from a linear combination of Hermite polynomials. The Hermite polynomials, therefore, form an orthogonal basis, and further, form an orthonormal basis when they are normalized.

The infinite set of unit vectors is orthonormal and is complete. The infinite set of sines and cosines used for the infinite square well is orthogonal so can be made orthonormal and is complete. The infinite set of Hermite polynomials used for the SHO is orthogonal so can be made orthonormal and is complete. The infinite sets of Associated Laguerre polynomials, Legendre functions, spherical harmonic functions, and numerous other sets of polynomials and functions are orthogonal so can be made orthonormal and complete. Each of these infinite sets form a basis in the same sense as the unit vectors of chapter 1 form a basis. These infinite sets of polynomials and functions generally require a weighting function to demonstrate orthogonality which, again, is a fact that is not always stated explicitly.

---

19. Given a simple harmonic oscillator potential, graph the first six eigenenergies on an energy versus position plot and superimpose the the first six eigenfunctions on corresponding eigenenergies on the same plot. Plot the probability densities of the first six eigenfunctions in the same manner.

Examine the two graphs below.

Eigenenergies and Eigenfunctions

Eigenenergies and Probability Density

Postscript: Like the infinite square well, it is conventional to graph energy versus position for the eigenenergies and the eigenfunctions are conventionally located at the level of the corresponding eigenenergies where each horizontal line represents zero amplitude for that eigenfunction.

Unlike the infinite square well, the eigenfunctions do not have an amplitude of zero at the boundaries. The eigenfunctions approach zero asymptotically outside the potential well. Further, eigenenergies for the SHO are evenly spaced, the ground state is at $\hbar\omega/2$, and each successive eigenenergy is $\hbar\omega$ higher than the last. Also in contrast, the eigenenergies of the infinite square well scale as n^2, that is $E_1 = E_g$, $E_2 = 4\,E_g$, $E_3 = 9\,E_g$, $E_4 = 16\,E_g$, etc.

Like the infinite square well, the probability densities are non-negative everywhere, there are points inside the well where the probability density is zero, and there are regions of maximal and minimal probability. Unlike the infinite square well and an additional non–classical feature of the SHO is that there are regions of non-zero probability density outside the potential well—there is a finite probability of finding the particle outside of the potential well. That there is a non-zero probability density outside the wall is a feature of all but infinite, vertical potential walls.

---

20. Sketch the linear combination $\Psi(x) = 2\,\psi_0(x) + \psi_1(x)$ for the SHO.

A general wavefunction of the SHO is a superposition or linear combination of its eigenfunctions,

$$\Psi(x) = c_0\,\psi_0(x) + c_1\,\psi_1(x) + c_2\,\psi_2(x) + c_3\,\psi_3(x) + \cdots = \sum_{n=0}^{\infty} c_n\,\psi_n(x)\,, \quad \text{in general, or}$$

$$|\,\Psi> = c_0\,|\,0> + \,c_1\,|\,1> + \,c_2\,|\,2> + \,c_3\,|\,3> + \cdots = \sum_{n=0}^{\infty} c_n\,|\,n> \quad \text{for the SHO.}$$

The linear combination given is combined graphically below.

A General State Vector that is the Superposition of two Eigenstates.

Postscript: The c_n are constants that provide the relative contributions of each eigenfunction. The c_n can be any scalars so $\Psi(x)$ can have any shape. As before, if the general wavefunction is normalized, $\Psi(x) = 1$, the relative magnitudes of the c_n are fixed. Also as before, the orthogonality of the eigenfunctions of the SHO ensures that the c_n are unique.

Problems 21 through 26 use the linear combination of two eigenstates,

$$|\,\Psi> = A\left[\;2\,|\,0> + \,5\,|\,2>\;\right],$$

which is the general linear combination of eigenstates for $c_0 = 2\,,\; c_2 = 5\,,$ and all other $c_n = 0\,.$

---

21. (a) Normalize the wavefunction $|\,\Psi>$ using row and column vectors, and

(b) using Dirac notation.

(a)
$$< \Psi \,|\, \Psi > = 1 \;\;\Rightarrow\;\; (2,\; 0,\; 5,\; 0,\; 0, \cdots)^* \, A^* \, A \begin{pmatrix} 2 \\ 0 \\ 5 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = 1$$

$$\Rightarrow \;\; (4+25)\,|\,A\,|^2 = 1 \;\;\Rightarrow\;\; A = \frac{1}{\sqrt{29}} \;\;\Rightarrow\;\; |\,\Psi > = \frac{1}{\sqrt{29}} \Big[\, 2\,|\,0> + \, 5\,|\,2> \Big] .$$

(b)
$$< \Psi \,|\, \Psi > = 1 \;\;\Rightarrow\;\; \Big[< 0\,|\,2^* + < 2\,|\,5^* \Big] \, A^* \, A \Big[2\,|\,0> + 5\,|\,2> \Big] = 1$$

$$\Rightarrow \;\; |\,A\,|^{\,2} \Big[4 < 0\,|\,0> + \, 10 < 0\,|\,2> + \, 10 < 2\,|\,0> + \, 25 < 2\,|\,2> \Big] = 1 \, .$$

Orthonormality, $<i\,|\,j> = \delta_{ij} \;\;\Rightarrow\;\; <0\,|\,0> \, = \, <2\,|\,2> = 1\,,$ and $<0\,|\,2> \, = \, <2\,|\,0> = 0\,,$ so

$$|\,A\,|^{\,2} \Big[4 + 25 \Big] = 1 \;\;\Rightarrow\;\; A = \frac{1}{\sqrt{29}} \;\;\Rightarrow\;\; |\,\Psi > = \frac{1}{\sqrt{29}} \Big[\, 2\,|\,0> + \, 5\,|\,2> \Big] .$$

Postscript: The row and column vector representation is likely easier to visualize, but normalizing $A \big[\, 5\,|\,0> + \; 6\,|\,100> \big]$ or any large system using the row and column method would be awkward. Both of these approaches are significantly easier than the same calculation in position space that requires evaluation of at least two integrals.

A portion of the utility of abstract Hilbert space is that calculations are dramatically simpler than in any specific representation, such as position space. Abstract Hilbert space also allows maximum generality in that you can represent your work in any appropriate basis at any time. Working in Hilbert space until a representation is necessary is the norm. There is no reason to represent the results of the last examples in any specific basis so they remain in Hilbert space.

