Chapter 2

Baseball! This job ain't so bad after all! He gave the ticket to an usher because he didn't recognize the seat. The usher started down the stairs so he restlessly followed as they kept stepping down, well past any seat he would buy for himself. The usher seated him in a two person box seat two rows back and just to the right of home plate. "Geez, I'm callin' balls and strikes!" He could hear the conversation as the managers waited for the umpire to exchange lineup cards... wives and kids, the squeeze bunt in yesterday's game, the merits of sun flower seeds versus chewing gum, and the brunette in back of the third base dugout. His reverie faded as a man in a white suit and a white Panama hat took the other seat of the box germinating the thought "Whoa...I'm watching a game with Mr. Clean ..." The stranger immediately enjoined with a genial smile brighter than his white suit "Think Schrodinger will play today?"

Postulates and Probability

Just as Euclidean geometry is founded upon postulates, quantum mechanics is founded upon postulates. You should not proceed beyond chapter 2 without knowing the postulates of quantum mechanics. You should perceive that there are essentially only two questions that can be asked of a measurement; these are what are the possibilities and what are the probabilities of each possibility? Quantum mechanics uses other statistical measures commonly encountered in discussions concerning probability, namely: **expectation value** and **standard deviation** or **uncertainty**. Notice that the Schrodinger equation is a postulate of quantum mechanics. The solutions to the spatial portion of the Schrodinger equation are known as **stationary states**. You should learn the time evolution of stationary states in this chapter. The postulates introduce the **state vector** as a description of a system. A complete set of commuting observables is required to uniquely determine the eigenstates of a state vector when one operator is degenerate. You should understand the terms **probability density** and **probability amplitude**. You should understand how to form a quantum mechanical **Hamiltonian operator**. You will need to use the mathematics presented in chapter 1 repeatedly.

1. What are the postulates of quantum mechanics? Compare and contrast each of the postulates to analogous statements from classical mechanics where a comparison is possible.

The postulates are the dominant reason for most of the mathematics of chapter 1 and are the basis either directly or indirectly for everything that follows.

- 1. The state of a system is represented by a vector $|\psi\rangle$ in Hilbert space.
- 2. Every observable quantity is represented by a Hermitian operator.
- 3. A measurement of an observable quantity represented by the operator \mathcal{A} can yield only the eigenvalues of \mathcal{A} .

4. If a system is in state $|\psi\rangle$, a measurement of the observable quantity represented by the operator \mathcal{A} that yields the eigenvalue α does so with the probability

$$P(\alpha) \propto \left| < \alpha \right| \psi > \left|^{2},$$

where $|\alpha\rangle$ is the eigenstate corresponding to the eigenvalue α .

5. A measurement of the observable quantity represented by the operator \mathcal{A} with the result α changes the state of the system to the eigenstate $|\alpha\rangle$.

6. The state vector obeys the time dependent Schrodinger equation

$$\mathcal{H} \,|\, \psi \!> = i\hbar \frac{d}{dt} \,|\, \psi \!> ,$$

where \mathcal{H} is the quantum mechanical Hamiltonian operator.

Postulate 1: The state vector $|\psi\rangle$ replaces the positions and momentums (or velocities) of classical mechanics.

Postulate 2: That observable quantities are described by operators is fundamentally different than the classical description that every dynamical variable is described by a function of position and momentum, for instance.

Postulate 3: Any real value is a possible result of a classical measurement. The only possible result of a quantum mechanical measurement is an eigenvalue of the operator representing the quantity being measured.

Postulate 4: There is no classical analogy. The interpretation of the square of an inner product as a probability is unique to quantum mechanics.

Postulate 5: There is also no classical analogy for postulate 5. Classically, measurement of a system does not affect the system.

Postulate 6: The time rate of change of the state variables of a classical system are governed by Hamilton's equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}$$
 and $\frac{dp}{dt} = -\frac{\partial H}{\partial x}$.

The time evolution of a quantum mechanical system is governed by Schrodinger's equation.

Postscript: Postulate 1 includes the principle of superposition. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible states of the system, then the linear combination, $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$, is also a possible state of the system. The $|\psi_i\rangle$ are generally the eigenstates of the system. The general state vector is the superposition or linear combination of all eigenstates, *i.e.*,

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + c_3 |\psi_3\rangle + \dots = \sum_{i=1}^{\infty} c_i |\psi_i\rangle$$

in the case of an infinity of eigenstates. Each of the coefficients c_i are scalars that indicate the relative "amount" of eigenstate $|\psi_i\rangle$ in the superposition that is the state vector $|\psi\rangle$ and are often called **probability amplitudes** because the probability of measuring the corresponding eigenvalue is often $|c_i|^2$. A coefficient can be zero meaning that the corresponding eigenstate is absent from that state vector.

Observable quantities are those that can be physically measured. There are two essential properties intrinsic to operators that are Hermitian. First, they have eigenvalues that are real numbers. Numbers used to describe physical quantities are necessarily real numbers. This fact is essential to postulate 3. Also, the eigenvectors of Hermitian operators are orthogonal, therefore, the eigenvectors form a basis that can be made orthonormal. This fact is essential to postulate 4.

The observable quantities of position and momentum remain focal. An extension of postulate 2 is that the matrix elements of the position operator \mathcal{X} and momentum operator \mathcal{P} in the position basis are represented

$$\langle x | \mathcal{X} | x' \rangle = x \, \delta \left(x - x' \right), \quad \text{and} \quad \langle x | \mathcal{P} | x' \rangle = -i\hbar \, \delta' \left(x - x' \right).$$

Postulate 3 is self explanatory but definitely non-classical. The *only* possible result of a measurement is an *eigenvalue* of the operator representing the physical quantity being measured.

A proportionality is used in postulate 4. The proportionality is replaced by an equality by dividing by the inner product of the unnormalized state vector,

$$P(\alpha) \propto \left| < \alpha \left| \psi > \right|^2 \Rightarrow P(\alpha) = \frac{\left| < \alpha \left| \psi > \right|^2}{< \psi \left| \psi > \right|},$$

or equivalently, by normalizing the state vector before calculating the inner product. Postulate 4 is the reason for the process of normalization.

