## Chapter 3

What are game birds? What does a quail look like? How can that dog be so ugly? Camouflage and a shooting vest, likely from an exclusive men's store, replaced the white suit and Panama hat. The hound froze abruptly. "Panama" approached deliberately, barked an order that only a dog could understand, the hound crept forward in a series of short lurches...wings exploded skyward...Panama fired once...the eager hound loped toward a spot over which feathers wafted on the breeze. As Panama cracked the engraved and inlayed over-under to replace the spent shell, he muttered like he was talking to the shotgum. Was he talking to me? Well. ..I'm not here for my good looks,... "Did you say grouse?" "No sir, no sir, Gauss,...I said Gauss. .."

## Gaussian Wavefunctions and the Uncertainty Principle

A free particle experiences no forces, or equivalently, the potential energy function is zero. Besides being useful in itself, the free particle can be used as a first approximation to many problems where the potential energy function is non-zero. The free particle problem is among the easiest that quantum mechanics can address so lends itself well to solution by different methods because of this simplicity. You should look for insight by studying the two methods (problems 2 and 3 ) used to solve this problem. Both methods are employed in future chapters.

You should appreciate the concept of the wave packet that incorporates both a particle and wave nature. The Gaussian wave packet is dominantly the most popular form though other forms are possible. It "spreads" or distributes itself more broadly in space as time changes. The assumption that the wavefunction is Gaussian is inherent in many common results.

The Heisenberg uncertainty relations are of fundamental importance to quantum mechanics. They concern canonical commutation relations that are also of central importance. Canonical commutators are so fundamental that they can be used to convey the postulates of quantum mechanics. Notice that the $\triangle$ 's in the product $\triangle x \Delta p$ are uncertainties that have the same meaning as standard deviation. Notice also that $\triangle x \triangle p$ is a statement in position space; the general uncertainty relation is a statement made in Hilbert space with abstract operators. Uncertainties of Gaussian wavefunctions can be calculated readily. The Gaussian wavefunction saturates the inequality of the Heisenberg uncertainty relation making it a mathematical equality.

1. Show that every operator commutes with itself and powers of itself.

This statement is not obvious though it is important. It will be used in realistic applications as we encounter an increasing amount of commutator algebra. The proof is short and simple. First you need an operator, say $\mathcal{A}$, and a commutator of $\mathcal{A}$ with an arbitrary power of $\mathcal{A}$, say $\left[\mathcal{A}, \mathcal{A}^{n}\right]$. You need to show that $\left[\mathcal{A}, \mathcal{A}^{n}\right]=0$, for all integral $n$. We solve this problem both intuitively and then by induction. The intuitive method is not a proof but the intent is that you believe the conjecture so may be the more valuable. Expand the commutators $\left[\mathcal{A}, \mathcal{A}^{1}\right],\left[\mathcal{A}, \mathcal{A}^{2}\right]$, $\left[\mathcal{A}, \mathcal{A}^{3}\right]$, and observe a pattern that should satisfy your intuition that $\left[\mathcal{A}, \mathcal{A}^{n}\right]$ must be zero because the pattern has to be identical regardless of $n$.

To show that $\left[\mathcal{A}, \mathcal{A}^{n}\right]=0$, first show that it is true for $n=1$, meaning

$$
[\mathcal{A}, \mathcal{A}]=\mathcal{A} \mathcal{A}-\mathcal{A} \mathcal{A}=\mathcal{A}^{2}-\mathcal{A}^{2}=0
$$

For $n=2$ and $n=3$,

$$
\begin{aligned}
& {\left[\mathcal{A}, \mathcal{A}^{2}\right]=\mathcal{A} \mathcal{A}^{2}-\mathcal{A}^{2} \mathcal{A}=\mathcal{A}^{3}-\mathcal{A}^{3}=0,} \\
& {\left[\mathcal{A}, \mathcal{A}^{3}\right]=\mathcal{A} \mathcal{A}^{3}-\mathcal{A}^{3} \mathcal{A}=\mathcal{A}^{4}-\mathcal{A}^{4}=0 .}
\end{aligned}
$$

The pattern does not change because the power changes so

$$
\left[\mathcal{A}, \mathcal{A}^{n}\right]=\mathcal{A} \mathcal{A}^{n}-\mathcal{A}^{n} \mathcal{A}=\mathcal{A}^{n+1}-\mathcal{A}^{n+1}=0 .
$$

More formally, the induction hypothesis is $\left[\mathcal{A}, \mathcal{A}^{n}\right]=0$ and we have shown $[\mathcal{A}, \mathcal{A}]=0$, which is the $n=1$ case. To relate the cases $n$ and $n+1$,

$$
\left[\mathcal{A}, \mathcal{A}^{n+1}\right]=\mathcal{A} \mathcal{A}^{n+1}-\mathcal{A}^{n+1} \mathcal{A}=\mathcal{A} \mathcal{A} \mathcal{A}^{n}-\mathcal{A} \mathcal{A}^{n} \mathcal{A}=\mathcal{A}\left(\mathcal{A} \mathcal{A}^{n}-\mathcal{A}^{n} \mathcal{A}\right)=\mathcal{A}\left[\mathcal{A}, \mathcal{A}^{n}\right]=\mathcal{A}(0)=0
$$

if the induction hypothesis is correct. The last line says that the commutator must be zero for the next power $n+1$ if commutator is zero for an arbitrary power of $n$. Since the induction hypothesis is true for $n=1$, it is then, necessarily true for all higher powers.
2. Find the eigenvalues of a free particle in abstract Hilbert space. Interpret the result.

The free particle is among the easiest problems that quantum mechanics can address. Focus on the concepts and rationale that are more important than the result. The free particle is one on which there are no forces, i.e., the free particle is devoid of interactions. No forces means

$$
F=-\frac{d V(x)}{d x}=0 \Rightarrow V(x)=\text { constant. Choose } V(x)=0
$$

from which the Hamiltonian follows. Use the result of the previous problem that every operator commutes with itself and powers of itself, and this means that $[\mathcal{H}, \mathcal{P}]=0$, so $\mathcal{H}$ and $\mathcal{P}$ share a common eigenbasis. Use the time-independent Schrodinger equation $\mathcal{H}\left|E>=E_{n}\right| E>$.