---

22. Find the normalized wavefunction $|\,\Psi>$ in position space.

Use the ground state and procedures similar to those seen in problem 17.

$$\psi_2\,(x) = \left(\frac{m\omega}{4\pi\hbar} \right)^{1/4} \left(2 \, \frac{m\omega}{\hbar} \, x^2 - 1 \right) e^{-m\omega x^2/\,2\hbar}$$

using table 8–1. Combining with the ground state from problem 17,

$$\Psi\,(x) = \frac{1}{\sqrt{29}} \left[2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-m\omega x^2/\,2\hbar} + 5 \left(\frac{m\omega}{4\pi\hbar} \right)^{1/4} \left(2 \, \frac{m\omega}{\hbar} \, x^2 - 1 \right) e^{-m\omega x^2/\,2\hbar} \right]$$
$$= \frac{1}{\sqrt{29}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \left[2 + \frac{5\sqrt{2}}{2} \left(2 \, \frac{m\omega}{\hbar} \, x^2 - 1 \right) \right] e^{-m\omega x^2/\,2\hbar} .$$

---

22. Calculate the probability of measuring $E = \frac{5}{2}\hbar\omega$ for the $|\,\Psi>$ given.

Postulates 3 and 4. The only possible results of a measurement of energy are the eigenvalues $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. The state vector contains only the two eigenstates $|\,n=0>$ and $|\,n=2>$, so $E_0 = \frac{1}{2}\hbar\omega$ and $E_2 = \frac{5}{2}\hbar\omega$ are the only possible results. Then $P\,(E = E_n) = |<n\,|\,\Psi>|^2$.

$$P\,(E = E_2) = \left| (0,\ 0,\ 1,\ 0, \cdots)\ \frac{1}{\sqrt{29}} \begin{pmatrix} 2 \\ 0 \\ 5 \\ 0 \\ \vdots \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{29}}\,(0 + 0 + 5 + 0 + \cdots) \right|^2 = \left| \frac{5}{\sqrt{29}} \right|^2 = \frac{25}{29},$$

using unit vector notation. The same calculation in Dirac notation looks like

$$P\,(E = E_2) = |<2\,|\,\Psi>|^2 = \left| <2\,|\,\frac{1}{\sqrt{29}}\Big(2\,|\,0> + 5\,|\,2>\Big) \right|^2$$

$$= \frac{1}{29}\,|\,2<2\,|\,0> + 5<2\,|\,2>|^2 = \frac{1}{29}\,|\,5\,|^2 = \frac{25}{29},$$

where the inner products are $<i\,|\,j> = \delta_{ij}$, meaning $<2\,|\,0> = 0$ and $<2\,|\,2> = 1$.

---

23. Calculate the expectation value of energy for the $|\,\Psi>$ given.

This problem demonstrates the calculation using the chapter 1 mathematics and is also intended to highlight the relative ease of calculations using matrix and Dirac notation. Setup the calculation in position space just to demonstrate the relative degree of difficulty.

$$<E> = <\mathcal{H}>_\psi = <\psi\,|\,\mathcal{H}\,|\,\psi> = \frac{1}{\sqrt{29}}\big(2,\ 0,\ 5,\ \cdots\big)^* \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{3}{2} & 0 & \cdots \\ 0 & 0 & \frac{5}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \hbar\omega\ \frac{1}{\sqrt{29}} \begin{pmatrix} 2 \\ 0 \\ 5 \\ \vdots \end{pmatrix}$$

$$= \frac{\hbar\omega}{29}\,\big(2,\ 0,\ 5,\ \cdots\big) \begin{pmatrix} \frac{1}{2}(2) \\ \frac{3}{2}(0) \\ \frac{5}{2}(5) \\ \vdots \end{pmatrix} = \frac{\hbar\omega}{29}\left(2 + \frac{125}{2}\right) = \frac{129}{58}\,\hbar\omega.$$

Notice that we attain the same result if the calculation is done in a three dimensional subspace. The same calculation in Dirac notation using the direct action of the Hamiltonian is

$$
\begin{aligned}
<E> &= <\psi\,|\,\mathcal{H}\,|\,\psi> \\
&= \left[\frac{1}{\sqrt{29}}\left(<0\,|\,2^* + <2\,|\,5^*\right)\right]\mathcal{H}\left[\frac{1}{\sqrt{29}}\left(2\,|\,0> + 5\,|\,2>\right)\right] && (1)\\
&= \frac{1}{29}\Big[<0\,|\,2 + <2\,|\,5\Big]\left[2\left(\frac{1}{2}\,\hbar\omega\right)|\,0> + 5\left(\frac{5}{2}\,\hbar\omega\right)|\,2>\right] && (2)\\
&= \frac{1}{29}\left[2\cdot 2\cdot\left(\frac{1}{2}\,\hbar\omega\right)<0\,|\,0> + 5\cdot 5\cdot\left(\frac{5}{2}\,\hbar\omega\right)<2\,|\,2>\right] && (3)\\
&= \frac{\hbar\omega}{29}\left[2 + \frac{125}{2}\right] = \frac{129}{58}\,\hbar\omega,
\end{aligned}
$$

where the Hamiltonian operating to the right on the two eigenstates in equation (1) results in the eigenvalues times the corresponding eigenstate in equation (2), and the orthonormality of eigenstates results in equation (3) where the inner product of terms that are known to be zero. Excluding known zeros is the norm, and in fact, eigenstates such as $<0\,|\,0> = <2\,|\,2> = 1$ that are known to be one are normally not explicitly written.

Postscript: The calculation of expectation value for the given $|\,\Psi>$ in position space would be

$$
\int_{-\infty}^{\infty}\frac{1}{\sqrt{29}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[2 + \frac{5\sqrt{2}}{2}\left(2\,\frac{m\omega}{\hbar}\,x^2 - 1\right)\right]e^{-m\omega x^2/2\hbar}\left[\frac{1}{2m}\left(-i\hbar\,\frac{d}{dx}\right)^2 + \frac{1}{2}\,kx^2\right]
$$

$$
\times\frac{1}{\sqrt{29}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[2 + \frac{5\sqrt{2}}{2}\left(2\,\frac{m\omega}{\hbar}\,x^2 - 1\right)\right]e^{-m\omega x^2/2\hbar}\,dx\,,
$$

which is a statement that implies that matrix methods and Dirac notation are worthwhile, in other words, if you want to do this integral, go ahead, we chose to avoid it in favor of modern techniques.

