## Chapter 7 The Heisenberg Uncertainty Relations

Werner Heisenberg, after studying under Sommerfeld, Born, and Bohr, originated matrix mechanics in 1925. He is most well know for the uncertainty principle to which his name is often attached, developed in 1927. Notably, after Chadwick discovered the neutron in 1932, Heisenberg was first to propose a nucleus composed of protons and neutrons was preferable to the then reigning theory of the nucleus composed of protons and electrons.

The form of the Heisenberg uncertainty principle with which we will be most interested concerns canonically conjugate operators. Canonically conjugate commutators are of the form

$$
[\mathcal{A}, \mathcal{B}]=i \hbar \mathcal{I}=i \hbar .
$$

The Heisenberg uncertainty principle for canonically conjugate operators is

$$
\triangle \mathcal{A} \triangle \mathcal{B} \geq \frac{\hbar}{2}
$$

The primary purpose of this chapter is to develop this relation.

## The General Uncertainty Relation

Consider the commutator of two arbitrary Hermitian operators $\mathcal{A}$ and $\mathcal{B}$, where

$$
[\mathcal{A}, \mathcal{B}]=i \mathcal{C} .
$$

Is $\mathcal{C}$ Hermitian? The answer is yes, because

$$
[\mathcal{A}, \mathcal{B}]^{\dagger}=(\mathcal{A B}-\mathcal{B} \mathcal{A})^{\dagger}=(\mathcal{A B})^{\dagger}-(\mathcal{B} \mathcal{A})^{\dagger}=\mathcal{B}^{\dagger} \mathcal{A}^{\dagger}-\mathcal{A}^{\dagger} \mathcal{B}^{\dagger}=\mathcal{B} \mathcal{A}-\mathcal{A B}=-[\mathcal{A}, \mathcal{B}]=-i \mathcal{C}
$$

so

$$
\begin{equation*}
(i \mathcal{C})^{\dagger}=i^{\dagger} \mathcal{C}^{\dagger}=-i \mathcal{C}^{\dagger}=-i \mathcal{C} \Longleftrightarrow \mathcal{C}=\mathcal{C}^{\dagger}, \tag{7-1}
\end{equation*}
$$

and $\mathcal{C}$ is Hermitian. We will use this result momentarily.
Recalling equation (5-19) from chapter 5,

$$
\begin{equation*}
\triangle \mathcal{A}_{\psi}=\langle\psi|(\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I})^{2}|\psi\rangle^{1 / 2} \tag{7-2}
\end{equation*}
$$

with which we can calculate standard deviation. We define $\widehat{\mathcal{A}}=\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I}$ just to keep the notation clean.

The commutator

$$
\begin{equation*}
[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]=[\mathcal{A}, \mathcal{B}] . \tag{7-3}
\end{equation*}
$$

Example 7-1: Prove equation (7-3).
Remember expectation value is a scalar so commutes with everything, and the identity operator also commutes with everything. The commutator is

$$
\begin{aligned}
{[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}] } & =\widehat{\mathcal{A}} \widehat{\mathcal{B}}-\widehat{\mathcal{B}} \widehat{\mathcal{A}}=(\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I})(\mathcal{B}-\langle\mathcal{B}\rangle \mathcal{I})-(\mathcal{B}-\langle\mathcal{B}\rangle \mathcal{I})(\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I}) \\
& =\mathcal{A B}-\mathcal{A}\langle\mathcal{B}\rangle \mathcal{I}-\langle\mathcal{A}\rangle \mathcal{I} \mathcal{B}+\langle\mathcal{A}\rangle \mathcal{I}\langle\mathcal{B}\rangle \mathcal{I}-\mathcal{B} \mathcal{A}+\mathcal{B}\langle\mathcal{A}\rangle \mathcal{I}+\langle\mathcal{B}\rangle \mathcal{I} \mathcal{A}-\langle\mathcal{B}\rangle \mathcal{I}\langle\mathcal{A}\rangle \mathcal{I} \\
& =\mathcal{A B}-\mathcal{B} \mathcal{A}+(-\langle\mathcal{B}\rangle \mathcal{I} \mathcal{A}+\langle\mathcal{B}\rangle \mathcal{I} \mathcal{A})+(-\langle\mathcal{A}\rangle \mathcal{I} \mathcal{B}+\langle\mathcal{A}\rangle \mathcal{I} \mathcal{B})+\left(\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle \mathcal{I}^{2}-\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle \mathcal{I}^{2}\right)
\end{aligned}
$$