A probability of "1" means a certainty and a probability of "0" means the absence of a possibility. The probability of measuring each eigenvalue of an observable quantity must be between 0 and 1 inclusive, or $0 \le P(\alpha) \le 1$. The sum of the probabilities of individual measurements resulting in all of the possible eigenvalues must be 1. The probabilistic interpretation of postulate 4 is the reason that two state vectors that are proportional represent the same physical state.

Postulate 5 describes what is often called the "collapse of the wave function." It is the statement that the observer interacts with the system; that the observer is part of the system. Regardless of how carefully a measurement is made, the process of measurement changes the system being measured. Further, the measurement changes the system in a specific way, the measurement forces the system into one of its eigenstates. Finally, once in that eigenstate, it remains in that eigenstate until it undergoes its next interaction which is its next "measurement."

Postulate 6 is the Schrödinger equation. It is not derived from the postulates of quantum mechanics, rather, the Schrödinger equation is one of the postulates of quantum mechanics.

Postulate 6 requires the quantum mechanical Hamiltonian operator, \mathcal{H} . The classical Hamiltonian operator is the total energy operator, H = T + V, where T is kinetic energy and V is potential energy. Each of the dynamical variables of classical mechanics is replaced by an operator for the transition to the quantum mechanical formulation, $x \to \mathcal{X}$, and $p \to \mathcal{P}$. The classical Hamiltonian then goes to the quantum mechanical Hamiltonian, $H \to \mathcal{H}$. These quantum

mechanical operators are basis independent. They can be represented using matrix operators or differential operators in any basis for further calculation.

The Schrodinger equation is not a postulate in the path integral formulation of quantum mechanics, rather, it is derived from the postulates. This indicates that there is something more fundamental about the path integral formulation of quantum mechanics. The path integral formulation is not always satisfying because it is difficult to apply to calculations for even simple systems. We will address the path integral formulation including its postulates in future chapters.

2. A system is represented by the normalized state vector $|\psi\rangle$. It is composed solely of two orthonormal eigenstates, $|\psi_1\rangle$ and $|\psi_2\rangle$.

- (a) Write $|\psi\rangle$ as the superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$.
- (b) What is the probability that the state is $|\psi_1\rangle$ following a measurement?
- (c) What is the probability that the state is $|\psi_2\rangle$ following a measurement?
- (d) Show that the probabilities of parts (b) and (c) sum to 1.

This problem amplifies postulates 1, 4, and 5. The discussion of postulate 1 in the postscript of problem 1 describes superposition. Postulate 5 indicates that a measurement changes the state vector to one of the eigenstates. If the first eigenstate is the new state vector, that is the one to use in the inner product in the probability calculation described in postulate 4. The proportionality of postulate 4 can be replaced by an equality because $|\psi\rangle$ is given to be normalized. Use the superposition of part (a) for $|\psi\rangle$ in parts (b) through (d), and you need to recognize the orthonormality of the eigenstates. For part (d), what is the inner product of $|\psi\rangle$ with its bra?

(a) $|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$ is the desired statement of superposition.

(b)

$$P_{1} = \left| \langle \psi_{1} | (c_{1} | \psi_{1} \rangle + c_{2} | \psi_{2} \rangle) \right|^{2}$$

$$= \left| \langle \psi_{1} | c_{1} | \psi_{1} \rangle + \langle \psi_{1} | c_{2} | \psi_{2} \rangle \right|^{2}$$

$$= \left| c_{1} \langle \psi_{1} | \psi_{1} \rangle + c_{2} \langle \psi_{1} / \psi_{2} \rangle \right|^{2} = \left| c_{1} \right|^{2},$$

because $\langle \psi_1 | \psi_1 \rangle = 1$ and $\langle \psi_1 | \psi_2 \rangle = 0$ due to the given orthonormality of eigenstates.

(c)

$$P_{2} = \left| \langle \psi_{2} | (c_{1} | \psi_{1} \rangle + c_{2} | \psi_{2} \rangle) \right|^{2}$$

$$= \left| \langle \psi_{2} | c_{1} | \psi_{1} \rangle + \langle \psi_{2} | c_{2} | \psi_{2} \rangle \right|^{2}$$

$$= \left| c_{1} \langle \psi_{2} / \psi_{1} \rangle + c_{2} \langle \psi_{2} | \psi_{2} \rangle \right|^{2} = \left| c_{2} \right|^{2}$$

because $\langle \psi_2 | \psi_1 \rangle = 0$ and $\langle \psi_2 | \psi_2 \rangle = 1$ due to the orthonormality of eigenstates.

(d) $\langle \psi | \psi \rangle = 1$ because $| \psi \rangle$ is given to be normalized. Then

$$\begin{aligned} <\psi \,|\,\psi> &= \left(\,<\psi_1\,|\,c_1^*\,+<\psi_2\,|\,c_2^*\right)\left(c_1\,|\,\psi_1>+\,c_2\,|\,\psi_2>\right) \\ &= <\psi_1\,|\,c_1^*\,c_1\,|\,\psi_1> + <\psi_1\,|\,c_1^*\,c_2\,|\,\psi_2> + <\psi_2\,|\,c_2^*\,c_1\,|\,\psi_1> + <\psi_2\,|\,c_2^*\,c_2\,|\,\psi_2> \\ &= \,|\,c_1\,|^{\,2} <\psi_1\,|\,\psi_1> + \,c_1^*\,c_2 <\psi_1 \middle/\!\!\!/\psi_2> + \,c_2^*\,c_1 <\psi_2 \middle/\!\!\!/\psi_1> + \,|\,c_2\,|^{\,2} <\psi_2\,|\,\psi_2> \\ &= \,|\,c_1\,|^{\,2} + \,|\,c_2\,|^{\,2} = 1\,, \end{aligned}$$

where $\langle \psi_1 | \psi_1 \rangle = \langle \psi_2 | \psi_2 \rangle = 1$ and other inner products are zero because of orthonormality.

Postscript: The $|c_i|^2$ are probabilities, thus the c_i are known as probability amplitudes.