---

24. Calculate the expectation value of energy for the $|\,\Psi\,>$ given using the ladder operator representation of the Hamiltonian.

This problem is an explicit example of ladder operators calculations. It is not the easiest way to calculate an expectation value. The ladder operators though, are prototypes for the creation and annihilation operators used in field theoretic descriptions of photons, electrons, phonons, etc. In other words, the raising and lowering operators become increasingly important as you progress.

$$
\mathcal{H} = \left(a^{\dagger}a + \frac{1}{2}\right)\hbar\omega,\quad a^{\dagger}\,|\,n> = \sqrt{n+1}\,|\,n+1>,\quad a\,|\,n> = \sqrt{n}\,|\,n-1>,\quad <E> = <\psi\,|\,\mathcal{H}\,|\,\psi>
$$

$$\Rightarrow \; <E> \; = \left[\frac{1}{\sqrt{29}} \left(<0\,|\,2^* + \; <2\,|\,5^* \right) \right] \left[\left(a^\dagger a + \frac{1}{2} \right) \hbar\omega \right] \left[\frac{1}{\sqrt{29}} \left(2\,|\,0> + 5\,|\,2> \right) \right]$$

$$= \frac{\hbar\omega}{29} \Big[2 <0\,| \, + \, 5 <2\,| \; \Big] \left[2\,a^\dagger a\,|\,0> + \, 5\,a^\dagger a\,|\,2> + \frac{2}{2}\;|\,0> + \frac{5}{2}\;|\,2> \right]$$

$$= \frac{\hbar\omega}{29} \Big[2 <0\,| \, + \, 5 <2\,| \; \Big] \left[2\,\cancel{a}^\dagger\,\vec{0} + \, 5\,a^\dagger \sqrt{2}\,|\,1> + \,|\,0> + \frac{5}{2}\,|\,2> \right] \tag{1}$$

$$= \frac{\hbar\omega}{29} \Big[2 <0\,| \, + \, 5 <2\,| \; \Big] \left[\, 5\sqrt{2} \cdot \sqrt{2}\,|\,2> + \,|\,0> + \frac{5}{2}\,|\,2> \right]$$

As before, the raising operator acting on the zero vector is zero so that term is struck in equation (1). The only non-zero contributions are from the $<0\,|\,0>$ and $<2\,|\,2>$ terms since these inner products are 1, the $<0\,|\,2>$ and $<2\,|\,0>$ products are 0, therefore,

$$<E> = \frac{\hbar\omega}{29} \left[2 <0\,|\,0> + \, 5{\cdot}5{\cdot}2 <2\,|\,2> + \, 5{\cdot}\frac{5}{2} <2\,|\,2> \right] = \frac{\hbar\omega}{29} \left[2 + 5{\cdot}5{\cdot}2 + 5{\cdot}\frac{5}{2} \right] = \frac{129}{58}\,\hbar\omega\,.$$

---

25. Express the state of the system described by $|\,\Psi>$ at time t,

(a) in Hilbert space in terms of the abstract $|\,n>$'s,

(b) in position space in terms of the $\psi_n\,(x)$'s,

(c) in momentum space in terms of the $\widehat{\psi}_n\,(p)$'s, and

(d) in energy space in terms of the $\widetilde{\psi}_n\,(E)$'s. Do not evaluate the specific $|\,n>$'s, $\psi_n\,(x)$'s, $\widehat{\psi}_n\,(p)$'s and $\widetilde{\psi}_n\,(E)$'s.

Stationary state time dependence is

$$|\,\psi\,(t)> = \sum_n^{\infty} |\,n><n\,|\,\psi\,(0)> \; e^{-iE_n t/\hbar} \quad \text{where} \quad |\,\psi\,(0)> = \frac{1}{\sqrt{29}} \Big[2\,|\,0> + \, 5\,|\,2> \Big] \quad \text{here.}$$

$$\text{(a)} \quad |\,\psi\,(t)> \; = \frac{1}{\sqrt{29}} \Big[\,|\,0><0\,| \left(2\,|\,0> + 5\,|\,2> \right) e^{-iE_0 t/\hbar} + \,|\,2><2\,| \left(2\,|\,0> + 5\,|\,2> \right) e^{-iE_2 t/\hbar} \Big]$$

$$= \frac{1}{\sqrt{29}} \Big[\,|\,0> \left(2 <0\,|\,0> + 5 <0\cancel{|}2>, \right) e^{-iE_0 t/\hbar} + \,|\,2> \left(2 <2\cancel{|}0> + \, 5 <2\,|\,2> \right) e^{-iE_2 t/\hbar} \Big]$$

$$= \frac{1}{\sqrt{29}} \Big[\, 2\,e^{-iE_0 t/\hbar}\,|\,0> + 5\,e^{-iE_2 t/\hbar}\,|\,2> \Big]$$

retaining only the non-zero terms. Using the eigenenergies $E_n = \left(n + \frac{1}{2} \right) \hbar\omega$,

$$|\,\psi\,(t)> = \frac{1}{\sqrt{29}} \Big[\, 2\,|\,0> e^{-i\omega t/2} + 5\,|\,2> e^{-i5\omega t/2} \Big] \,.$$

Parts (b) through (d) require only operation on both sides of this equation with the appropriate bra so are essentially exercises in notation.

(b) Remembering that $<x\,|\,\psi\,(t)> = \psi\,(x,t)\,,$

$$
\begin{aligned}
<x\,|\,\psi\,(t)> \ &= \frac{1}{\sqrt{29}}\Big[2<x\,|\,0>e^{-i\omega t/2} + \ 5<x\,|\,2>e^{-i5\omega t/2}\Big] \\
\Rightarrow \quad \psi\,(x,t) &= \frac{1}{\sqrt{29}}\Big[2\;\psi_0\,(x)\;e^{-i\omega t/2} + \ 5\;\psi_2\,(x)\;e^{-i5\omega t/2}\Big].
\end{aligned}
$$

(c) To transition to momentum space, $<p\,|\,\psi\,(t)> = \widehat{\psi}\,(p,t)\,,$

$$
\begin{aligned}
<p\,|\,\psi\,(t)> \ &= \frac{1}{\sqrt{29}}\Big[2<p\,|\,0>e^{-i\omega t/2} + \ 5<p\,|\,2>e^{-i5\omega t/2}\Big] \\
\Rightarrow \quad \widehat{\psi}\,(p,t) &= \frac{1}{\sqrt{29}}\Big[2\;\widehat{\psi}_0\,(p)\;e^{-i\omega t/2} + \ 5\;\widehat{\psi}_2\,(p)\;e^{-i5\omega t/2}\Big].
\end{aligned}
$$