The classical Hamiltonian for a particle is

$$
H=\frac{p^{2}}{2 m}+V(x) \Rightarrow \mathcal{H}=\frac{\mathcal{P}^{2}}{2 m}+\mathcal{V}(\mathcal{X})
$$

is the quantum mechanical Hamiltonian. The potential energy function for a free particle is chosen to be $V(x)=0 \Rightarrow \mathcal{V}(\mathcal{X})=0 \Rightarrow \mathcal{H}=\frac{\mathcal{P}^{2}}{2 m}$. The time-independent Schrodinger equation, $\mathcal{H}\left|E>=E_{n}\right| E>\Rightarrow \frac{\mathcal{P}^{2}}{2 m}\left|E>=E_{n}\right| E>$. This Hamiltonian is a form of a power, proportional to a square, of the momentum operator for a free particle. Since $\left[\mathcal{P}^{2}, \mathcal{P}\right]=0$, $[\mathcal{H}, \mathcal{P}]=0, \quad$ which means that $\mathcal{H}$ and $\mathcal{P}$ share a common eigenbasis. Do not attempt to identify the eigenbasis, be concerned only that it exists. The statement that the energy and momentum operators share a common eigenbasis is $|E\rangle=|p\rangle$, since the eigenstates of $\mathcal{H}$ are $\mid E>$. Therefore, $\frac{\mathcal{P}^{2}}{2 m}|p\rangle=E_{n}|p\rangle$. The eigenvalue of the momentum operator acting on a momentum eigenvector is momentum,

$$
\mathcal{P}|p>=p| p>\Rightarrow \frac{\mathcal{P}^{2}}{2 m}\left|p>=\frac{p^{2}}{2 m}\right| p>\Rightarrow \frac{p^{2}}{2 m}\left|p>=E_{n}\right| p>
$$

$$
\Rightarrow \quad \frac{p^{2}}{2 m}=E_{n} \quad \Rightarrow \quad p= \pm \sqrt{2 m E} \quad \Rightarrow \quad E_{+}=\frac{(+p)^{2}}{2 m}, \quad E_{-}=\frac{(-p)^{2}}{2 m}
$$

are the eigenvalues. The problem is solved without identifying the eigenbasis.
The state vector is a superposition of the eigenstates, i.e.,

$$
\left|E>=c_{1}\right|+p^{2} / 2 m>+c_{2} \mid-p^{2} / 2 m>.
$$

Both energies or momenta are possible until we measure. When a measurement is made, the free particle is found to be moving to the right with momentum $p=+\sqrt{2 m E}$ or to the left with momentum $p=-\sqrt{2 m E}$. The result of a measurement is consistent with classical mechanics.
3. Show that $\psi(x, t)=A e^{i(k x-\omega t)}+B e^{-i(k x+\omega t)}$ is a solution to the time-dependent Schrodinger equation for a free particle in one-dimensional position space,

$$
-\frac{\hbar^{2}}{2 m} \frac{\partial^{2}}{\partial x^{2}} \psi(x, t)=i \hbar \frac{\partial}{\partial t} \psi(x, t),
$$

and show that this result is consistent with the result of the previous problem.

The necessary step of representation, in this case in position space, is comparable to using arithmetic when you can use algebra. Nevertheless, representations are the grounds where most applications are addressed. This problem asks you to calculate the necessary derivatives of the given wavefunction. You need the de Broglie relation, $p=h / \lambda$, to show equivalence with the previous problem. Remember that $k$ is wavenumber, $k=2 \pi / \lambda$, and that $E=h \nu=\hbar \omega$.

$$
\begin{aligned}
\frac{\partial}{\partial x} \psi(x, t) & =\frac{\partial}{\partial x}\left(A e^{i(k x-\omega t)}+B e^{-i(k x+\omega t)}\right) \\
& =A(i k) e^{i(k x-\omega t)}+B(-i k) e^{-i(k x+\omega t)}, \\
\Rightarrow \quad \frac{\partial^{2}}{\partial x^{2}} \psi(x, t) & =\frac{\partial}{\partial x}\left(A(i k) e^{i(k x-\omega t)}+B(-i k) e^{-i(k x+\omega t)}\right) \\
& =A(i k)^{2} e^{i(k x-\omega t)}+B(-i k)^{2} e^{-i(k x+\omega t)} \\
& =-k^{2}\left(A e^{i(k x-\omega t)}+B e^{-i(k x+\omega t)}\right)=-k^{2} \psi(x, t), \quad \text { and } \\
\frac{\partial}{\partial t} \psi(x, t) & =\frac{\partial}{\partial t}\left(A e^{i(k x-\omega t)}+B e^{-i(k x+\omega t)}\right) \\
& =A(-i \omega) e^{i(k x-\omega t)}+B(-i \omega) e^{-i(k x+\omega t)} \\
& =(-i \omega)\left(A e^{i(k x-\omega t)}+B e^{-i(k x+\omega t)}\right)=-i \omega \psi(x, t)
\end{aligned}
$$

Using these in the time-dependent Schrodinger equation in position space,

$$
\begin{equation*}
-\frac{\hbar^{2}}{2 m}\left(-k^{2} \psi(x, t)\right)=i \hbar(-i \omega \psi(x, t)) \quad \Rightarrow \quad \frac{\hbar^{2} k^{2}}{2 m} \psi(x, t)=\hbar \omega \psi(x, t) . \tag{1}
\end{equation*}
$$

Using the de Broglie relation,

$$
p=\frac{h}{\lambda}=\frac{h}{2 \pi} \frac{2 \pi}{\lambda}=\hbar k \quad \Rightarrow \quad \hbar^{2} k^{2}=p^{2}, \quad \text { and } \quad \hbar \omega=\frac{h}{2 \pi} 2 \pi \nu=h \nu=E,
$$

equation (1) is equivalent to

$$
\frac{p^{2}}{2 m} \psi(x, t)=E \psi(x, t) \quad \Rightarrow \quad E=\frac{p^{2}}{2 m} \quad \Rightarrow \quad E_{+}=\frac{(+p)^{2}}{2 m}, \quad \text { and } \quad E_{-}=\frac{(-p)^{2}}{2 m},
$$

which are the eigenvalues. This is consistent with what has previously been shown.

Postscript: The time-dependent Schrodinger equation is a special form of the wave equation. Recognizing this allows assumption of the given wavefunction which is a known solution.

The $\psi(x, t)$ given is a superposition of plane waves. $A e^{i(k x-\omega t)}$ is a plane wave moving in the positive $x$ direction and $B e^{-i(k x+\omega t)}$ is a plane wave moving in the negative $x$ direction. Plane waves are not acceptable as wavefunctions because they cannot be normalized. Combining plane waves with a substantially localized envelope, however, does yield an acceptable wavefunction.
4. Show that the integral over all space of a Gaussian function, $e^{-a x^{2}}$, is $(\pi / a)^{1 / 2}$.

A particle is localized in space and time. As Feynman says, "They come in lumps." The Schrodinger equation is a wave equation so its solution necessarily has a wave nature. A wave packet uses a functional form as an envelope to contain a wave whose amplitude varies with position within the envelope. The Gaussian function $f(x)=A e^{-a x^{2}}, \quad a>0$, is dominantly the most popular functional form used to build such an envelope.

This problem is the second step in building a Gaussian wave packet. A plane wave is not acceptable as a wavefunction because it cannot be normalized. Equivalently, a plane wave cannot be localized because it is evenly distributed throughout all space. That the integral of the Gaussian function over all space is finite demonstrates that a plane wave contained within a Gaussian envelope can yield an acceptable wavefunction.