3. A system is represented by the unnormalized state vector $|\psi\rangle$. It is composed solely of the two orthonormal eigenstates, $|\psi_1\rangle$ and $|\psi_2\rangle$.

- (a) Show that the proportionality in postulate 4 is replaced by an equality when $|\langle \psi_i | \psi \rangle|^2$ is divided by the inner product of the unnormalized state vector.
- (b) Show that $P_i = |\langle \psi_i | \psi \rangle|^2$ when $|\psi\rangle$ is normalized prior to the probability calculation.
- (c) Explain why two state vectors that are proportional represent the same physical state.

Postulate 4 indicates that an inner product of two state vectors is a probability. The normalization condition, $\langle \psi | \psi \rangle = 1$, says only that it is certain that the system exists. It may be easier to see in a position space statement, $\int_{-\infty}^{\infty} \psi^*(x) \psi(x) dx = 1$. If the system exists, it is certain that the system is between $-\infty$ and $+\infty$. Certainty is expressed by a probability of 1. The same condition $\langle \psi | A^* A | \psi \rangle = 1$ for an unnormalized $| \psi \rangle$ also says only that the system exists with certainty. The normalization condition is simply an application of the probabilistic interpretation.

Since the state vector is given to be unnormalized, attach a proportionality constant, that is $A | \psi \rangle = A c_1 | \psi_1 \rangle + A c_2 | \psi_2 \rangle$. You need to use the condition $\langle \psi | A^* A | \psi \rangle = 1$ to find

$$P_1 = \frac{|c_1|^2}{\langle \psi | \psi \rangle}$$
 and $P_2 = \frac{|c_2|^2}{\langle \psi | \psi \rangle}$ for part (a).

The calculations of parts (a) and (b) are fairly duplicative—this is because the normalization condition is just a statement of the probabilistic condition of certainty. Familiarity with this duplication should provide insight into part (c), which is actually the point of this problem,.

(a) Since the state vector is given to be unnormalized, we attach a normalization constant, that is $A | \psi \rangle = A c_1 | \psi_1 \rangle + A c_2 | \psi_2 \rangle$. Then, remembering that $| \psi_1 \rangle$ and $| \psi_2 \rangle$ are orthonormal,

$$P_{1} \propto \left| <\psi_{1} | \left(A c_{1} | \psi_{1} > + A c_{2} | \psi_{2} > \right) \right|^{2} = \left| <\psi_{1} | A c_{1} | \psi_{1} > + <\psi_{1} | A c_{2} | \psi_{2} > \right|^{2} \\ = \left| A c_{1} <\psi_{1} | \psi_{1} > + A c_{2} <\psi_{1} / | \psi_{2} > \right|^{2} = |A c_{1}|^{2} = |A|^{2} |c_{1}|^{2}.$$

$$P_{2} \propto \left| \langle \psi_{2} | (A c_{1} | \psi_{1} \rangle + A c_{2} | \psi_{2} \rangle) \right|^{2} = \left| \langle \psi_{2} | A c_{1} | \psi_{1} \rangle + \langle \psi_{2} | A c_{2} | \psi_{2} \rangle \right|^{2} \\ = \left| A c_{1} \langle \psi_{2} / \psi_{1} \rangle + A c_{2} \langle \psi_{2} | \psi_{2} \rangle \right|^{2} = \left| A c_{2} \right|^{2} = \left| A \right|^{2} \left| c_{2} \right|^{2}.$$

The sum of all the possibilities, in this case the two possibilities, must be 1. Then the sum

$$|A|^{2} |c_{1}|^{2} + |A|^{2} |c_{2}|^{2} = |A|^{2} (|c_{1}|^{2} + |c_{2}|^{2}) = 1.$$

The condition that the system exists with certainty is

$$\begin{aligned} <\psi \,|\, A^* \,A \,|\, \psi > = 1 \quad \Rightarrow \quad |\, A \,|^2 <\psi \,|\, \psi > = 1 \quad \Rightarrow \quad |\, A \,|^2 = \frac{1}{<\psi \,|\, \psi >} \\ \Rightarrow \quad \frac{|\, c_1 \,|^2 + |\, c_2 \,|^2}{<\psi \,|\, \psi >} = 1 \,, \quad \text{and} \quad P_1 = \frac{|\, c_1 \,|^2}{<\psi \,|\, \psi >} \quad \text{and} \quad P_2 = \frac{|\, c_2 \,|^2}{<\psi \,|\, \psi >} \,, \\ \text{Therefore,} \quad P_i = \frac{|\, c_i \,|^2}{<\psi \,|\, \psi >} = \frac{|\, <\psi_i \,|\, \psi > \,|^2}{<\psi \,|\, \psi >} \quad \text{in general.} \end{aligned}$$

(b) The normalization constant is found

$$\langle \psi | A^* A | \psi \rangle = 1 \quad \Rightarrow \quad |A|^2 = \frac{1}{\langle \psi | \psi \rangle} \quad \Rightarrow \quad A = \frac{1}{\sqrt{\langle \psi | \psi \rangle}}$$

$$P_{1} \propto \left| <\psi_{1} \left| \left(A c_{1} \mid \psi_{1} > + A c_{2} \mid \psi_{2} > \right) \right|^{2} = \left| <\psi_{1} \mid \frac{c_{1}}{\sqrt{<\psi \mid \psi >}} \mid \psi_{1} > + <\psi_{1} \mid \frac{c_{2}}{\sqrt{<\psi \mid \psi >}} \mid \psi_{2} > \right|^{2}$$
$$= \left| \frac{c_{1}}{\sqrt{<\psi \mid \psi >}} <\psi_{1} \mid \psi_{1} > + \frac{c_{2}}{\sqrt{<\psi \mid \psi >}} <\psi_{1} / / \psi_{2} > \right|^{2} = \left| \frac{c_{1}}{\sqrt{<\psi \mid \psi >}} \right|^{2} = \frac{|c_{1}|^{2}}{|\psi >|\psi >|^{2}}$$

which is the same as P_1 from part (a). A similar calculation shows that P_2 is also the same as found in part (a). Since the probabilities are the same, particularly now that we know the origin of normalization condition, the procedure of using a normalized state vector allows the use of the relation of equality in postulate 4.