where all the grouped terms sum to zero, so

$$
[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]=\mathcal{A} \mathcal{B}-\mathcal{B} \mathcal{A}=[\mathcal{A}, \mathcal{B}]
$$

Consider the product

$$
\begin{aligned}
(\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} & =\langle\psi|(\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I})^{2}|\psi\rangle\langle\psi|(\mathcal{B}-\langle\mathcal{B}\rangle \mathcal{I})^{2}|\psi\rangle \\
& =\langle\psi| \widehat{\mathcal{A}}^{2}|\psi\rangle\langle\psi| \widehat{\mathcal{B}}^{2}|\psi\rangle \\
& =\left\langle\widehat{\mathcal{A}}^{\dagger} \psi \mid \widehat{\mathcal{A}} \psi\right\rangle\left\langle\widehat{\mathcal{B}}^{\dagger} \psi \mid \widehat{\mathcal{B}} \psi\right\rangle \\
& =\langle\widehat{\mathcal{A}} \psi \mid \widehat{\mathcal{A}} \psi\rangle\langle\widehat{\mathcal{B}} \psi \mid \widehat{\mathcal{B}} \psi\rangle
\end{aligned}
$$

where the last step uses the fact the operators are given to be Hermitian. This can be written

$$
\begin{equation*}
\left.\left.(\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2}=| | \widehat{\mathcal{A}} \psi\right\rangle\left.\right|^{2}| | \widehat{\mathcal{B}} \psi\right\rangle\left.\right|^{2} \tag{7-4}
\end{equation*}
$$

Example 7-2: Prove that if $\mathcal{A}$ is Hermitian, $\widehat{\mathcal{A}}$ is Hermitian.
The definition is

$$
\begin{gathered}
\widehat{\mathcal{A}}=\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I} \\
\Rightarrow \quad \widehat{\mathcal{A}}^{\dagger}=(\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I})^{\dagger}=\mathcal{A}^{\dagger}-\langle\mathcal{A}\rangle^{*} \mathcal{I}^{\dagger} .
\end{gathered}
$$

Now $\mathcal{I}^{\dagger}=\mathcal{I}$ because of the nature of the identity, and $\mathcal{A}^{\dagger}=\mathcal{A}$ because it is given to be Hermitian. The expectation value $\langle\mathcal{A}\rangle$ is an expectation value of a Hermitian operator so must be a real number. The complex conjugate of a real number is the same real number, that is $\langle\mathcal{A}\rangle^{*}=\langle\mathcal{A}\rangle$, so

$$
\mathcal{A}^{\dagger}-\langle\mathcal{A}\rangle^{*} \mathcal{I}^{\dagger}=\mathcal{A}-\langle\mathcal{A}\rangle \mathcal{I}=\widehat{\mathcal{A}},
$$

therefore

$$
\widehat{\mathcal{A}}^{\dagger}=\widehat{\mathcal{A}}
$$

and $\widehat{\mathcal{A}}$ is Hermitian.