Problem 20 of part 3 of chapter 1 normalizes a Gaussian function. Below we present a second method to attain the same result. Form the square of $I=\int_{-\infty}^{\infty} e^{-a x^{2}} d x$, change to polar coordinates where the square of the integral is easily evaluated because of the symmetry of the Gaussian function, and then a square root yields the value of the original integral.

$$
\text { Let } \quad I=\int_{-\infty}^{\infty} e^{-a x^{2}} d x \Rightarrow I^{2}=\int_{-\infty}^{\infty} e^{-a x^{2}} d x \int_{-\infty}^{\infty} e^{-a y^{2}} d y=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a\left(x^{2}+y^{2}\right)} d x d y
$$

We can express this in polar coordinates, where $x^{2}+y^{2}=r^{2}$ and $d x d y=r d r d \phi$, so

$$
\begin{aligned}
I^{2} & =\int_{0}^{\infty} \int_{0}^{2 \pi} e^{-a r^{2}} r d r d \phi=\int_{0}^{\infty} r e^{-a r^{2}} d r \int_{0}^{2 \pi} d \phi \\
& =\int_{0}^{\infty} r e^{-a r^{2}} d r[\phi]_{0}^{2 \pi}=2 \pi \int_{0}^{\infty} r e^{-a r^{2}} d r \\
& =2 \pi\left[-\frac{1}{2 a} e^{-a r^{2}}\right]_{0}^{\infty}=\frac{2 \pi}{2 a}\left[-\phi^{-a(\infty)^{2}}+e^{0}\right]=\frac{\pi}{a} \quad \Rightarrow \quad I=\sqrt{\frac{\pi}{a}} .
\end{aligned}
$$

Postscript: We use $\alpha$ to denote full width at half maximum height and $\Delta$ to denote uncertainty/standard deviation. Notation varies. Standard deviation is often denoted $\sigma$. Uncertainty is related to full width at half maximum as

$$
2 \triangle^{2}=\alpha^{2} \quad \text { or } \quad \triangle=\frac{\alpha}{\sqrt{2}} .
$$

The normalized Gaussian function in terms of full width at half maximum is

$$
f(x)=\frac{e^{-x^{2} / 2 \alpha^{2}}}{\left(\pi \alpha^{2}\right)^{1 / 4}}
$$

Multiplying the plane wave by this Gaussian function results in the upper and lower boundary of the plane wave being defined by the Gaussian function. This is the meaning of a Gaussian wave packet. The amplitude of the wave varies with position within this envelope.

Uncertainty and full width at half maximum have the same units as the independent variable. Specify variables/quantities when more that one is being addressed, $\triangle x$ and $\triangle p$, for example.
5. (a) Show that the classical velocity of a free particle is given by the group velocity.
(b) Show that the classical velocity of a wave within a wave packet is given by the phase velocity.

There are two velocities associated with a wave packet, a group velocity which is the speed of the wave packet through space, and a phase velocity which is the speed of the wave within the packet. Both velocities are commonly described in terms of angular frequency and wavenumber,

$$
v_{\text {group }}=\frac{d \omega}{d k} \quad \text { and } \quad v_{\text {phase }}=\frac{\omega}{k} .
$$

These can easily be stated in terms of frequency and wavelength. Total energy is kinetic energy for a free particle. Express total energy in terms of momentum, differentiate with respect to momentum, use the de Broglie relation, remember $E=h \nu=\hbar \omega$, and caste a form that you can recognize as classical velocity. The speed of a wave in a medium is $v=\lambda \nu$ for part (b).
(a) A free particle has total energy $T=E=\frac{1}{2} m v^{2} \quad \Rightarrow \quad E=\frac{1}{2} m v^{2}=\frac{p^{2}}{2 m} \quad \Rightarrow \quad \frac{d E}{d p}=\frac{p}{m}$. The de Broglie wavelength associated with a particle is $\lambda=\frac{h}{p} \Rightarrow p=\frac{h}{\lambda}=\frac{h k}{2 \pi}=\hbar k$. The classical velocity of a particle is $\quad v=\frac{m v}{m}=\frac{p}{m}=\frac{d E}{d p}=\frac{d(\hbar \omega)}{d(\hbar k)}=\frac{\hbar}{\hbar} \frac{d \omega}{d k}=\frac{d \omega}{d k}$.
(b) The speed of a wave in a medium is $\quad v=\lambda \nu=\frac{2 \pi}{k} \frac{\omega}{2 \pi}=\frac{\omega}{k}$.

Postscript: The relation between $\omega$ and $k$ is known as a dispersion relation. The dispersion relation is particularly important in fields that use quantum mechanics, such as solid state physics. Graphs are usually plotted $\omega$ versus $k$. A commonly plotted dispersion relation is essentially a
graph in the momentum basis since $p=\hbar k$. Further, since energy is proportional to angular frequency, $E=\hbar \omega$, the common graph of a dispersion relation is also proportional to a graph of energy versus momentum.

The solid curve of $\omega=\hbar k^{2} / 2 m$ indicates all energies and momenta. The eigenvalues, $E_{+}=\frac{(+p)^{2}}{2 m}$ and $E_{-}=\frac{(-p)^{2}}{2 m}$, are the functional values at $\pm k= \pm p / \hbar$ on the $k$-axis.

Note also that the slope at any point, say $+\hbar k$, is $d \omega / d k$, the group velocity, and the chord from the origin to that point is $\omega / k$, the phase velocity.
6. Find the function in wavenumber space that corresponds to the normalized Gaussian wavefunction in position space.

That a plane wave solution is limited by a constant wavenumber is its second shortcoming. This problem is a step toward rectifying the limitation of constant wavenumber/momentum. A relation $g(k)$ that models a distribution in momenta, or for all practical purposes equivalently, in wavenumber is the Fourier transform of $f(x)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-x^{2} / 2 \alpha^{2}}$. Use form 3.323.2 from Gradshteyn and Ryzhik, $\quad \int_{-\infty}^{\infty} e^{-p^{2} x^{2} \pm q x} d x=\frac{\sqrt{\pi}}{p} e^{q^{2} / 4 p^{2}}$. You should find $g(k)=\left(\frac{\alpha^{2}}{\pi}\right)^{1 / 4} e^{-k^{2} \alpha^{2} / 2}$.

The general Fourier transform is

$$
g(k)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} f(x) e^{-i k x} d x \Rightarrow g(k)=\frac{1}{\sqrt{2 \pi}} \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} \int_{-\infty}^{\infty} e^{-x^{2} / 2 \alpha^{2}-i k x} d x
$$

Using form 3.323.2 from Gradshteyn and Ryzhik, $p^{2}=1 / 2 \alpha^{2}$ and $q=i k$,

$$
g(k)=\frac{1}{\sqrt{2 \pi}} \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} \alpha \sqrt{2 \pi} e^{(i k)^{2} / 4\left(1 / 2 \alpha^{2}\right)}=\left(\frac{\alpha^{2}}{\pi}\right)^{1 / 4} e^{-k^{2} \alpha^{2} / 2} .
$$

Postscript: The function $g(k)$ is in wavenumber space. Elements in wavenumber space are proportional to elements momentum space. Wavenumber and momentum are so closely related that the preference is often to work in terms of wavenumber, particularly where waves are involved.

The function $g(k)$ models all wavenumber/momenta in a Gaussian distribution. Multiplying the plane wave solution by $g(k)$ provides the Gaussian wave packet shape.
7. Derive the dispersion relation for a free particle.