(d) To find the wavefunction in energy space, remember that $<E\,|\,\psi\,(t)> = \widetilde{\psi}\,(E,t)\,,$

$$
<E\,|\,\psi\,(t)> = \frac{1}{\sqrt{29}}\Big[2<E\,|\,0>\ e^{-i\omega t/2} + \ 5<E\,|\,2>\ e^{-i5\omega t/2}\Big]
$$

$$
\Rightarrow \quad \widetilde{\psi}\,(E,t) = \frac{1}{\sqrt{29}}\Big[2\,\widetilde{\psi}_0\,(E,t)\;e^{-i\omega t/2} + \ 5\,\widetilde{\psi}_2\,(E,t)\;e^{-i5\omega t/2}\Big].
$$

---

26. Calculate the probability of measuring $E = \frac{5}{2}\hbar\omega$ at time t for the $|\,\Psi>$ given.

Likely the best choice or representation is the abstract state vector. You will find that the same calculation is much more arduous in position space in one of the problems at the end of this part. Remember that the square of a magnitude is the product of the complex conjugates.

Since the state vector is normalized, probability is $P\;(E = E_n) = |<n\,|\,\psi\,(t)>|^2$, then

$$
\begin{aligned}
P\;(E = E_2) &= \left|\;<2\,|\,\frac{1}{\sqrt{29}}\Big[2\,|\,0>e^{-i\omega t/2} + \ 5\,|\,2>e^{-i5\omega t/2}\Big]\;\right|^2 \\
&= \frac{1}{29}\Big|\,2\ \cancel{<2\,|\,0>}\,e^{-i\omega t/2} + 5\ <2\,|\,2>e^{-i5\omega t/2}\,\Big|^2,
\end{aligned}
$$

where the first term is struck because the eigenstates of the SHO are orthonormal so the inner product of unlike vectors is zero. Orthonormality also means that $<2\,|\,2> = 1\,.$ Then

$$
P\;(E = E_2) = \frac{1}{29}\left|\,5\,e^{-i5\omega t/2}\,\right|^2 = \frac{1}{29}\Big(5\,e^{i5\omega t/2}\Big)\Big(5\,e^{-i5\omega t/2}\Big) = \frac{1}{29}\Big(25\,e^0\Big) = \frac{25}{29}\,,
$$

necessarily identical to the time independent case for all startionary state probability calculations.

---

27. Calculate $<\mathcal{P}>$ using the state vector

$$|\,\Phi\,(t)> \;=\; \frac{1}{\sqrt{30}}\left[2\,|\,0>e^{-i\omega t/2}+\,|\,1>e^{-i3\omega t/2}+\;5\,|\,2>e^{-i5\omega t/2}\right].$$

Time never appears in any probability of a stationary state. The exponential stationary state time dependence does affect other calculations, however, such as an expectation value of a dynamic variable. We pick $<\mathcal{P}>\,=\,<\Phi\,(t)\,|\,\mathcal{P}\,|\,\Phi\,(t)>$ to demonstrate both (a) vector/matrix operator and (b) Dirac notation/ladder operator calculation of an expectation value that is time dependent. The principles are the same for $<\mathcal{H}>$, $<\mathcal{X}>$, $<\mathcal{P}^2/2m>$, and $<m\omega^2\mathcal{X}^2/2>$, or other classically dynamical variables. The state vector is changed to the three-dimensional $|\,\Phi>$ because the two-dimensional $|\,\Psi>$ used for the last six problems leads to too many zeros to be a good example.

(a) Using the matrix operator representation of $\mathcal{P}$ for the SHO found earlier,

$$<\mathcal{P}>=<\Phi(t)\,|\,i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & -\sqrt{5} & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix} 2e^{-i\omega t/2} \\ e^{-i3\omega t/2} \\ 5e^{-i5\omega t/2} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$=\frac{i}{30}\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(2\,e^{+i\omega t/2},\;e^{+i3\omega t/2},\;5\,e^{+i5\omega t/2},\;0,\cdots\right)\begin{pmatrix} -e^{-i3\omega t/2} \\ 2\,e^{-i\omega t/2}-5\sqrt{2}\,e^{-i5\omega t/2} \\ \sqrt{2}\,e^{-i3\omega t/2} \\ 5\sqrt{3}\,e^{-i5\omega t/2} \\ \vdots \end{pmatrix}$$

$$=\frac{i}{30}\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(-2\,e^{-i\omega t}+2\,e^{+i\omega t}-5\sqrt{2}\,e^{-i\omega t}+5\sqrt{2}\,e^{+i\omega t}+0\right)$$

$$=\frac{i}{30}\left(\frac{m\omega\hbar}{2}\right)^{1/2}2i\left(2\frac{(e^{+i\omega t}-e^{-i\omega t})}{2i}+5\sqrt{2}\,\frac{(e^{+i\omega t}-e^{-i\omega t})}{2i}\right)$$

$$=-\frac{1}{15}\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(2\sin(\omega t)+5\sqrt{2}\sin(\omega t)\right)=-\left(\frac{2+5\sqrt{2}}{15}\right)\left(\frac{m\omega\hbar}{2}\right)^{1/2}\sin(\omega t)\,.$$