(c) The interpretation of an inner product as a probability renders the "length" of a state vector immaterial because the length is necessarily adjusted so that the probability of certainty is 1. The probabilities of individual possibilities of any measurement are necessarily identical using $|\psi\rangle$ or $A |\psi'\rangle$ because of this adjustment. The conclusion is two state vectors that are proportional represent the same physical state.

Physicists will not usually write any symbology that differentiates two state vectors that are proportional. For instance, $|\psi\rangle$ and $A|\psi\rangle$ where A is a scalar are both appropriate descriptions of the state vector for the same photon. In fact, you may see equations like $|\psi\rangle = A|\psi\rangle$ which

Postscript: Insisting state vectors are normalized prior to calculating probabilities usually leads to shorter and cleaner calculations. You are likely going to be most efficient if you consider the statement "normalize all state vectors prior to calculating probabilities" as a corollary to postulate 4 because the proportionality is then replaced by an equality.

is a true statement for all values of A if $|\psi\rangle$ is a state vector. If $|\psi\rangle$ is not a state vector, then A = 1 is the only value of A that will make the statement true. If $|\psi\rangle$ is a state vector, it will be adjusted so that the sum of all possible probabilities is 1.

The concept of this two-dimensional problem extends to arbitrary dimensions.

4. The " \mathcal{A} -ness" of a particle or system in the state

$$|\psi\rangle = \begin{pmatrix} 1\\2\\3 \end{pmatrix}$$
 where $\mathcal{A} = \begin{pmatrix} 3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5 \end{pmatrix}$ is measured.

- (a) Normalize $|\psi\rangle$.
- (b) What are the possible outcomes of a measurement of the " \mathcal{A} -ness" of $|\psi\rangle$?
- (c) Calculate the probability of each possibility.
- (d) Show that the sum of the probabilities is 1.
- (e) A measurement of the " \mathcal{A} -ness" of $|\psi\rangle$ yields the value 4. A second measurement of the " \mathcal{A} -ness" of $|\psi\rangle$ is then made. What are the possibilities and probabilities of the outcomes of this second measurement?

This problem is meant to amplify postulates 3, 4, and 5. Per the postscript to problem 3, we will routinely do the normalization of part (a) prior to any probability calculations. The only possible results of a measurement are the eigenvalues of \mathcal{A} per postulate 3. You need the eigenvectors to calculate probabilities. You should solve for the eigenvalues and eigenvectors of this diagonal matrix by inspection. Of course, if the three probabilities do not sum to 1 you have made an error. Postulate 5 tells you how to approach part (e).

(a) The normalized state vector is found

$$1 = (1, 2, 3) A^* A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = |A|^2 (1+4+9) = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{14}} \quad \Rightarrow \quad |\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(b) The possible results of a measurement are the eigenvalues 3, 4, and 5. The elements on the principal diagonal of a diagonal matrix are the eigenvalues. (c) The eigenvectors are found by inspection to be

$$|3\rangle = \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \quad |4\rangle = \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \quad |5\rangle = \begin{pmatrix} 0\\0\\1 \end{pmatrix}.$$
$$P(ev = 3) = \left| (1, 0, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (1+0+0) \right|^2 = \left| \frac{1}{\sqrt{14}} \right|^2 = \frac{1}{14},$$

$$P(ev = 4) = \left| (0, 1, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (0 + 2 + 0) \right|^2 = \left| \frac{2}{\sqrt{14}} \right|^2 = \frac{4}{14},$$
$$P(ev = 5) = \left| (0, 0, 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (0 + 0 + 3) \right|^2 = \left| \frac{3}{\sqrt{14}} \right|^2 = \frac{9}{14}.$$
$$(d) \quad \sum_{1}^{3} P_i = \frac{1}{14} + \frac{4}{14} + \frac{9}{14} = \frac{14}{14} = 1.$$

(e) Postulate 5 says that the state vector of the system is the eigenstate of ev = 4 following the first measurement, so the state vector is $|\psi'\rangle = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$. The probability of attaining ev = 4 as a result of the second measurement is 1.

The probability of measuring either of the other two eigenvalues is zero. The probability of measuring ev = 3 or ev = 5 is

$$\left| \begin{pmatrix} 1, 0, 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0, 0, 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0+0+0 \end{pmatrix} \right|^2 = \left| 0 \right|^2 = 0.$$

Postscript: Postulates 1 and 2 are sometimes overlooked because of an emphasis on calculation. Notice that the state vector $|\psi\rangle$ satisfies postulate 1 and that \mathcal{A} is a Hermitian operator that satisfies postulate 2.

Notice also that the state vector is not pertinent to the possible results of a measurement. Possibilities are determined soley by the operator. The state vector is pertinent only to the probabilities of the possible results of a measurement.

5. What are the possible results and the probability of attaining each possible result of a measurement of the " \mathcal{B} -ness" of a system where

$$\mathcal{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and the state is} \quad |\psi\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The question asks only for possibilities and probabilities. The possibilities are the eigenvalues, and the probabilities follow from the inner product of the eigenvector and the state vector. You normalized this state vector in problem 4. You should find

$$|2> = \begin{pmatrix} 1\\0\\0 \end{pmatrix}, |3> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\1 \end{pmatrix}, |1> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\-1 \end{pmatrix}.$$

If your probabilities do not sum to 1, you have made an error.

Using the normalized state vector, the probabilities of each possibility are

$$P(ev = 2) = \left| (1, 0, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (1+0+0) \right|^2 = \left| \frac{1}{\sqrt{14}} \right|^2 = \frac{1}{14},$$

$$P(ev = 3) = \left| \frac{1}{\sqrt{2}} (0, 1, 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{28}} (0+2+3) \right|^2 = \left| \frac{5}{\sqrt{28}} \right|^2 = \frac{25}{28},$$

$$P(ev = 1) = \left| \frac{1}{\sqrt{2}} (0, 1, -1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{28}} (0+2-3) \right|^2 = \left| \frac{-1}{\sqrt{28}} \right|^2 = \frac{1}{28},$$
and $\frac{2}{28} + \frac{25}{28} + \frac{1}{28} = \frac{28}{28} = 1,$ as it must.