Though given in the postscript to problem 5, this is an important piece used in the next problem to arrive at a quantum mechanical Gaussian wavefunction. Total energy is kinetic energy for a free particle. Use this fact and the de Broglie relation to arrive at $\omega(k)$.

$$
E=\frac{1}{2} m v^{2}=\frac{p^{2}}{2 m} \quad \Rightarrow \quad \hbar \omega=\frac{\hbar^{2} k^{2}}{2 m} \quad \Rightarrow \quad \omega=\omega(k)=\frac{\hbar k^{2}}{2 m} .
$$

8. Assume that $B=0$ in the plane wave solution to the Schrodinger equation given in problem 3. Find the resulting quantum mechanical Gaussian wavefunction, $\psi(x, t)$.

This problem puts the parts of the last three problems together with the plane wave solution to the Schrodinger equation in an acceptable wavefunction in position space. The plane wave solution is

$$
\psi(x, t)=A e^{i(k x-\omega t)}+B e^{-i(k x+\omega t)} \text { and } B=0 \Rightarrow \psi(x, t)=A e^{i(k x-\omega t)} .
$$

$B=0$ is assumed to simplify the development. Multiply by the Gaussian distribution in wavenumber space, $g(k)$. Use the dispersion relation in place of $\omega$ in $e^{i(k x-\omega t)}$ to complete this product wavefunction in terms of all wavenumber. Integrate this product over all wavenumber to establish the functional relation in position space. Use the integral given in problem 6. Reduce the result to

$$
\psi(x, t)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1+\frac{i \hbar}{m \alpha^{2}} t\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}+\frac{i \hbar}{m} t\right)}\right]
$$

after multiplying by the symmetrical factor of $1 / \sqrt{2 \pi}$ to complete the Fourier integral (since the Fourier transform was not done in both directions).

$$
\begin{aligned}
\psi(x, t) & =\int_{-\infty}^{\infty} g(k)\left(A e^{i(k x-\omega t)}\right) d k=A \int_{-\infty}^{\infty} g(k) e^{i\left(k x-\hbar k^{2} t / 2 m\right)} d k \\
& =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty}\left(\frac{\alpha^{2}}{\pi}\right)^{1 / 4} e^{-k^{2} \alpha^{2} / 2} e^{i k x-i \hbar k^{2} t / 2 m} d k \\
& =\frac{1}{\sqrt{2 \pi}}\left(\frac{\alpha^{2}}{\pi}\right)^{1 / 4} \int_{-\infty}^{\infty} \exp \left[-\left(\frac{\alpha^{2}}{2}+\frac{i \hbar t}{2 m}\right) k^{2}+i x k\right] d k
\end{aligned}
$$

and we can use the same integral as used in problem 6 with $p^{2}=\left(\frac{\alpha^{2}}{2}+\frac{i \hbar t}{2 m}\right)$, and $q=i x$,

$$
\begin{aligned}
\psi(x, t) & =\frac{1}{\sqrt{2 \pi}}\left(\frac{\alpha^{2}}{\pi}\right)^{1 / 4}\left(\frac{\pi}{\frac{\alpha^{2}}{2}+\frac{i \hbar}{2 m} t}\right)^{1 / 2} \exp \left[\frac{(i x)^{2}}{4\left(\frac{\alpha^{2}}{2}+\frac{i \hbar}{2 m} t\right)}\right] \\
& =\frac{1}{\sqrt{2 \pi}}\left(\frac{1}{\pi}\right)^{1 / 4} \alpha^{1 / 2} \sqrt{2 \pi}\left(\frac{1}{\alpha^{2}+\frac{i \hbar}{m} t}\right)^{1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}+\frac{i \hbar}{m} t\right)}\right]
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{\pi^{1 / 4}} \frac{1}{\alpha^{1 / 2}}\left(\frac{\alpha^{2}}{\alpha^{2}+\frac{i \hbar}{m} t}\right)^{1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}+\frac{i \hbar}{m} t\right)}\right] \\
& =\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1+\frac{i \hbar}{m \alpha^{2}} t\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}+\frac{i \hbar}{m} t\right)}\right]
\end{aligned}
$$

Postscript: Problem 8 is the culmination of problems 4 through 7. The Gaussian function is normalizable. Normalization is required to be consistent with postulate 4. A wavefunction with fixed wavenumber/momentum is unrealistic so the Gaussian form $g(k)$ is developed to include all wavenumbers/momenta. Multiplying the plane wave solution by $g(k)$ yields the Gaussian wave packet. The dispersion relation is used to imbed variable wavenumber in the plane wave. Integrating over all wavenumbers results in the Gaussian wavefunction in position space.

An alternative view of the factor $1 / \sqrt{2 \pi}$ is that it completes the process of normalization. The Gaussian wavefunction derived above is normalized.

Also keep in mind that $\alpha$ is full width at half maximum of the Gaussian envelope. The uncertainty of a Gaussian form is related to the full width at half maximum by $\triangle=\alpha / \sqrt{2}$.
9. Find the uncertainty in position of the normalized Gaussian function $\frac{e^{-x^{2} / 2 a^{2}}}{\left(\pi a^{2}\right)^{1 / 4}}$.

This problem derives the final statement of the last postscript. Find $<x>=\int \psi^{*}(x) x \psi(x) d x$. Then find the second moment similarly using $x^{2}$. Form 3.461.2 from Gradshteyn and Ryzhik,

$$
\int_{0}^{\infty} x^{2 n} e^{-p x^{2}} d x=\frac{(2 n-1)!!}{2(2 p)^{n}} \sqrt{\frac{\pi}{p}}
$$

should be useful. Uncertainty is $\triangle x=\left(<x^{2}>-<x>^{2}\right)^{1 / 2}$. Remember that the integral of an odd function between symmetric limits is zero, and that the integral of an even function between symmetric limits is twice the same integral between zero and the upper limit.

$$
<x>=\int_{-\infty}^{\infty} \psi^{*}(x) x \psi(x) d x=\int_{-\infty}^{\infty} \frac{e^{-x^{2} / 2 a^{2}}}{\left(\pi a^{2}\right)^{1 / 4}}(x) \frac{e^{-x^{2} / 2 a^{2}}}{\left(\pi a^{2}\right)^{1 / 4}} d x=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-x^{2} / a^{2}} d x=0
$$

because the integral of an odd function between symmetric limits is zero.

$$
\begin{aligned}
<x^{2}> & =\int_{-\infty}^{\infty} \psi^{*}(x) x^{2} \psi(x) d x=\int_{-\infty}^{\infty} \frac{e^{-x^{2} / 2 a^{2}}}{\left(\pi a^{2}\right)^{1 / 4}}\left(x^{2}\right) \frac{e^{-x^{2} / 2 a^{2}}}{\left(\pi a^{2}\right)^{1 / 4}} d x=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty} x^{2} e^{-x^{2} / a^{2}} d x \\
& =\frac{2}{a \sqrt{\pi}} \int_{0}^{\infty} x^{2} e^{-x^{2} / a^{2}} d x=\frac{2}{a \sqrt{\pi}} \frac{(2-1)!!}{2\left(2\left(1 / a^{2}\right)\right)^{1}} \sqrt{\frac{\pi}{1 / a^{2}}}=\frac{2}{a \sqrt{\pi}} \frac{a^{2}}{4} a \sqrt{\pi}=\frac{a^{2}}{2}
\end{aligned}
$$

where we have used the fact that an even function between symmetric limits is twice the same integral between zero and the upper limit.

$$
\Delta x=\left(<x^{2}>-<x>^{2}\right)^{1 / 2}=\left(\frac{a^{2}}{2}-0\right)^{1 / 2}=\frac{a}{\sqrt{2}} .
$$

Postscript: This problem was completed using an unspecified parameter $a$ instead of the full width at half maximum. Uncertainty is the square root of one fourth of the denominator of the argument of the exponential when calculated using a normalized Gaussian wavefunction. Similarly, the uncertainty is $\triangle x=a / 2$ for $A e^{-x^{2} / a^{2}}$. The uncertainty is $\triangle x=\sqrt{a} / 2$ for $A e^{-x^{2} / a}$. The uncertainty is $\triangle x=1 / 2 \sqrt{a}$ for $A e^{-a x^{2}}$. The given wavefunction must be normalized to apply this insight. The Gaussian wavefunction of problem 8 is normalized per problem 30 .