(b) The momentum operator is $\mathcal{P}=i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^\dagger-a)$ in terms of the ladder operators, so

$$\begin{aligned}<\mathcal{P}>\;&=\;<\Phi\,(t)\,\big|\,\mathcal{P}\,\big|\Phi\,(t)>\\ &=<\Phi\,(t)\,|\,i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^\dagger-a)\big|\frac{1}{\sqrt{30}}\left(2\,|\,0>e^{-i\omega t/2}+\,|\,1>e^{-i3\omega t/2}+5\,|\,2>e^{-i5\omega t/2}\right)\\ &=<\Phi\,(t)\,|\,i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\frac{1}{\sqrt{30}}\Big(2\,a^\dagger\,|\,0>e^{-i\omega t/2}-2\,a\,|\,0>e^{-i\omega t/2}+a^\dagger\,|\,1>e^{-i3\omega t/2}\\ &\qquad\qquad -a\,|\,1>e^{-i3\omega t/2}+5\,a^\dagger\,|\,2>e^{-i5\omega t/2}-5\,a\,|\,2>e^{-i5\omega t/2}\Big)\end{aligned}$$

$$= <\Phi(t)\,|\, i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\frac{1}{\sqrt{30}}\Big(2\sqrt{1}\,|\,1>\,e^{-i\omega t/2} - 0 + \sqrt{2}\,|\,2>\,e^{-i3\omega t/2} - \sqrt{1}\,|\,0>\,e^{-i3\omega t/2}$$
$$+\,5\sqrt{3}\,|\,3>\,e^{-i5\omega t/2} - 5\sqrt{2}\,|\,1>\,e^{-i5\omega t/2}\Big)$$
$$= <\Phi(t)\,|\, i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\frac{1}{\sqrt{30}}\Big(-\,|\,0>\,e^{-i3\omega t/2} + |\,1>\left[2\,e^{-i\omega t/2} - 5\sqrt{2}\,e^{-i5\omega t/2}\right]$$
$$+\,|\,2>\,\sqrt{2}\,e^{-i3\omega t/2} + |\,3>\,5\sqrt{3}\,e^{-i5\omega t/2}\Big)$$
$$= \frac{i}{30}\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(2<0\,|\,e^{+i\omega t/2} + <1\,|\,e^{+i3\omega t/2} + 5<2\,|\,e^{+i5\omega t/2}\right)\times$$
$$\left(|\,0>\left[-e^{-i3\omega t/2}\right] + |\,1>\left[2e^{-i\omega t/2} - 5\sqrt{2}\,e^{-i5\omega t/2}\right] + |\,2>\left[\sqrt{2}\,e^{-i3\omega t/2}\right] + |\,3>\left[5\sqrt{3}\,e^{-i5\omega t/2}\right]\right)$$

$$= \frac{i}{30}\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(-2e^{-i\omega t} + 2\,e^{+i\omega t} - 5\sqrt{2}\,e^{-i\omega t} + 5\sqrt{2}\,e^{+i\omega t}\right) \tag{1}$$
$$= \frac{i}{30}\left(\frac{m\omega\hbar}{2}\right)^{1/2} 2i\left(2\,\frac{(e^{+i\omega t} - e^{-i\omega t})}{2i} + 5\sqrt{2}\,\frac{(e^{+i\omega t} - e^{-i\omega t})}{2i}\right)$$
$$= -\left(\frac{2+5\sqrt{2}}{15}\right)\left(\frac{m\omega\hbar}{2}\right)^{1/2}\sin(\omega t)\,.$$

Postscript: This calculation is made shorter by dropping all but the $|\,0>$, $|\,1>$, and $|\,2>$ terms right after letting the ladder operators act. Only these terms will survive when the inner products are calculated. Orthonormality of eigenstates is used to attain equation (1) though other economies are ignored in order to show the new "machinery" explicitly in this example.

Practice Problems

28. Show that the time-independent Schrodinger Equation for the SHO can be written

$$\hbar\omega\left(a\,a^{\dagger} - \frac{1}{2}\right)|\,\psi> = E_n\,|\,\psi>\ .$$

Practice in using the raising and lowering operators. See problem 2.

---

29. Show that a and $a^{\dagger}$ are not Hermitian.

More practice in using the raising and lowering operators. Does $a^\dagger = a$?

---

30. Show that $\mathcal{H}\, a = a\,\mathcal{H} - a\hbar\omega$.

Parallel problem 4.

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31. Normalize $a\,|\,n> = C\,(n)\,|\,n-1>$.

See problem 10.

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32. Show that $\mathcal{H}\,|\,3> = \frac{7}{2}\hbar\omega\,|\,3>$ by explicit matrix multiplication.

Use the unit vector $|\,3>$ and matrix representation of $\mathcal{H}$ given in problem 12.

---

33. Express the matrix elements (a) $<n\,|\,a\,|\,m>$, (b) $<n\,|\,a^\dagger|\,m>$, (c) $<n\,|\,\mathcal{X}\,|\,m>$, (d) $<n\,|\,\mathcal{P}\,|\,m>$, and (e) $<n\,|\,\mathcal{H}\,|\,m>$, in terms of quantum numbers and Kronecker deltas.

The condition of orthonormality, $<i\,|\,j> = \delta_{i,\,j}$, can also be expressed $<n\,|\,m-1> = \delta_{n,\,m-1}$. The action of $a^\dagger$ on a general ket $|\,m>$ is given by $a^\dagger|\,m> = \sqrt{m+1}\,|\,m+1>$. Then consider the inner product with the bra $<n\,|$ to obtain

$$<n\,|\,a^\dagger|\,m> = <n\,|\,\sqrt{m+1}\;|\,m+1> = \sqrt{m+1}\,<n\,|\,m+1> = \sqrt{m+1}\;\delta_{n,\,m+1}\,.$$

In other words, $<n\,|\,a^\dagger|\,m> = \sqrt{m+1}$ for $n = m+1$, and $<n\,|\,a^\dagger|\,m> = 0$ otherwise. Part (b) is now done for you. Part (a) is done in problem 13. Parts (c) through (e) are similar in concept because all operators can be expressed in terms of the ladder operators. You should find

$$<n\,|\,\mathcal{X}\,|\,m> = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\sqrt{m+1}\;\delta_{n,\,m+1} + \sqrt{m}\;\delta_{n,\,m-1}\right) \quad \text{for instance.}$$

---

34. Use the results of the previous problem to calculate the matrix elements (a) $<7\,|\,a\,|\,8>$, (b) $<7\,|\,a^\dagger|\,8>$, (c) $<7\,|\,\mathcal{X}\,|\,8>$, (d) $<7\,|\,\mathcal{P}\,|\,8>$, and (e) $<14\,|\,\mathcal{H}\,|\,14>$.

This problem illustrates that it may be convenient to have skills using Kronecker deltas. It is impractical to build matrices large enough to attain the requested elements. See problem 13.

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35. (a) Develop the matrix representation of the raising operator for the SHO.

(b) Raise the third excited state of the SHO using explicit matrix multiplication, and

(c) demonstrate the equivalence $a^\dagger \,|\, n> = \sqrt{n+1} \,|\, n+1>.$

Parallel the procedures in problem 13. Also see problem 33. The matrix operator representation of the raising operator is seen in many places in this chapter.