6. The operator \mathcal{A} is as given in problem 4 and the operator \mathcal{B} is as given in problem 5. The state vector of the system is initially $|\psi\rangle = \begin{pmatrix} 1\\2\\3 \end{pmatrix}$ for the following questions.

- (a) If we measure a value of 3 for A, what are the possible values of successive measurements of B, A, B, etc., and their respective probabilities?
- (b) If we measure \mathcal{A} and get 4, what are the possible values of a subsequent measurement of \mathcal{B} and the probabilities of each possibility?
- (c) Having measured a value of 4 for \mathcal{A} , and then measured a value of 3 for \mathcal{B} , what are the possible values of another measurement of \mathcal{A} and their respective probabilities?
- (d) Having measured a value of 4 for \mathcal{A} , and then measured a value of 1 for \mathcal{B} , what are the possible values of a measurement of \mathcal{A} and their respective probabilities?
- (e) If we measure 5 for \mathcal{A} , what are the possible values of subsequent measurements of \mathcal{B} and their respective probabilities?

This problem should help to focus postulate 5. The state vector following a measurement is the eigenstate corresponding to the eigenvalue measured. Use the new state vector determined by the measurement in probability calculations consistent with postulate 4 to answer all five questions.

(a) If we measure \mathcal{A} and get ev = 3, the state vector of the system is now in the eigenstate of \mathcal{A} corresponding to this eigenvalue, *i.e.*, $|\psi'\rangle = \begin{pmatrix} 1\\0\\0 \end{pmatrix}$. This is identical to the eigenstate

of \mathcal{B} corresponding to ev = 2. The eigenvectors of \mathcal{B} corresponding to ev = 3 or ev = 1 both have zero as their first component, so $\langle ev_{\mathcal{B}} = 3 | \psi' \rangle = \langle ev_{\mathcal{B}} = 1 | \psi' \rangle = 0$. We will measure ev = 2 for \mathcal{B} with probability of $\langle ev_{\mathcal{B}} = 2 | \psi' \rangle = 1$. Subsequent measurements of \mathcal{A} , \mathcal{B} , \mathcal{A} , etc., yield ev = 3, ev = 2, ev = 3, etc., where the probability for each measurement is 1.

(b) From the given measurement of \mathcal{A} , the state vector of the system is $|\psi'\rangle = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$. The

eigenvector corresponding to the eigenvalue measured is now the eigenstate of the system. Both of the eigenvectors |ev = 3 > and |ev = 1 > of operator \mathcal{B} are non-zero in the same component that $|\psi'\rangle$ is non-zero while the corresponding component of |ev = 2 > is zero. From that fact alone, we can conclude that P(ev = 2) = 0. Nevertheless, the probabilities of all possibilities are

$$P(ev = 2) = \left| \begin{pmatrix} 1, 0, 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0 + 0 + 0 \end{pmatrix} \right|^2 = \left| 0 \right|^2 = 0,$$
$$P(ev = 3) = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0, 1, 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 + 1 + 0 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},$$
$$P(ev = 1) = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0, 1, -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 + 1 + 0 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

(c) If 3 was found for the measurement of \mathcal{B} , the new state vector is $|\psi'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\1 \end{pmatrix}$. Then the probabilities for a subsequent measurement of \mathcal{A} follow from postulate 4,

$$P(ev = 3) = \left| (1, 0, 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} (0+0+0) \right|^2 = |0|^2 = 0,$$

$$P(ev = 4) = \left| (0, 1, 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} (0+1+0) \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},$$

$$P(ev = 5) = \left| (0, 0, 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} (0+0+1) \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

(d) If we found 1 for the measurement of \mathcal{B} , the state vector is $|\psi'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\ 1\\ -1 \end{pmatrix}$. In a calculation that is similar to part (c), we find for a measurement of \mathcal{A} , P(ev = 3) = 0, P(ev = 4) = 1/2, and P(ev = 5) = 1/2.

(e) The state vector of the system is $\begin{pmatrix} 0\\0\\1 \end{pmatrix}$ after the given measurement. Again, both |ev = 3 > and |ev = 1 > of the operator \mathcal{B} have corresponding components that are non-zero but |ev = 2 > does not. The probabilities are P(ev = 2) = 0, P(ev = 3) = 1/2, and P(ev = 1) = 1/2 for a subsequent measurement of \mathcal{B} .

7. Find the possibilities and probabilities of a measurement of \mathcal{L}_y for a system in the state

$$|\psi\rangle = \begin{pmatrix} 1\\2\\3 \end{pmatrix}$$
 where $\mathcal{L}_y = \begin{pmatrix} 0 & -i & 0\\i & 0 & -i\\0 & i & 0 \end{pmatrix}$.

Some of the eigenvectors of \mathcal{L}_y have imaginary components so that the probability calculations require use of complex numbers. The complex number facet is the only difference from problems 4 or 5. You found that \mathcal{L}_y is Hermitian in problem 9 of part 2 of chapter 1, and in problem 19 of part 2 of chapter 1, that the eigenvalues and eigenvectors are

$$|-\sqrt{2}\rangle = \frac{1}{2} \begin{pmatrix} 1\\ -\sqrt{2}i\\ -1 \end{pmatrix}, |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix}, \text{ and } |\sqrt{2}\rangle = \frac{1}{2} \begin{pmatrix} 1\\ \sqrt{2}i\\ -1 \end{pmatrix}.$$