Quantum mechanics requires the field of complex numbers. The Gaussian wavefunction of problem 8 contains an imaginary component in the argument of the exponential, for instance. Values representing observable quantities including uncertainties, however, are limited to the field of real numbers. Uncertainty is properly calculated in such a circumstance from the probability density for which the field of real numbers is sufficient. The process of forming a probability density from a Gaussian wavefunction results in doubling the real part of the argument of the exponential. Uncertainty calculated from such a probability density is, therefore, the square root of one half of the denominator of the argument of the exponential.
10. (a) Find the probability density associated with the Gaussian wavefunction of problem 8 .
(b) Use this result in conjunction with the result of problem 9 to interpret the uncertainty in position as a function of time.

Probability density is $|\psi(x, t)|^{2}=\psi^{*}(x, t) \psi(x, t)$. For part (a), you should find that

$$
|\psi(x, t)|^{2}=\frac{1}{\alpha \sqrt{\pi}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{\alpha^{2}+\frac{\hbar^{2}}{m^{2} \alpha^{2}} t^{2}}\right] .
$$

Per the postscript to problem 9, uncertainty is properly derived from probability density for the Gaussian wavefunction containing imaginary components in the exponential. Probability density and uncertainty are both intrinsically real quantities. Apply the result of problem 9 to find

$$
\triangle x(t)=\frac{\alpha}{\sqrt{2}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{1 / 2}
$$

What does is mean that uncertainty is a function of time?
(a) $|\psi(x, t)|^{2}=\psi^{*}(x, t) \psi(x, t)$

$$
=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1-\frac{i \hbar}{m \alpha^{2}} t\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}-\frac{i \hbar}{m} t\right)}\right] \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1+\frac{i \hbar}{m \alpha^{2}} t\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}+\frac{i \hbar}{m} t\right)}\right]
$$

$$
=\frac{1}{\alpha \sqrt{\pi}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{2\left(\alpha^{2}-\frac{i \hbar}{m} t\right)}-\frac{x^{2}}{2\left(\alpha^{2}+\frac{i \hbar}{m} t\right)}\right]
$$

and the remaining task is to simplify the argument of the exponential. Factoring and multiplying each rational expression by its complex conjugate, the argument of the exponential is

$$
\begin{gathered}
{\left[-\frac{x^{2}}{2}\left(\frac{\alpha^{2}+\frac{i \hbar}{m} t}{\alpha^{4}+\frac{\hbar^{2}}{m^{2}} t^{2}}+\frac{\alpha^{2}-\frac{i \hbar}{m} t}{\alpha^{4}+\frac{\hbar^{2}}{m^{2}} t^{2}}\right)\right]=\left[-\frac{x^{2}}{2}\left(\frac{\alpha^{2}+\frac{i \hbar}{m} t+\alpha^{2}-\frac{i \hbar}{m} t}{\alpha^{4}+\frac{\hbar^{2}}{m^{2}} t^{2}}\right)\right]} \\
=\left[-\frac{x^{2}}{2}\left(\frac{2 \alpha^{2}}{\alpha^{4}+\frac{\hbar^{2}}{m^{2}} t^{2}}\right)\right]=\left[-\frac{x^{2}}{\alpha^{2}+\frac{\hbar^{2}}{m^{2} \alpha^{2}} t^{2}}\right] \\
\Rightarrow|\psi(x, t)|^{2}=\frac{1}{\alpha \sqrt{\pi}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{-1 / 2} \exp \left[-\frac{x^{2}}{\alpha^{2}+\frac{\hbar^{2}}{m^{2} \alpha^{2}} t^{2}}\right]
\end{gathered}
$$

The result of problem 9 is $\Delta x=\frac{a}{\sqrt{2}}$ for the normalized Gaussian function. Therefore,

$$
\Delta x=\triangle x(t)=\frac{1}{\sqrt{2}}\left(\alpha^{2}+\frac{\hbar^{2}}{m^{2} \alpha^{2}} t^{2}\right)^{1 / 2}=\frac{\alpha}{\sqrt{2}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{1 / 2} .
$$

The uncertainty in position is a function of time. Notice from the result of problem 8 that the Gaussian wave packet "spreads" and "flattens" as time advances. The uncertainty resultingly grows as time advances...or as time is retarded. There is a minimum uncertainty in position at time $t=0$. This is the meaning of the phrase "free space is dispersive."

Postscript: The distribution in wavenumber/momentum is not time dependent. The average momentum and the uncertainty in momentum are constants for a free particle. See problem 13.
11. Find the uncertainty in position of $\psi(x)=e^{-(x-b)^{2} / 2 a^{2}}$.

You will encounter this Gaussian form in problem 12. (a) Normalize the wave function, (b) calculate $\langle x\rangle$, (c) calculate $\left\langle x^{2}\right\rangle$, and then (d) calculate $\Delta x=\left(\left\langle x^{2}\right\rangle-\langle x\rangle^{2}\right)^{1 / 2}$. Change variables to $y=x-b$ to make the result of problem 4 clearly applicable in parts (a) through (c). You need the integral used in problem 9 for part (c). The uncertainty is the same as problem 9.
(a)

$$
1=\int_{-\infty}^{\infty} A^{*} e^{-(x-b)^{2} / 2 a^{2}} A e^{-(x-b)^{2} / 2 a^{2}} d x=|A|^{2} \int_{-\infty}^{\infty} e^{-(x-b)^{2} / a^{2}} d x
$$