36. Find the matrix representation of $\mathcal{P}$ for the SHO.

See problem 14, and in particular, see the postscript to problem 14, and problem 33.

37. Use explicit matrix operations to find the matrix representation of the Hamiltonian operator.

Use the matrix representations of $a^\dagger$ and a in $\mathcal{H} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$ to attain the same result as problem 12. Remember that a scalar in a matrix equation means multiply that scalar by the identity matrix of the same dimension as the matrices with which this "scalar" is intended to operate. An infinite dimensional identity matrix is appropriate for the scalar "1/2" in this problem.

38. Normalize the ground state eigenfunction of the SHO in position space.

Apply the normalization condition to the result of problem 15 to attain the result presented in the postscript of problem 15. Gaussian integrals have been addressed in both chapters 1 and 3.

Problems 39 through 47 concern a simple harmonic oscillator that is initially in the state

$$|\, \psi\,(t=0)\, > = N \left[3\,|\,0> + \,2\,|\,1> + \,1\,|\,5> \right].$$

39. Calculate the normalization constant using (a) row and column vectors, and (b) Dirac notation.

See problem 21.

40. Energy is measured at $t = 0$. What results are possible and what is the probability of each possibility? Calculate probabilities using (a) row and column vectors, and (b) Dirac notation.

What are postulates 3 and 4. See problem 22.

41. (a) Find the expectation value $<E>$ by calculating $<\psi\,|\,\mathcal{H}\,|\,\psi>$ using matrix methods.

(b) Show that $a^\dagger a\,|\,n\!> = n\,|\,n\!>$.

(c) Use the result of part (b) to calculate $<\!E\!>$ using ladder operator methods.

Use the matrix representation of the Hamiltonian derived in problem 12 for part (a). Operate on $a\,|\,n\!> = \sqrt{n}\,|\,n-1\!>$ with $a^\dagger$ for part (b). Remember that $a^\dagger\,|\,n\!> = \sqrt{n+1}\,|\,n+1\!>$. The operator $a^\dagger a$ is often called "the number operator," often denoted $\mathcal{N}$, because it's eigenvalues are the index numbers of the SHO eigenvectors. Problem 23 should also be helpful.

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42. (a) Find the standard deviation of energy using matrix methods.

(b) Show that $\mathcal{N}^2\,|\,n\!> = n^2\,|\,n\!>$ where $\mathcal{N}^2 = a^\dagger a\, a^\dagger a$.

(c) Use the result of part (b) to find the standard deviation of energy by ladder operator methods.

Remember that $\Delta E = \sqrt{<\!\psi\,|\,(\mathcal{H}-<\!\mathcal{H}\!>)^2\,|\,\psi\!>}$ for part (a). Multiply $<\!\mathcal{H}\!>$ from the previous problem by the identity matrix to attain the matrix form of $<\!\mathcal{H}\!>$. Subtract this from $\mathcal{H}$ and square the result. You should find

$$(\mathcal{H}-\,<\!\mathcal{H}\!>)^2 = \begin{pmatrix} \frac{81}{196} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{25}{196} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{361}{196} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1089}{196} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{2209}{196} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{3721}{196} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \hbar^2\omega^2 .$$

Next calculate $\Delta E^2 = <\!\psi\,|\,(\mathcal{H}-\,<\!\mathcal{H}\!>)^2\,|\,\psi\!>$, and the square root to find

$$\Delta E = \sqrt{<\!\psi\,|\,(\mathcal{H}-\,<\!\mathcal{H}\!>)^2\,|\,\psi\!>} = \sqrt{\frac{325}{196}\hbar^2\omega^2} \approx 1.288\,\hbar\omega .$$

You have previously shown $\mathcal{N}\,|\,n\!> = n\,|\,n\!>$. Operate on both sides with the lowering operator, then operate on both sides of the result with the raising operator to complete part (b). You need to determine $\mathcal{H}^2$ in terms of $\mathcal{N}$ so that you can calculate $<\!\psi\,|\,\mathcal{H}^2\,|\,\psi\!>$ for part (c). Since

$$\mathcal{H} = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega = \left(\mathcal{N}+\frac{1}{2}\right)\hbar\omega, \quad \text{you should find that} \quad \mathcal{H}^2 = \left(\mathcal{N}^2+\mathcal{N}+\frac{1}{4}\right)\hbar^2\omega^2.$$

You should then find that $<\!\psi\,|\,\mathcal{H}^2\,|\,\psi\!> = 83\,\hbar^2\omega^2/28$ for the given wavefunction. All the pieces for $\Delta E = \left[\,<\!\psi\,|\,\mathcal{H}^2\,|\,\psi\!> - <\!\psi\,|\,\mathcal{H}\,|\,\psi\!>^2\,\right]^{1/2}$ are present. It must, of course, agree with part (a).

---

43. Verify your earlier results by calculating

$$\text{(a)} \quad <\!E\!> = \sum_i P(E_i)\,E_i \qquad \text{and} \qquad \text{(b)} \quad \Delta E = \sqrt{\sum_i P(E_i)\left[\,E_i - <\!E\!>\,\right]^2}.$$

Another way to get $<\!E\!>$, also denoted $<\!\mathcal{H}\!>$, is to multiply each of the possible eigenvalues by the probability of measuring that eigenvalue and add the products. Calculating standard deviation using probabilities is common. Of course, you must attain the same answers.

---

44. Plot your calculated probabilities $P(E_i)$ versus E. Show your expectation value and standard deviation on the plot. Discuss what this plot shows, *i.e.*, explain how the expectation value and the standard deviation are related to the outcome of a series of energy measurements.

The range of probability is 0 to 1 on the vertical axis. Use units of $\hbar\omega$ on the horizontal axis. The requested graph has only three discrete spikes at the energies of the eigenvalues, a discrete value at $<E>$, and a discrete value at ΔE. Expectation value and standard deviation mean the same as they did when presented in chapter 2. It is important that you not lose the foundational elements as they are applied to new systems, like the SHO.

---

45. Calculate the state of the system at time t. Express your answer

(a) in the Hilbert space in terms of the abstract $|n>$'s,

(b) in position space in terms of the $\psi_n(x)$'s,

(c) in momentum space in terms of the $\widehat{\psi}_n(p)$'s, and

(d) in energy space in terms of the $\widetilde{\psi}_n(E)$'s. Do not evaluate the specific $|n>$'s, $\psi_n(x)$'s, $\widehat{\psi}_n(p)$'s or $\widetilde{\psi}_n(E)$'s.