Using the normalized state vector from previous problems,

$$P(ev = -\sqrt{2}) = \left| \frac{1}{2} (1, \sqrt{2}i, -1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{2\sqrt{14}} (1 + 2\sqrt{2}i - 3) \right|^2$$
$$= \frac{1}{4 \cdot 14} \left| -2 + 2\sqrt{2}i \right|^2 = \frac{1}{56} (-2 - 2\sqrt{2}i) (-2 + 2\sqrt{2}i)$$
$$= \frac{1}{56} (4 + 8) = \frac{12}{56} = \frac{3}{14},$$
$$P(ev = 0) = \left| \frac{1}{\sqrt{2}} (1, 0, 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{28}} (1 + 0 + 3) \right|^2 = \left| \frac{4}{\sqrt{28}} \right|^2 = \frac{16}{28} = \frac{8}{14},$$
$$P(ev = \sqrt{2}) = \left| \frac{1}{2} (1, -\sqrt{2}i, -1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \right|^2 = \left| \frac{1}{2\sqrt{14}} (1 - 2\sqrt{2}i - 3) \right|^2$$
$$= \frac{1}{4 \cdot 14} \left| -2 - 2\sqrt{2}i \right|^2 = \frac{1}{56} (-2 + 2\sqrt{2}i) (-2 - 2\sqrt{2}i)$$
$$= \frac{1}{56} (4 + 8) = \frac{12}{56} = \frac{6}{28} = \frac{3}{14},$$

and the sum of the probabilities is $\frac{3}{14} + \frac{8}{14} + \frac{3}{14} = \frac{14}{14} = 1$, as is necessary.

Postscript: Remember that for any scalar α , $|\alpha|^2 = \alpha^* \alpha$, the product of conjugate quantities, so $|\alpha|^2$ is the sum of the squares of the real and imaginary parts of the complex scalar.

8. Find the expectation values of \mathcal{A} , \mathcal{B} , and \mathcal{L}_y using the operators given and the probabilities calculated in problems 4, 5, and 7, respectively.

An **expectation value** is the anticipated average of many measurements made on an ensemble of identical systems. An expectation value is defined

$$<\mathcal{A}>_{\psi}=\sum_{i}P\left(\alpha_{i}\right)\alpha_{i}.$$

It is the sum of the products of the eigenvalue and the probability of measuring that eigenvalue. An expectation value is simply a weighted average.

$$<\mathcal{A}>_{\psi} = \sum_{i} P(\alpha_{i}) \alpha_{i} = \frac{1}{14} (3) + \frac{4}{14} (4) + \frac{9}{14} (5) = \frac{64}{14} = 4\frac{4}{7}.$$

$$<\mathcal{B}>_{\psi} = \sum_{i} P(\beta_{i}) \beta_{i} = \frac{2}{28} (2) + \frac{25}{28} (3) + \frac{1}{28} (1) = \frac{80}{28} = 2\frac{6}{7}.$$

$$<\mathcal{L}_{y}>_{\psi} = \frac{3}{14} (-\sqrt{2}) + \frac{8}{14} (0) + \frac{3}{14} (\sqrt{2}) = 0.$$

Postscript: Probabilities are dependent upon the state vector, therefore, expectation values that are computed using probabilities are also dependent upon the state vector. The expectation value symbols have been subscripted with ψ to emphasize this dependence.

9. Use a normalized state vector $|\psi\rangle$ to show that $\langle \Omega \rangle_{\psi} = \langle \psi | \Omega | \psi \rangle$.

The expression on the right is an alternative method of calculating an expectation value without having to complete the eigenvalue/eigenvector problem. It is also a good method to check calculations concerning small dimensional operators like \mathcal{A} , \mathcal{B} , and \mathcal{L}_y .

This problem is a good exercise in applying some of the concepts and notation encountered previously. Start with the definition of expectation value given in the last problem. Then successively use postulate 4, the definition of a norm, the property that scalars commute, the eigenvalue/eigenvector equation, and the completeness relation to arrive at the desired expression.

$$<\Omega>_{\psi} = \sum_{i} P(\omega_{i}) \omega_{i} = \sum_{i} |<\omega_{i}| \psi > |^{2} \omega_{i} = \sum_{i} <\psi |\omega_{i}><\omega_{i}| \psi > \omega_{i}$$
(1)

$$=\sum_{i} \langle \psi | \omega_{i} | \omega_{i} \rangle \langle \omega_{i} | \psi \rangle$$
(2)

$$=\sum_{i} \langle \psi | \Omega | \omega_{i} \rangle \langle \omega_{i} | \psi \rangle$$
(3)

$$= \langle \psi | \Omega | \left(\sum_{i} | \omega_{i} \rangle \langle \omega_{i} | \right) | \psi \rangle = \langle \psi | \Omega | \mathcal{I} | \psi \rangle = \langle \psi | \Omega | \psi \rangle .$$

$$\tag{4}$$

Line (1) is the definition of an expectation value, application of postulate 4, and the definition of a norm. Eigenvalues are scalars so can be moved into the braket in line (2). The eigenvalue equation, $\Omega | \omega_i \rangle = \omega_i | \omega_i \rangle$, is used to arrive at line (3). The unsubscripted vectors $\langle \psi | \Omega |$ and $| \psi \rangle$ are not pertinent to the sum so are removed from the summation to arrive at the first expression in line (4). The summation remaining in the parenthesis is the completeness relation which is a statement of the identity.

10. Check that $\langle \psi | \Omega | \psi \rangle$ yields the expectation values of \mathcal{A} , \mathcal{B} , and \mathcal{L}_y calculated previously.

Remember that an expectation value is dependent upon the state vector. You must use the state vector from problem 8, $|\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix}$, to attain the same expectation values as problem 8.

$$\langle \mathcal{A} \rangle_{\psi} = \langle \psi | \mathcal{A} | \psi \rangle = \frac{1}{\sqrt{14}} (1, 2, 3) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} (1, 2, 3) \begin{pmatrix} 3 \\ 8 \\ 15 \end{pmatrix} = \frac{1}{14} (3 + 16 + 45) = \frac{64}{14} = 4\frac{4}{7}.$$

$$\langle \mathcal{B} \rangle_{\psi} = \langle \psi | \mathcal{B} | \psi \rangle = \frac{1}{\sqrt{14}} (1, 2, 3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} (1, 2, 3) \begin{pmatrix} 2 \\ 4 + 3 \\ 2 + 6 \end{pmatrix} = \frac{1}{14} (1, 2, 3) \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix} = \frac{1}{14} (2 + 14 + 24) = \frac{40}{14} = 2\frac{6}{7}.$$

$$\langle \mathcal{L}_{y} \rangle_{\psi} = \langle \psi | \mathcal{L}_{y} | \psi \rangle = \frac{1}{\sqrt{14}} (1, 2, 3) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} (1, 2, 3) \begin{pmatrix} -2i \\ i - 3i \\ 2i \end{pmatrix} = \frac{1}{14} (1, 2, 3) \begin{pmatrix} -2i \\ -2i \\ 2i \end{pmatrix} = \frac{1}{14} (-2i - 4i + 6i) = 0.$$

Postscript: The subscript ψ is rarely appended to expectation values. The expectation value of Ω appears as $\langle \Omega \rangle$, which is conventional. Remember, nevertheless, that an expectation value is dependent upon a state vector.

