \hbar\,\beta_{\text{max}}|\alpha, \beta_{\text{max}}> = 0 \\
\Rightarrow\quad &\big(\alpha - \beta_{\text{max}}^2 - \hbar\,\beta_{\text{max}}\big)|\alpha, \beta_{\text{max}}> = 0 \\
\Rightarrow\quad &\alpha - \beta_{\text{max}}^2 - \hbar\,\beta_{\text{max}} = 0 \\
\Rightarrow\quad &\alpha = \beta_{\text{max}}^2 + \hbar\,\beta_{\text{max}}. && (9-22)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathcal{L}_+\mathcal{L}_-|\alpha, \beta_{\text{min}}> = \vec{0} \\
\Rightarrow\quad &\alpha = \beta_{\text{max}}^2 - \hbar\,\beta_{\text{max}}. && (9-23)
\end{aligned}$$

Equating equations (9–22) and (9–23), we get

$$\beta_{\text{max}}^2 + \hbar\beta_{\text{max}} - \beta_{\text{min}}^2 + \hbar\beta_{\text{min}} = 0.$$

This is quadratic in both β_{max} and β_{min}, and to solve the equation, we will use the quadratic formula to solve for β_{max}, or

$$\begin{aligned}
\beta_{\text{max}} &= -\frac{1}{2}\hbar \pm \frac{1}{2}\sqrt{\hbar^2 - 4(-\beta_{\text{min}}^2 + \hbar\,\beta_{\text{min}})} \\
&= -\frac{1}{2}\hbar \pm \frac{1}{2}\sqrt{4\beta_{\text{min}}^2 - 4\hbar\,\beta_{\text{min}} + \hbar^2} \\
&= -\frac{1}{2}\hbar \pm \frac{1}{2}\sqrt{(2\beta_{\text{min}} - \hbar)^2} \\
&= -\frac{1}{2}\hbar \pm \frac{1}{2}(2\beta_{\text{min}} - \hbar) \\
\Rightarrow\quad \beta_{\text{max}} &= -\beta_{\text{min}}, \quad \beta_{\text{min}} - \hbar. && (9-24)
\end{aligned}$$

The case $\beta_{\text{max}} = -\beta_{\text{min}}$ is the maximum separation case. It gives us the top and bottom of the ladder. We assume the rungs of the ladder are separated by $\hbar$, because that is the amount of change indicated by the raising and lowering operators. The picture corresponds to figure 9–1. If there is other than minimum separation, say there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 9–1 we expect

$$2\beta_{\max} = n\hbar \quad \Rightarrow \quad \beta_{\max} = \frac{n\hbar}{2}.$$

Using this in equation (9–22),

$$\begin{aligned} \alpha &= \beta_{\max}\big(\beta_{\max} + \hbar\big) \\ &= \frac{n\hbar}{2}\left(\frac{n\hbar}{2} + \hbar\right) \\ &= \hbar^2\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right). \end{aligned}$$

We are going to re–label, letting $j = n/2$, so

$$\alpha = \hbar^2\, j\big(j+1\big). \tag{9 – 25}$$

Wait a minute.... The fact $j = \hbar/2$ vice just $\hbar$ does not appear consistent with the assumption that the rungs of the ladder are separated by $\hbar$...and it isn't. It appears the rungs of the ladder are separated by $\hbar/2$ vice $\hbar$.

What has occurred is that we have actually solved a more general problem than intended. Because of symmetry, the linear algebra arguments have given us the solution for **total angular momentum**. Total angular momentum is

$$\vec{J} = \vec{L} + \vec{S}, \tag{9 – 26}$$

where $\vec{L}$ is **orbital angular momentum**, $\vec{S}$ is **spin angular momentum** or just **spin**. We posed the problem for orbital angular momentum, but because total angular momentum and spin obey analogous commutation relations to orbital angular momentum, we arrive at the solution for total angular momentum. Equations (9–7) indicated components of orbital angular momentum do not commute,

$$\big[\,\mathcal{L}_x,\,\mathcal{L}_y\,\big] = i\hbar\,\mathcal{L}_z, \qquad \big[\,\mathcal{L}_y,\,\mathcal{L}_z\,\big] = i\hbar\,\mathcal{L}_x, \qquad \text{and} \qquad \big[\,\mathcal{L}_z,\,\mathcal{L}_x\,\big] = i\hbar\,\mathcal{L}_y,$$

and for the ladder operator solution, we formed $\mathcal{L}_\pm = \mathcal{L}_x \pm i\mathcal{L}_y$. The commutation relations among the components of total angular momentum and spin angular momentum are exactly the same, *i.e.*,

$$\big[\,\mathcal{J}_x,\,\mathcal{J}_y\,\big] = i\hbar\,\mathcal{J}_z, \qquad \big[\,\mathcal{J}_y,\,\mathcal{J}_z\,\big] = i\hbar\,\mathcal{J}_x, \qquad \text{and} \qquad \big[\,\mathcal{J}_z,\,\mathcal{J}_x\,\big] = i\hbar\,\mathcal{J}_y,$$

and

$$\big[\,\mathcal{S}_x,\,\mathcal{S}_y\,\big] = i\hbar\,\mathcal{S}_z, \qquad \big[\,\mathcal{S}_y,\,\mathcal{S}_z\,\big] = i\hbar\,\mathcal{S}_x, \qquad \text{and} \qquad \big[\,\mathcal{S}_z,\,\mathcal{S}_x\,\big] = i\hbar\,\mathcal{S}_y.$$

If we had started out with $\mathcal{J}_\pm = \mathcal{J}_x \pm i\mathcal{J}_y$, or $\mathcal{S}_\pm = \mathcal{S}_x \pm i\mathcal{S}_y$, we would have come out with exactly the same result. In fact, this is the problem we solved, except using the symbol $\mathcal{L}$ vice $\mathcal{J}$ or $\mathcal{S}$.