Notation, how things are written and what the symbols mean, can be formidable in any endeavor, particularly when the subject material is new. This problem is designed to help you assimilate the notation of quantum mechanics using an SHO state vector as a vehicle. It is largely an exercise in notation. Per postulate 1, the state of a system is described by a state vector. This problem is asking only for the form of the state vector in different spaces. See problem 25.

---

46. Calculate the following time-dependent expectation values for (a) $<\mathcal{H}>$, (b) $<\mathcal{P}>$, (c) $<\mathcal{X}>$, (d) $<\mathcal{P}^2/2m>$, (e) $<m\omega^2\mathcal{X}^2/2>$.

This problem demonstrates that the phase, the factor $e^{-iE_nt/\hbar}$ in this case, can be consequential even for stationary states. There are multiple options to calculate the five requested expectation values. Matrix arguments are likely the most straightforward. Part (a) is $<\psi(t)|\mathcal{H}|\psi(t)>$ where $\mathcal{H}$ is seen in problem 12. In ket form, the state vector is

$$\psi(t) = \frac{1}{\sqrt{14}} \begin{pmatrix} 3\,e^{-i\omega t/2} \\ 2\,e^{-i3\omega t/2} \\ 0 \\ 0 \\ 0 \\ e^{-i11\omega t/2} \\ \vdots \end{pmatrix}.$$

$<\mathcal{H}>$ is independent of time and the same as previously calculated. You should find that $<\mathcal{P}^2/2m> = <m\omega^2\mathcal{X}^2/2> = 4\hbar\omega/7$. You will also find

$$<\mathcal{P}> = -\frac{6}{7}\left(\frac{m\hbar\omega}{2}\right)^{1/2} \sin(\omega t) \quad \text{and} \quad <\mathcal{X}> = \frac{6}{7}\left(\frac{\hbar}{2m\omega}\right)^{1/2} \cos(\omega t)$$

See problem 27. See problem 14 for matrices to use matrix methods for part (b) through (e).

---

47. (a) Show that this state vector obeys the quantum mechanical virial theorem for the SHO, which states that the expectation value of the kinetic energy is equal to the expectation value of the potential energy, $<T> = <V>$.

(b) Show that your values of $<T>$ and $<V>$ are consistent with previous calculations of $<E>$.

The general virial theorem is deeper than the statement given here, but this is a simple example that the expectation value of the kinetic energy will equal the expectation value of the potential energy in this circumstance. Examine the parts (d) and (e) of the last problem. If you can recognize kinetic and potential energy in those expectation values, you have essentially completed both parts (a) and (b). Part (b) means that the expectation values of kinetic and potential energies must sum to the expectation value of the total energy which you have from problem 41.

---

48. (a) Write the time-independent Schrodinger equation for the SHO in position space.

(b) Write the time-independent Schrodinger equation for the SHO in momentum space.

Again, notation and representation are important. You have done this problem and problems like it in chapter 2. It is also preparation for part 2 of this chapter for which the position space representation of the SHO is central. Remember that in position space,

$$\mathcal{P} \longrightarrow -i\hbar\frac{\partial}{\partial x} \qquad \text{and} \qquad \mathcal{X} \longrightarrow x\,,$$

the time-independent Schrodinger equation is $\mathcal{H}\,|\,\psi> = E\,|\,\psi>$ where $|\,\psi> \rightarrow \psi\,(x)$ in position space, and that the Hamiltonian is the total energy operator $\mathcal{H} = \mathcal{T} + \mathcal{V}$. In momentum space,

$$\mathcal{P} \longrightarrow p \qquad \text{and} \qquad \mathcal{X} \longrightarrow i\hbar\frac{\partial}{\partial p} \qquad \text{and} \qquad |\,\psi> \longrightarrow \widehat{\psi}\,(p).$$

---

49. Calculate the ground state energy for a 1 gram mass on a $k = 0.1\ N/m$ spring. Calculate the approximate quantum number of this oscillator if its energy is $k_B T/2$ where $T = 300\ K$ and k_B is Boltzmann's constant. At what temperature would this oscillator be in its ground state?

This numerical problem should give you some appreciation of the magnitudes involved. Ground state energy, also known as zero point energy, means $n = 0$ for an SHO. You should find that the ground state energy is on the order of $10^{-34}\ J$ for this system, and that ground state temperature is on the order of $10^{-11}\ K$. $T = 300\ K$ is approximately room temperature where the quantum number is just less than 2×10^{12}. Notice that n is inversely proportional to $\hbar\omega$ to calculate the quantum number at large n, *i.e.*, at 300 K. MKS units are likely easiest given that the spring constant is in N/m. $k_B = 1.38 \times 10^{-23}\ J/K$ and $\hbar = 1.06 \times 10^{-34}\ J \cdot s$ in MKS units. The assumption that the energy is $k_B T/2$ is a statement of the equipartition theorem[2]. Finally, remember that $\omega = \sqrt{k/m}$, $E = k_B T/2$, and $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

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[2] Atkins *Quanta* (Oxford University Press, Oxford, 1991), pp 111-112.

50. (a) Find $\psi_2(x)$ for the SHO using the Hermite polynomials given in Table 8–1.

(b) Verify this eigenfunction using the generating function of problem 16.

Hermite polynomials are well known so part (a) is likely an easier method to attain an excited state of the SHO in position space than using the generating function. See problem 17.

Nevertheless, the generating function for the second excited state in position space is

$$\psi_2(y) = \frac{1}{\sqrt{2!}} \left(\frac{1}{\sqrt{2}} \left(y - \frac{d}{dy} \right) \right)^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-y^2/2} = \frac{1}{2\sqrt{2}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \left(y - \frac{d}{dy} \right)^2 e^{-y^2/2}.$$

The two terms in the operator $\left(y - \dfrac{d}{dy} \right)^2$ do not commute. The operator can be applied as

$$\left(y - \frac{d}{dy} \right) \left(y - \frac{d}{dy} \right) \quad \text{or} \quad y^2 - y\frac{d}{dy} - \frac{d}{dy}y + \frac{d}{dy}\frac{d}{dy}, \quad \text{but not as} \quad y^2 - 2y\frac{d}{dy} - \frac{d^2}{dy^2}.$$

Remember that $y = \sqrt{\dfrac{m\omega}{\hbar}}\, x$ to attain the final form. Parts (a) and (b) must agree, of course.