Let $y=x-b \quad \Rightarrow \quad d x=d y \quad \Rightarrow \quad 1=|A|^{2} \int_{-\infty}^{\infty} e^{-y^{2} / a^{2}} d y=|A|^{2} a \sqrt{\pi} \quad \Rightarrow \quad A=\frac{1}{\left(\pi a^{2}\right)^{1 / 4}}$.
(c)
(d)

$$
\begin{gather*}
<x>=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-(x-b)^{2} / a^{2}} d x=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty}(y+b) e^{-y^{2} / a^{2}} d y  \tag{b}\\
=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty} y / e^{-y^{2} / a^{2}} d y+\frac{b}{a \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^{2} / a^{2}} d y=\frac{b}{a \sqrt{\pi}} a \sqrt{\pi}=b . \\
<x^{2}>=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty} x^{2} e^{-(x-b)^{2} / a^{2}} d x=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty}(y+b)^{2} e^{-y^{2} / a^{2}} d y \\
=\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{\infty} y^{2} e^{-y^{2} / a^{2}} d y+\frac{2 b}{a \sqrt{\pi}} \int_{-\infty}^{\infty} y / e^{-y^{2} / a^{2}} d y+\frac{b^{2}}{a \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^{2} / a^{2}} d y \\
=\frac{2}{a \sqrt{\pi}} \frac{a^{2}}{2(2)} a \sqrt{\pi}+\frac{b^{2}}{a \sqrt{\pi}} a \sqrt{\pi}=\frac{a^{2}}{2}+b^{2} . \\
\triangle x=\left(<x^{2}>-\left\langle x>^{2}\right)^{1 / 2}=\left(\frac{a^{2}}{2}+b^{2}-b^{2}\right)^{1 / 2}=\frac{a}{\sqrt{2}} .\right.
\end{gather*}
$$

12. The normalized distribution of a plane wave within a wave packet is given to be

$$
\psi(x)=\frac{e^{i k_{0} x} e^{-x^{2} / 2 \alpha^{2}}}{\left(\pi \alpha^{2}\right)^{1 / 4}}
$$

(a) Find the normalized distribution in wave number space
(b) and the wavefunction as a function of position and time.
(c) Show that

$$
\psi(x, t)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \exp \left[i k_{0}\left(x-\frac{\hbar k_{0}^{2} t}{2 m}\right)\right] \exp \left[\frac{-\left(x-\hbar k_{0} t / m\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m}\right]
$$

(d) Find the wave function at $t=0$,
(e) the probability density as a function of position and time,
(f) and the uncertainty in position as a function of time.

The normalized Gaussian distribution $\quad \psi(x)=\frac{e^{-x^{2} / 2 \alpha^{2}}}{\left(\pi \alpha^{2}\right)^{1 / 4}} \quad$ provides the simplest development of a Gaussian wavefunction. It has the physical limitation that the resulting distribution in wavenumber space is centered at $k=0 \Rightarrow p=0$. Thinking classically for a moment, this would mean that you are traveling with the particle. The given distribution is centered at $k=k_{0}$, or $p=p_{0}$,
which is a more realistic reference frame. You will usually encounter the distribution given above when a Gaussian wavefunction is used explicitly. The amount of algebra required, particularly for the focal part (c), conspires to make this problem more difficult than the concepts involved.

This problem parallels the development of the Gaussian wavefunction centered at $k_{0}=0$. Find the Fourier transform of the given distribution which is the distribution in wavenumber, $g(k)$. Using the same integral and procedure as used in problem 6, you should find

$$
\begin{gathered}
g(k)=\frac{\alpha}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\left(k-k_{0}\right)^{2} \alpha^{2} / 2} \quad \text { for part (a). Parallel problem } 8 \text { to attain } \\
\psi(x, t)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\alpha^{2} k_{0}^{2} / 2}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \exp \left[\frac{\left(i x+\alpha^{2} k_{0}\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m}\right]
\end{gathered}
$$

for part (b). Part (c) is a substantial algebra problem. Reduce the amount of algebra by letting

$$
b=2 \alpha^{2}+\frac{2 i \hbar t}{m} \quad \text { and } \quad c=\frac{\hbar k_{0} t}{m} \text {. Express the exponentials }
$$

in terms of one argument. Complete the square of that argument by multiplying by 1 in the form

$$
\exp \left[\frac{2 \hbar k_{0} t x}{m b}-\frac{2 \hbar k_{0} t x}{m b}-\frac{\hbar^{2} k_{0}^{2} t^{2}}{m^{2} b}+\frac{\hbar^{2} k_{0}^{2} t^{2}}{m^{2} b}\right]
$$

There are now eight terms forming the argument of the exponential. The term in $x^{2}$ and two of the terms you introduced to complete the square are $\frac{-\left(x-\hbar k_{0} t / m\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m}$. This is the argument of the second exponential desired for part (c). The two remaining terms that are linear in $x$ reduce to $i k_{0} x$. The other three terms reduce to $-i \hbar k_{0}^{2} t / 2 m$. These combine to be the argument of the first exponential desired for part (c). $\psi(x, 0)$ should now be easy to calculate, and you will find it to be the same as the given distribution function. Part (e) is very similar to problem 10 (a). The result of problem 11 leads to the answer for part (f).

$$
\begin{align*}
& \psi(x)=\frac{e^{i k_{0} x} e^{-x^{2} / 2 \alpha^{2}}}{\left(\pi \alpha^{2}\right)^{1 / 4}} \Rightarrow g(k)=\frac{1}{\sqrt{2 \pi}} \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} \int_{-\infty}^{\infty} e^{i k_{0} x} e^{-x^{2} / 2 \alpha^{2}} e^{-i k x} d x  \tag{a}\\
&=\frac{1}{\sqrt{2 \pi}} \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} \int_{-\infty}^{\infty} e^{-x^{2} / 2 \alpha^{2}} e^{-i\left(k-k_{0}\right) x} d x \\
& \int_{-\infty}^{\infty} e^{-p^{2} x^{2} \pm q x} d x=\frac{\sqrt{\pi}}{p} e^{q^{2} / 4 p^{2}} \quad \text { so using } \quad p^{2}=\frac{1}{2 \alpha^{2}}, \quad q=i\left(k-k_{0}\right), \\
& \Rightarrow \quad g(k)=\frac{1}{\sqrt{2 \pi}} \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} \alpha \sqrt{2 \pi} e^{\left[i\left(k-k_{0}\right)\right]^{2} / 4\left(1 / 2 \alpha^{2}\right)}=\frac{\alpha}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\left(k-k_{0}\right)^{2} \alpha^{2} / 2}
\end{align*}
$$

(b) The integral over wavenumber of a plane wave weighted by the dispersion relation, which is the distribution in wavenumber, yields the wavefunction.

$$
\begin{aligned}
\psi(x, t) & =\frac{1}{\sqrt{2 \pi}} \frac{\alpha}{\left(\pi \alpha^{2}\right)^{1 / 4}} \int_{-\infty}^{\infty} e^{-\left(k-k_{0}\right)^{2} \alpha^{2} / 2} e^{i k x-i \hbar k^{2} t / 2 m} d k \\
& =\frac{1}{\sqrt{2 \pi}} \frac{\alpha}{\left(\pi \alpha^{2}\right)^{1 / 4}} \int_{-\infty}^{\infty} \exp \left[-\frac{\alpha^{2}}{2} k^{2}+\alpha^{2} k_{0} k-\frac{\alpha^{2} k_{0}^{2}}{2}+i k x-\frac{i \hbar k^{2} t}{2 m}\right] d k \\
& =\frac{1}{\sqrt{2 \pi}} \frac{\alpha}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\alpha^{2} k_{0}^{2} / 2} \int_{-\infty}^{\infty} \exp \left[-\left(\frac{\alpha^{2}}{2}+\frac{i \hbar k^{2} t}{2 m}\right) k^{2}+\left(i x+\alpha^{2} k_{0}\right) k\right] d k .
\end{aligned}
$$