We want to use the Schwarz Inequality ${ }^{2}$, which is

$$
\left.\left|v_{1}\right|^{2}\left|v_{2}\right|^{2} \geq\left|<v_{1}\right| v_{2}\right\rangle\left.\right|^{2}
$$

The equality is saturated if and only if the two vectors are proportional, that is $\left|v_{1}\right\rangle=c\left|v_{2}\right\rangle$. Applying the Schwarz Inequality, equation (7-4) means

$$
\begin{equation*}
\left.\left.\left.(\triangle \mathcal{A})^{2}(\Delta \mathcal{B})^{2}=| | \widehat{\mathcal{A}} \psi\right\rangle\left.\right|^{2}| | \widehat{\mathcal{B}} \psi\right\rangle\left.\right|^{2} \geq|\langle\psi| \widehat{\mathcal{A}} \widehat{\mathcal{B}}| \psi\right\rangle\left.\right|^{2} \tag{7-5}
\end{equation*}
$$

We also want to use the anti-commutator defined

$$
[\mathcal{A}, \mathcal{B}]_{+}=\mathcal{A B}+\mathcal{B} \mathcal{A}
$$

We can express the operator product in the last term of equation (7-5) as a sum of an anticommutator and commutator by "adding zero," or

$$
\begin{aligned}
\widehat{\mathcal{A}} \widehat{\mathcal{B}} & =\frac{1}{2} \widehat{\mathcal{A}} \widehat{\mathcal{B}}+\frac{1}{2} \widehat{\mathcal{A}} \widehat{\mathcal{B}}+\frac{1}{2} \widehat{\mathcal{B}} \widehat{\mathcal{A}}-\frac{1}{2} \widehat{\mathcal{B}} \widehat{\mathcal{A}} \\
& =\frac{1}{2}(\widehat{\mathcal{A}} \widehat{\mathcal{B}}+\widehat{\mathcal{B}} \widehat{\mathcal{A}})+\frac{1}{2}(\widehat{\mathcal{A}} \widehat{\mathcal{B}}-\widehat{\mathcal{B}} \widehat{\mathcal{A}}) \\
& =\frac{1}{2}[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+}+\frac{1}{2}[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}] .
\end{aligned}
$$

The inequality of relation (7-5) is

$$
\begin{align*}
& \left.(\triangle \mathcal{A})^{2}(\Delta \mathcal{B})^{2} \geq\left|\langle\psi| \frac{1}{2}[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+}+\frac{1}{2}[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]\right| \psi\right\rangle\left.\right|^{2} \\
\Rightarrow & \left.(\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq \frac{1}{4}\left|\langle\psi|[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+}+[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]\right| \psi\right\rangle\left.\right|^{2} . \tag{7-6}
\end{align*}
$$

Now the anti-commutator portion is Hermitian, i.e.

$$
\begin{aligned}
{[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+}^{\dagger} } & =(\widehat{\mathcal{A}} \widehat{\mathcal{B}}+\widehat{\mathcal{B}} \widehat{\mathcal{A}})^{\dagger} \\
& =(\widehat{\mathcal{A}} \widehat{\mathcal{B}})^{\dagger}+(\widehat{\mathcal{B}} \widehat{\mathcal{A}})^{\dagger} \\
& =\widehat{\mathcal{B}}^{\dagger} \widehat{\mathcal{A}}^{\dagger}+\widehat{\mathcal{A}}^{\dagger} \widehat{\mathcal{B}}^{\dagger} \\
& =\widehat{\mathcal{B}} \widehat{\mathcal{A}}+\widehat{\mathcal{A}} \widehat{\mathcal{B}}
\end{aligned}
$$

because the operators are Hermitian, so

$$
[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+}^{\dagger}=\widehat{\mathcal{B}} \widehat{\mathcal{A}}+\widehat{\mathcal{A}} \widehat{\mathcal{B}}=\widehat{\mathcal{A}} \widehat{\mathcal{B}}+\widehat{\mathcal{B}} \widehat{\mathcal{A}}=[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+},
$$

therefore the anti-commutator is Hermitian. A Hermitian operator is analogous to a real number. We assume the commutator portion of the Hermitian operators is of the form $[\mathcal{A}, \mathcal{B}]=i \mathcal{C}$. And