---

51. Show that $H_1(\xi)$ is orthogonal to $H_3(\xi)$ when both are weighted with the factor $e^{-\xi^2/2}$.

To be useful for quantum mechanics, a basis must be orthonormal. If the basis is orthogonal, it can be made orthonormal. The intent of this problem is to demonstrate orthogonality for a selected pair of Hermite polynomials. When weighted by $e^{-\xi^2/2}$, the recurrence relation immediately below table 8–1 leads to a differential equation that is self-adjoint and the solutions to a self-adjoint differential equation are orthogonal[3]. Byron and Fuller show by methods of integration that Hermite polynomials are orthogonal in general when weighted by $e^{-\xi^2/2}$ per footnote 1.

Per problem 18, weighting each Hermite polynomial by $e^{-\xi^2/2}$ means to multiply both $H_1(\xi)$ and $H_3(\xi)$ by $e^{-\xi^2/2}$ in the orthogonality condition which may be written

$$\int_{-\infty}^{\infty} H_1(\xi)\, e^{-\xi^2/2}\, H_3(\xi)\, e^{-\xi^2/2}\, d\xi .$$

This results in a difference of two integrals that are both even. An even integral is equal to twice the same integrand from the limits zero to infinity. Both integrals can be evaluated using

$$\int_0^{\infty} x^{2n}\, e^{-p\xi^2}\, dx = \frac{(2n-1)!!}{2\,(2p)^n} \sqrt{\frac{\pi}{p}}\,, \quad p > 0\,, \quad n = 0\,, 1\,, 2\,, \ldots$$

---

52. Express the position space state function

$$\Psi(x,t) = \frac{1}{\sqrt{14}} \left[3\, \psi_0(x)\, e^{-i\omega t/2} + 2\, \psi_1(x)\, e^{-i3\omega t/2} + \psi_5(x)\, e^{-i11\omega t/2} \right]$$

[3] Arfken *Mathematical Methods for Physicists* (Academic Press, New York, 1970, 2nd ed.), pp 424-432 and 611.

in functional form in terms of the independent variable x.

The given $\Psi(x,t)$ is the answer to problem 45 (b). Express the three $\psi_n(x)$'s in terms of x. You have $\psi_0(x)$ and $\psi_1(x)$ from previous problems. You need only to find $\psi_5(x)$, which is likely easiest using table 8–1, and then substitute the three eigenfunctions into the given wavefunction. Though the calculation is not very difficult, you may be surprised at the length. You should find

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}\Bigg\{\frac{3}{\sqrt{14}}\,e^{-i\omega t/2} + \frac{4}{\sqrt{14}}\left(\frac{m\omega}{2\hbar}\right)^{1/2} x\,e^{-i3\omega t/2}$$
$$+\frac{1}{\sqrt{210}}\left[2\left(\frac{m\omega}{\hbar}\right)^{5/2}x^5 - 10\left(\frac{m\omega}{\hbar}\right)^{3/2}x^3 + \frac{15}{2}\left(\frac{m\omega}{\hbar}\right)^{1/2}x\right]e^{-i11\omega t/2}\Bigg\}.$$

---

53. Express the state function of the last problem in functional form in terms of the variable p.

The intent of this problem is to practice a change of basis from position space to momentum space. It should also demonstrate an application of the raising operator. The most obvious idea is to employ a quantum mechanical Fourier transform to change the solution of problem 52 to momentum space. The Fourier transform of $\Psi(x)$ is given by the integral

$$\widehat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)\,e^{-ipx/\hbar}\,dx\,.$$

Obtaining $\widehat{\psi}_0(p)$ from $\psi_0(x)$ this way is straight forward. You can also do the Fourier transform of $\psi_1(x)$, but this is actually a fairly difficult integration. If you want to Fourier transform $\psi_5(x)$, you likely want (or need) to use a software package like Maple or Mathematica, and validating the results of the software calculation may then be a problem.

A paper and pencil method is to use the raising operator in momentum space which is

$$a^\dagger = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\frac{\partial}{\partial p} - i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}p = i\left[\left(\frac{\alpha}{2}\right)^{1/2}\frac{\partial}{\partial p} - \left(\frac{1}{2\alpha}\right)^{1/2}p\right]$$

where $\alpha = m\omega\hbar$ is used to reduce clutter. Calculate $\widehat{\psi}_1(p)$ using this raising operator on $\widehat{\psi}_0(p)$,

$$\widehat{\psi}_1(p) = a^\dagger\,\widehat{\psi}_0(p) = i\left[\left(\frac{\alpha}{2}\right)^{1/2}\frac{\partial}{\partial p} - \left(\frac{1}{2\alpha}\right)^{1/2}p\right]\widehat{\psi}_0(p)$$

and then attain $\widehat{\psi}_2(p)$ from $\widehat{\psi}_1(p)$, $\widehat{\psi}_3(p)$ from $\widehat{\psi}_2(p)$, and so on until you arrive at $\widehat{\psi}_5(p)$. You should find

$$\widehat{\psi}_2(p) = \frac{\sqrt{2}}{(\pi)^{1/4}(m\omega\hbar)^{3/4}}\left[\left(\frac{m\omega\hbar}{2}\right)^{1/2} - \left(\frac{2}{m\omega\hbar}\right)^{1/2}p^2\right]e^{-p^2/2m\omega\hbar},\quad \text{for instance, and eventually}$$

$$\widehat{\psi}_5(p) = \frac{-i\sqrt{2}}{(\pi)^{1/4}(m\omega\hbar)^{3/4}}\left[\frac{4p^5}{(m\omega\hbar)^2} - \frac{20p^3}{m\omega\hbar} + 15p\right]e^{-p^2/2m\omega\hbar},\quad \text{so that}$$

$$\widehat{\Psi}(p,t) = \frac{e^{-p^2/2m\omega\hbar}}{\sqrt{14}\,(\pi m\omega\hbar)^{1/4}}\Bigg\{3\,e^{-i\omega t/2} - \frac{i\,2\sqrt{2}\,p}{(m\omega\hbar)^{1/2}}\,e^{-i3\omega t/2}$$
$$-\frac{i\sqrt{2}}{(m\omega\hbar)^{1/2}}\left[\frac{4\,p^5}{(m\omega\hbar)^2} - \frac{20\,p^3}{m\omega\hbar} + 15\,p\right]e^{-i11\omega t/2}\Bigg\}.$$