11. Find the uncertainty of \mathcal{A} using the operator and the state vector given in problem 4.

Two measures of central tendency are frequently encountered in quantum mechanics. The first is the previously discussed expectation value. The other is **uncertainty** or **standard deviation**.

Uncertainty or standard deviation is defined in terms of the expectation value,

$$\triangle \mathcal{A}_{\psi} = \langle \psi | \left(\mathcal{A} - \langle \mathcal{A} \rangle \mathcal{I} \right)^2 | \psi \rangle^{1/2},$$

where the standard deviation on the left is subscripted because it is dependent on the state vector $|\psi\rangle$. A state vector is needed to calculate the expectation value, therefore, a state vector is needed to calculate the uncertainty.

Use the operator $\mathcal{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ and the normalized state vector $|\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. The expectation value times the identity operator means

$$<\mathcal{A}>\mathcal{I}=4\frac{4}{7}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}4\frac{4}{7}&0&0\\0&4\frac{4}{7}&0\\0&0&4\frac{4}{7}\end{pmatrix}.$$
 You should find that $\bigtriangleup\mathcal{A}_{\psi}=\frac{\sqrt{19}}{7}\approx 0.62$.

$$\begin{split} \left(\bigtriangleup \mathcal{A}_{\psi} \right)^2 &= \langle \psi \mid \left(\mathcal{A} - \langle \mathcal{A} \rangle \mathcal{I} \right)^2 \mid \psi \rangle \\ &= \frac{1}{\sqrt{14}} \left(1, \ 2, \ 3 \right) \left[\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 4\frac{4}{7} & 0 & 0 \\ 0 & 4\frac{4}{7} & 0 \\ 0 & 0 & 4\frac{4}{7} \end{pmatrix} \right]^2 \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \frac{1}{14} \left(1, \ 2, \ 3 \right) \left[\begin{pmatrix} -1\frac{4}{7} & 0 & 0 \\ 0 & -\frac{4}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix} \right]^2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \frac{1}{14} \left(1, \ 2, \ 3 \right) \begin{pmatrix} -\frac{11}{7} & 0 & 0 \\ 0 & -\frac{4}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix} \begin{pmatrix} -\frac{11}{7} & 0 & 0 \\ 0 & -\frac{4}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix} \begin{pmatrix} -\frac{11}{7} & 0 & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \frac{1}{14} \left(1, \ 2, \ 3 \right) \begin{pmatrix} -\frac{11}{7} & 0 & 0 \\ 0 & -\frac{4}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix} \begin{pmatrix} -11/7 \\ -8/7 \\ 9/7 \end{pmatrix} = \frac{1}{14} \left(1, \ 2, \ 3 \right) \begin{pmatrix} 121/49 \\ 32/49 \\ 27/49 \end{pmatrix} \\ &= \frac{1}{14} \left(121/49 + 64/49 + 81/49 \right) = \frac{1}{14} \frac{266}{49} = \frac{2 \cdot 7 \cdot 19}{2 \cdot 7^3} = \frac{19}{49} \approx 0.39 \\ \Rightarrow & \bigtriangleup \mathcal{A}_{\psi} = \frac{\sqrt{19}}{7} \approx 0.62 \,. \end{split}$$

Postscript: The term "uncertainty" has the same meaning in quantum mechanics as the term "standard deviation" does in statistics. The quantity "uncertainty" calculated in this problem is the same quantity that is calculated for use in the **Heisenberg uncertainty principle**. We introduce the Heisenberg uncertainty relations in chapter 3.

The conventional way to write the definition of uncertainty is $\triangle \mathcal{A} = \langle (\mathcal{A} - \langle \mathcal{A} \rangle)^2 \rangle^{1/2}$, where the state vector and identity operator are implicit.

Variance is the square of standard deviation, or $(\Delta A_{\psi})^2 = \langle \psi | (\mathcal{A} - \langle \mathcal{A} \rangle \mathcal{I})^2 | \psi \rangle$. Variance is often a convenient intermediate result in a calculation of uncertainty.

12. Show that $\Delta \mathcal{A}_{\psi} = \langle \psi | \mathcal{A}^2 - \langle \mathcal{A} \rangle^2 \mathcal{I} | \psi \rangle^{1/2}$.

This alternative is often the most direct way to calculate an uncertainty. A useful theorem from an ordinary study of probability and statistics is that

$$<\mathcal{A} + \mathcal{B}> = <\mathcal{A}> + <\mathcal{B}>$$

We use this result, but refer you to Meyer¹ or your favorite book on probability and statistics for depth concerning this theorem.