Using the same integral employed in part (a) where $p^{2}=\left(\frac{\alpha^{2}}{2}+\frac{i \hbar t}{2 m}\right)$ and $q=i x+\alpha^{2} k_{0}$,
(c)

$$
\begin{align*}
\psi(x, t) & =\frac{1}{\sqrt{2 \pi}} \frac{\alpha}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\alpha^{2} k_{0}^{2} / 2} \frac{\sqrt{\pi}}{\left(\frac{\alpha^{2}}{2}+\frac{i \hbar t}{2 m}\right)^{1 / 2}} \exp \left[\frac{\left(i x+\alpha^{2} k_{0}\right)^{2}}{4\left(\frac{\alpha^{2}}{2}+\frac{i \hbar t}{2 m}\right)}\right] \\
& =\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\alpha^{2} k_{0}^{2} / 2}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \exp \left[\frac{\left(i x+\alpha^{2} k_{0}\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m}\right] . \\
\text { (c) } \quad \psi(x, t)= & \frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}} e^{-\alpha^{2} k_{0}^{2} / 2}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \exp \left[\frac{-x^{2}+2 i \alpha^{2} k_{0} x+\alpha^{4} k_{0}^{2}}{2 \alpha^{2}+2 i \hbar t / m}\right] . \tag{1}
\end{align*}
$$

Simplify the work with constants by letting $b=2 \alpha^{2}+\frac{2 i \hbar t}{m}$ and $c=\frac{\hbar k_{0} t}{m}$, then

$$
\psi(x, t)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \exp \left[-\frac{\alpha^{2} k_{0}^{2}}{2}\right] \exp \left[-\frac{x^{2}}{b}+\frac{2 i \alpha^{2} k_{0} x}{b}+\frac{\alpha^{4} k_{0}^{2}}{b}\right]
$$

Complete the square by multiplying by 1 in the form

$$
\exp \left[\frac{2 \hbar k_{0} t x}{m b}-\frac{2 \hbar k_{0} t x}{m b}-\frac{\hbar^{2} k_{0}^{2} t^{2}}{m^{2} b}+\frac{\hbar^{2} k_{0}^{2} t^{2}}{m^{2} b}\right]=\exp \left[\frac{2 c x}{b}-\frac{2 c x}{b}-\frac{c^{2}}{b}+\frac{c^{2}}{b}\right] .
$$

The overall argument of a single exponential is the eight terms

$$
\begin{gathered}
-\frac{\alpha^{2} k_{0}^{2}}{2}+\frac{\alpha^{4} k_{0}^{2}}{b}+\frac{2 i \alpha^{2} k_{0} x}{b}-\frac{2 c x}{b}+\frac{c^{2}}{b}-\frac{x^{2}}{b}+\frac{2 c x}{b}-\frac{c^{2}}{b} \\
=\left(-\frac{\alpha^{2} k_{0}^{2}}{2}+\frac{\alpha^{4} k_{0}^{2}}{b}+\frac{c^{2}}{b}\right)+\left(\frac{2 i \alpha^{2} k_{0} x}{b}-\frac{2 c x}{b}\right)+\left(-\frac{x^{2}}{b}+\frac{2 c x}{b}-\frac{c^{2}}{b}\right)
\end{gathered}
$$

where we have formed three groups. The last group is

$$
\begin{equation*}
-\frac{x^{2}}{b}+\frac{2 c x}{b}-\frac{c^{2}}{b}=\frac{-(x-c)^{2}}{b}=\frac{-\left(x-\hbar k_{0} t / m\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m} . \tag{2}
\end{equation*}
$$

This is the argument of the second exponential in the desired wavefunction. The middle group is

$$
\begin{equation*}
\frac{2 i \alpha^{2} k_{0} x}{b}-\frac{2 c x}{b}=\frac{2\left(i \alpha^{2} k_{0}-\hbar k_{0} t / m\right) x}{2 \alpha^{2}+2 i \hbar t / m}=\frac{\left(i \alpha^{2}-\hbar t / m\right) k_{0} x}{\alpha^{2}+i \hbar t / m}+\frac{i\left(\alpha^{2}+i \hbar t / m\right) k_{0} x}{\alpha^{2}+i \hbar t / m}=i k_{0} x \tag{3}
\end{equation*}
$$

which is a portion of the first exponential in the desired wavefunction. The first group is

$$
\begin{gathered}
-\frac{\alpha^{2} k_{0}^{2}}{2}+\frac{\alpha^{4} k_{0}^{2}}{b}+\frac{c^{2}}{b}=-\frac{\alpha^{2} k_{0}^{2}}{2}+\frac{\alpha^{4} k_{0}^{2}}{2 \alpha^{2}+2 i \hbar t / m}+\frac{\hbar^{2} k_{0}^{2} t^{2}}{m^{2}\left(2 \alpha^{2}+2 i \hbar t / m\right)} \\
=-\frac{\alpha^{2} k_{0}^{2}}{2}+\frac{\alpha^{4} k_{0}^{2} m^{2}}{2 \alpha^{2} m^{2}+2 i \hbar t m}+\frac{\hbar^{2} k_{0}^{2} t^{2}}{2 \alpha^{2} m^{2}+2 i \hbar t m}=\frac{-\alpha^{2} k_{0}^{2}\left(\alpha^{2} m^{2}+i \hbar t m\right)+\alpha^{4} k_{0}^{2} m^{2}+\hbar^{2} k_{0}^{2} t^{2}}{2 \alpha^{2} m^{2}+2 i \hbar t m}
\end{gathered}
$$

$$
\begin{equation*}
=\frac{-\alpha^{4} \not \not \ell_{0}^{2} m^{2}-i \alpha^{2} \hbar k_{0}^{2} t m+\alpha^{4} \not k_{0}^{2} m^{2}+\hbar^{2} k_{0}^{2} t^{2}}{2 \alpha^{2} m^{2}+2 i \hbar t m}=\frac{-i \hbar k_{0}^{2} t}{2 m}\left(\frac{\alpha^{2} m+i \hbar t}{\alpha^{2} m+i \hbar t}\right)=\frac{-i \hbar k_{0}^{2} t}{2 m} \tag{4}
\end{equation*}
$$

which is the other portion of the first exponential. Combining expressions (3) and (4),

$$
i k_{0} x-\frac{i \hbar k_{0}^{2} t}{2 m}=i k_{0}\left(x-\frac{\hbar k_{0} t}{2 m}\right)
$$

so the combination of all the exponentials in equation (1) may be written

$$
\begin{aligned}
& \exp \left[i k_{0}\left(x-\frac{\hbar k_{0}^{2} t}{2 m}\right)\right] \exp \left[\frac{-\left(x-\hbar k_{0} t / m\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m}\right] . \text { Using this in equation (1) yields } \\
& \psi(x, t)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \exp \left[i k_{0}\left(x-\frac{\hbar k_{0}^{2} t}{2 m}\right)\right] \exp \left[\frac{-\left(x-\hbar k_{0} t / m\right)^{2}}{2 \alpha^{2}+2 i \hbar t / m}\right]
\end{aligned}
$$