We will reinforce in chapter 13 that spin can have half integral values, or values of multiples of $\hbar/2$. Since spin can be half integral, values of total angular momentum can be half integral. When we use symbols such as $\mathcal{L}^2$ and $\mathcal{L}_i$, we get the information contained in the commutation relations, independent of whatever symbols we choose. Had we used explicit representations, such as equations (9–8) and (9–9), we would get the same information, however, limited by the representation. In that case, only integral values would be possible, though the form of the result analogous to equation (9–25) would remain the same. Using l as the quantum number for orbital angular momentum, the eigenvalue for orbital angular momentum squared is

$$\alpha = \hbar^2\, l(l+1). \qquad (9-27)$$

A comment about the picture and notation is appropriate. The first impression is that this is similar to classical mechanics. The earth orbits the sun and has orbital angular momentum in that regard, and also spins on its axis so has spin angular momentum as well. It is tempting to apply this picture to a quantum mechanical system, say an electron in a hydrogen atom. It simply does not apply. The electron is not a small ball spinning on its axis as it orbits the proton. Per the first chapter, an electron is not a particle, it is not a wave, it is an electron. There is no classical analogy for an electron, and many of the manifestations of quantum mechanical angular momentum are similarly not classical analogs.

Equation (9–26) says the total is the sum of the parts, but it is an operator equation which in Dirac notation is $|\mathcal{J}\!> = |\mathcal{L}\!> + |\mathcal{S}\!>$. Since earlier development was in this form, it may be useful to assist you to realize that each of these three operators has three components which are also each operators. Equation (9-26) is standard notation nevertheless.

Eigenvalue Solution for the Z Component of Orbital Angular Momentum

We have calculated the eigenvalue of $\mathcal{L}^2$, but still need to find the eigenvalue of $\mathcal{L}_z$. We know one of the possible eigenvalues of $\mathcal{L}_z$ is zero from the last of equations (9–8), the explicit representations, regardless of the eigenstate. We have also calculated

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\big) = \big(\beta+\hbar\big)\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\big).$$

If we start with an eigenstate that has the z component of angular momentum equal to zero,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha,\,0\!>\big) = \big(0+\hbar\big)\big(\mathcal{L}_+\big|\alpha,\,0\!>\big) = \hbar\big(\mathcal{L}_+\big|\alpha,\,0\!>\big),$$

so $\hbar$ is the next eigenvalue. Using $\hbar$ as the eigenvalue,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha,\,\hbar\!>\big) = \big(\hbar+\hbar\big)\big(\mathcal{L}_+\big|\alpha,\,\hbar\!>\big) = 2\hbar\big(\mathcal{L}_+\big|\alpha,\,\hbar\!>\big),$$

so $2\hbar$ is the next eigenvalue. If we use this as an eigenvalue,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha,\,2\hbar\!>\big) = \big(\hbar+2\hbar\big)\big(\mathcal{L}_+\big|\alpha,\,2\hbar\!>\big) = 3\hbar\big(\mathcal{L}_+\big|\alpha,\,2\hbar\!>\big),$$

and $3\hbar$ is the next eigenvalue up the ladder. We can continue, and will attain integral values of $\hbar$. But we cannot continue forever, because we determined β is bounded by the eigenvalue of $\mathcal{L}^2$. What is the maximum value? We go back to figure 9–1 and the result from this figure is

$$\beta_{\max} = \frac{n\hbar}{2},$$

where we want only integral values for the orbital angular momentum, so this becomes

$$\beta_{\max} = l\hbar.$$

Were we to do the same calculation with the lowering operator, that is

$$\mathcal{L}_z\big(\mathcal{L}_-\big|\alpha,\, 0>\,\big) = -\hbar\big(\mathcal{L}_-\big|\alpha,\, 0>\,\big),$$

we step down the ladder in increments of $-\hbar$ until we get to $\beta_{\min}$. Remember $\beta_{\min}$ also has a minimum, which is of the same magnitude but negative or

$$\beta_{\min} = -l\hbar.$$

So we have eigenvalues which climb to $l\hbar$ and drop to $-l\hbar$ in integral increments of $\hbar$. The eigenvalue of the z component of angular momentum is just an integer times $\hbar$, from minimum to maximum values. The symbol conventionally used to denote this integer is m, so

$$\mathcal{L}_z\big|\alpha,\, \beta> = m\hbar\big|\alpha,\, \beta>, \quad -l < m < l$$

is the eigenvalue/eigenvector equation for the z component of angular momentum. The quantum number m, occasionally denoted m_l, is known as the **magnetic quantum number**.

Eigenvalue/Eigenvector Equations for Orbital Angular Momentum

If we use l vice α to denote the state of total angular momentum, realizing l itself is not an eigenvalue of $\mathcal{L}^2$, and m to denote the state of the z component of angular momentum, realizing the eigenvalue of $\mathcal{L}_z$ is actually $m\hbar$, the eigenvalue/eigenvector equations for $\mathcal{L}^2$ and $\mathcal{L}_z$ are

$$\mathcal{L}^2\big|l,\, m> = \hbar^2\, l(l+1)\big|l,\, m>, \tag{9 – 28}$$

$$\mathcal{L}_z\big|l,\, m> = m\hbar\big|l,\, m>, \quad -l < m < l \tag{9 – 29}$$

which is the conventional form of the two eigenvalue/eigenvector equations for $\mathcal{L}^2$ and $\mathcal{L}_z$.