[^0]from the first calculation of this section, since $\mathcal{C}$ is Hermitian, also analogous to a real number; equation (7-6) is analogous to a complex number of the form $a+i b$. Equation (7-6) concerns the magnitude of a quantity analogous to a complex number. The magnitude of a complex number is $|a+i b|^{2}=a^{2}+b^{2}=a^{2}-(i)^{2} b$. Extending this plausibility argument to equation (7-6), we have
$$
(\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq \frac{1}{4}\langle\psi|[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_{+}|\psi\rangle^{2}-\frac{1}{4}\langle\psi|[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]|\psi\rangle^{2},
$$
where we have used a negative sign preceeding the last term. If $[\mathcal{A}, \mathcal{B}]=i \mathcal{C}$, consistent with the assumptions, we want the last equation to reflect the form $|a+i b|^{2}=a^{2}-(i)^{2} b$. From equation (7-3), we can remove the hats, so
\[

$$
\begin{equation*}
(\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq \frac{1}{4}\langle\psi|[\mathcal{A}, \mathcal{B}]_{+}|\psi\rangle^{2}-\frac{1}{4}\langle\psi|[\mathcal{A}, \mathcal{B}]|\psi\rangle^{2}, \tag{7-7}
\end{equation*}
$$

\]

which is the General Uncertainty Relation.

## Uncertainty Relations for Canonically Conjugate Operators

If we consider canonically conjugate operators, operators such that

$$
[\mathcal{A}, \mathcal{B}]=i \hbar=i \hbar \mathcal{I},
$$

we have a special case applicable to equation (7-7). In fact we have a case where $[\mathcal{A}, \mathcal{B}]=i \mathcal{C}$, where $\mathcal{C}=\hbar \mathcal{I}$. Equation (7-7) becomes

$$
\begin{equation*}
(\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq \frac{1}{4}\langle\psi|[\mathcal{A}, \mathcal{B}]_{+}|\psi\rangle^{2}-\frac{1}{4}\langle\psi| i \hbar \mathcal{I}|\psi\rangle^{2} . \tag{7-8}
\end{equation*}
$$

The first term on the right side of the inequality is Hermitian, so its magnitude is real, and we can conclude it is greater than or equal to zero, i.e.,

$$
\begin{aligned}
& \frac{1}{4}\langle\psi|[\mathcal{A}, \mathcal{B}]_{+}|\psi\rangle^{2} \geq 0 \\
\Rightarrow & (\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq-\frac{1}{4}\langle\psi| i \hbar \mathcal{I}|\psi\rangle^{2} \\
\Rightarrow & (\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq-\frac{1}{4}(i \hbar)^{2}\langle\psi| \mathcal{I}|\psi\rangle^{2} \\
\Rightarrow & (\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq \frac{1}{4} \hbar^{2}\langle\psi \mid \psi\rangle^{2} \\
\Rightarrow & (\triangle \mathcal{A})^{2}(\triangle \mathcal{B})^{2} \geq \frac{1}{4} \hbar^{2},
\end{aligned}
$$

where we have assumed the basis is orthonormal so $\langle\psi \mid \psi\rangle=1$, in the last step. A square root of both sides yields

$$
\begin{equation*}
(\triangle \mathcal{A})(\triangle \mathcal{B}) \geq \frac{1}{2} \hbar \tag{7-9}
\end{equation*}
$$

for canonically conjugate commutators for any $|\psi\rangle$ in an orthonormal basis.
The most common application of equation (7-9) is to the fundamental canonical commutator, equation (9-3),

$$
[\mathcal{X}, \mathcal{P}]=i \hbar,
$$

so

$$
(\triangle \mathcal{X})(\triangle \mathcal{P}) \geq \frac{1}{2} \hbar
$$


[^0]:    ${ }^{2}$ Byron and Fuller Mathematics of Classical and Quantum Physics (Dover Publications, New York, 1970), pp. 146-148.