Including the $e^{i k_{0} x}$ term adds substantially to the algebra required which is why we initially developed the Gaussian wavefunction without it. The $e^{i k_{0} x}$ term centers the Gaussian wave packet at $k_{0} \neq 0$ so is usually essential. The algebra is insignificant compared to the concept.

$$
\begin{equation*}
\psi(x, 0)=\frac{1}{\left(\pi \alpha^{2}\right)^{1 / 4}}(1+0)^{-1 / 2} \exp \left[i k_{0}(x-0)\right] \exp \left[\frac{-(x-0)^{2}}{2 \alpha^{2}+0}\right]=\frac{e^{i k_{0} x} e^{-x^{2} / 2 \alpha^{2}}}{\left(\pi \alpha^{2}\right)^{1 / 4}} \tag{d}
\end{equation*}
$$

which is the initial distribution, $\psi(x)$. It will often be seen $\psi(x, 0)=\psi(x)=\frac{e^{i p_{0} x / \hbar} e^{-x^{2} / 2 \alpha^{2}}}{\left(\pi \alpha^{2}\right)^{1 / 4}}$.
(e) This part is similar to problem 10 (a) with two exceptions. The first is that the product of the first exponential and its complex conjugate is 1 ,

$$
\exp \left[-i k_{0}\left(x-\frac{\hbar k_{0}^{2} t}{2 m}\right)\right] \exp \left[i k_{0}\left(x-\frac{\hbar k_{0}^{2} t}{2 m}\right)\right]=e^{0}=1
$$

so does not affect the final answer. The other is the numerator of the argument of the remaining exponential is a two-term square instead of the simpler $x^{2}$ term. All else is identical so

$$
|\psi(x, t)|^{2}=\frac{1}{\alpha \sqrt{\pi}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{-1 / 2} \exp \left[\frac{-\left(x^{2}-\hbar k_{0} t / m\right)^{2}}{\alpha^{2}+\frac{\hbar^{2}}{m^{2} \alpha^{2}} t^{2}}\right]
$$

(f) Using the result problem 11, the uncertainty is

$$
\Delta x=\triangle x(t)=\frac{1}{\sqrt{2}}\left(\alpha^{2}+\frac{\hbar^{2}}{m^{2} \alpha^{2}} t^{2}\right)^{1 / 2}=\frac{\alpha}{\sqrt{2}}\left(1+\frac{\hbar^{2}}{m^{2} \alpha^{4}} t^{2}\right)^{1 / 2}
$$

The uncertainty is increasing as time advances, or wave packet "spreading" is apparent.
13. Show that the uncertainty in momentum of the Gaussian wavefunction is independent of time.

The wave packet does not "spread" in momentum space as time advances. (a) Use a Fourier transform to change the result of problem 8 from position space into momentum space, and (b) use this result to form the probability density in momentum space. Then (c) the uncertainty follows from the probability density. You should find

$$
\widehat{\psi}(p, t)=\frac{\alpha}{\sqrt{\hbar}\left(\alpha^{2} \pi\right)^{1 / 4}} \exp \left[-p^{2}\left(\frac{\alpha^{2}}{2 \hbar^{2}}+\frac{i t}{2 \hbar m}\right)\right]
$$

for part (a). You need form 3.323.2 from Gradshteyn and Ryzhik for part (b) which we write

$$
\int_{-\infty}^{\infty} e^{-b^{2} x^{2} \pm q x} d x=\frac{\sqrt{\pi}}{b} e^{q^{2} / 4 b^{2}},
$$

to prevent confusion between constants and momentum. You should find that

$$
|\widehat{\psi}(p, t)|^{2}=\frac{\alpha}{\hbar \sqrt{\pi}} e^{-p^{2} \alpha^{2} / \hbar^{2}}
$$

which is independent of time. The uncertainty follows and is also independent of time.

$$
\begin{align*}
\widehat{\psi}(p, t) & =\frac{1}{\sqrt{2 \pi \hbar}} \int_{-\infty}^{\infty} \psi(x, t) e^{-i p x / \hbar} d x  \tag{a}\\
& =\frac{1}{\sqrt{2 \pi \hbar}} \frac{1}{\left(\alpha^{2} \pi\right)^{1 / 4}}\left(1+\frac{i \hbar t}{m \alpha^{2}}\right)^{-1 / 2} \int_{-\infty}^{\infty} \exp \left[-\left(\frac{1}{2 \alpha^{2}+2 i \hbar t / m}\right) x^{2}-\frac{i p}{\hbar} x\right] d x \\
& =\frac{1}{\sqrt{2 \pi \hbar}} \frac{\alpha}{\left(\alpha^{2} \pi\right)^{1 / 4}}\left(\alpha^{2}+\frac{i \hbar t}{m}\right)^{-1 / 2} \sqrt{2 \pi}\left(\alpha^{2}+\frac{i \hbar t}{m}\right)^{1 / 2} \exp \left[-\frac{p^{2}}{4 \hbar^{2}}\left(2 \alpha^{2}+\frac{2 i \hbar t}{m}\right)\right] \\
& =\frac{\alpha}{\sqrt{\hbar}\left(\alpha^{2} \pi\right)^{1 / 4}} \exp \left[-p^{2}\left(\frac{\alpha^{2}}{2 \hbar^{2}}+\frac{i t}{2 \hbar m}\right)\right]
\end{align*}
$$

(c)

$$
\begin{align*}
|\widehat{\psi}(p, t)|^{2}= & \left(\frac{\alpha}{\sqrt{\hbar}\left(\alpha^{2} \pi\right)^{1 / 4}}\right)^{2} \exp \left[-p^{2}\left(\frac{\alpha^{2}}{2 \hbar^{2}}-\frac{i t}{2 \hbar m}\right)\right] \exp \left[-p^{2}\left(\frac{\alpha^{2}}{2 \hbar^{2}}+\frac{i t}{2 \hbar m}\right)\right]  \tag{b}\\
=\frac{\alpha^{2}}{\hbar} \frac{1}{\alpha \sqrt{\pi}} e^{-p^{2} \alpha^{2} / \hbar^{2}} & =\frac{\alpha}{\hbar \sqrt{\pi}} e^{-p^{2} \alpha^{2} / \hbar^{2}} \\
& \Rightarrow \Delta p=\frac{\hbar}{\alpha \sqrt{2}}
\end{align*}
$$

14. Prove the Schwarz inequality.

Having developed uncertainty for a realistic wavefunction, the next step is to discuss in the Heisenberg uncertainty relations. The Schwarz inequality, $\left.\left|\left|v_{1}\right\rangle\right|\left|\left|v_{2}\right\rangle\right| \geq\left|<v_{1}\right| v_{2}\right\rangle \mid$, plays a major role in the derivation of the Heisenberg uncertainty relations.

Form a vector from two others such that $\left|3>=\left|1>-\frac{<2|1\rangle}{\left||2>|^{2}\right.}\right| 2>\right.$. Form the adjoint of $|3\rangle$. Then form the inner product $\langle 3 \mid 3\rangle$ which is necessarily non-negative meaning that