This problem is essentially an expansion, summation of like terms, and a reduction. Work from the variance and take a square root as the last step to get the uncertainty. You need to use the fact that $\langle \mathcal{A} + \mathcal{B} \rangle = \langle \mathcal{A} \rangle + \langle \mathcal{B} \rangle$ twice. You also need the fact that $\langle \psi | \mathcal{A} \langle \mathcal{A} \rangle \mathcal{I} | \psi \rangle = \langle \psi | \langle \mathcal{A} \rangle^2 \mathcal{I} | \psi \rangle$. Since this is not obvious and also to provide a sample of what is expected, this is true because

$$\begin{aligned} <\psi \,|\,\mathcal{A}<\mathcal{A}>\mathcal{I}\,|\,\psi> \ &= <\psi \,|\,\mathcal{A}<\mathcal{A}> \,|\,\psi> = <\mathcal{A}><\psi \,|\,\mathcal{A}\,|\,\psi> = <\mathcal{A}><\mathcal{A}> \\ &= <\mathcal{A}>^2 = <\mathcal{A}>^2 <\psi \,|\,\psi> = <\psi \,|\,<\mathcal{A}>^2 \,|\,\psi> = <\psi \,|\,<\mathcal{A}>^2 \,|\,\psi> = <\psi \,|\,<\mathcal{A}>^2 \,\mathcal{I}\,|\,\psi> \;. \end{aligned}$$

$$\left(\bigtriangleup \mathcal{A}_{\psi} \right)^{2} = \langle \psi | \left(\mathcal{A} - \langle \mathcal{A} \rangle \mathcal{I} \right)^{2} | \psi \rangle$$

$$= \langle \psi | \left(\mathcal{A} - \langle \mathcal{A} \rangle \mathcal{I} \right) \left(\mathcal{A} - \langle \mathcal{A} \rangle \mathcal{I} \right) | \psi \rangle$$

$$= \langle \psi | \mathcal{A}^{2} - \mathcal{A} \langle \mathcal{A} \rangle \mathcal{I} - \langle \mathcal{A} \rangle \mathcal{I} \mathcal{A} + \langle \mathcal{A} \rangle^{2} \mathcal{I} | \psi \rangle \qquad (1)$$

$$= \langle \psi | \mathcal{A}^{2} - 2\mathcal{A} \langle \mathcal{A} \rangle \mathcal{I} + \langle \mathcal{A} \rangle^{2} \mathcal{I} | \psi \rangle \qquad (2)$$

$$= \langle \psi | \mathcal{A}^{2} | \psi \rangle - \langle \psi | 2\mathcal{A} \langle \mathcal{A} \rangle \mathcal{I} | \psi \rangle + \langle \psi | \langle \mathcal{A} \rangle^{2} \mathcal{I} | \psi \rangle \qquad (3)$$

$$= \langle \psi | \mathcal{A} | \psi \rangle - 2 \langle \psi | \langle \mathcal{A} \rangle \mathcal{I} | \psi \rangle + \langle \psi | \langle \mathcal{A} \rangle \mathcal{I} | \psi \rangle$$

$$= \langle \psi | \mathcal{A}^{2} | \psi \rangle - \langle \psi | \langle \mathcal{A} \rangle^{2} \mathcal{I} | \psi \rangle$$
(4)

$$= \langle \psi | \mathcal{A}^{2} - \langle \mathcal{A} \rangle^{2} \mathcal{I} | \psi \rangle$$
(5)

$$\Rightarrow \quad \triangle \mathcal{A}_{\psi} = \langle \psi \, | \, \mathcal{A}^2 - \langle \mathcal{A} \, \rangle^2 \, \mathcal{I} \, | \, \psi \rangle^{1/2} \, .$$

¹ Meyer Introductory Probability and Statistical Applications (Addison-Wesley Publishing Co., Reading, Massachusetts, 1970), pp.123–136.

Line (1) uses the fact that $\mathcal{I}^2 = \mathcal{I}$. Line (2) can be written because an operator \mathcal{A} commutes with a scalar $\langle \mathcal{A} \rangle$ and the identity operator. Line (3) uses $\langle \mathcal{A} + \mathcal{B} \rangle = \langle \mathcal{A} \rangle + \langle \mathcal{B} \rangle$. Equation (4) depends on the fact that $\langle \psi | \mathcal{A} \langle \mathcal{A} \rangle \mathcal{I} | \psi \rangle = \langle \psi | \langle \mathcal{A} \rangle^2 \mathcal{I} | \psi \rangle$. Line (5) uses $\langle \mathcal{A} \rangle + \langle \mathcal{B} \rangle = \langle \mathcal{A} + \mathcal{B} \rangle$ again.

Postscript: You will encounter the same statement written $\Delta \Omega = (\Omega^2 - \langle \Omega \rangle^2)^{1/2}$, where the state vector and the identity operator are implied.

13. Calculate the uncertainty of the operator \mathcal{A} from problem 4 using the result of problem 12.

The operator \mathcal{A} , the state vector $|\psi\rangle$, and the necessary result are all stated in the comments that preface the solution to problem 11. Calculate $\mathcal{A}^2 - \langle \mathcal{A} \rangle^2 \mathcal{I}$, form the braket with the state vector, then take the square root.

$$\begin{split} \mathcal{A}^{2} - \langle \mathcal{A} \rangle^{2} \mathcal{I} &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{2} - \begin{pmatrix} 32 \\ 7 \end{pmatrix}^{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \frac{1024}{49} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix} - \begin{pmatrix} 1024/49 & 0 & 0 \\ 0 & 1024/49 & 0 \\ 0 & 0 & 1024/49 \end{pmatrix} \\ &= \begin{pmatrix} 441/49 & 0 & 0 \\ 0 & 784/49 & 0 \\ 0 & 0 & 1225/49 \end{pmatrix} - \begin{pmatrix} 1024/49 & 0 & 0 \\ 0 & 1024/49 & 0 \\ 0 & 0 & 1024/49 \end{pmatrix} \\ &= \begin{pmatrix} -583/49 & 0 & 0 \\ 0 & -240/49 & 0 \\ 0 & 0 & 201/49 \end{pmatrix}, \end{split}$$

$$\Rightarrow \quad \triangle \mathcal{A}_{\psi} = \langle \psi | \mathcal{A}^{2} - \langle \mathcal{A} \rangle^{2} \mathcal{I} | \psi \rangle^{1/2} \\ &= \begin{bmatrix} \frac{1}{\sqrt{14}} (1, 2, 3) \begin{pmatrix} -583/49 & 0 & 0 \\ 0 & -240/49 & 0 \\ 0 & 0 & 201/49 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{bmatrix}^{1/2} \\ &= \begin{bmatrix} \frac{1}{14} (1, 2, 3) \begin{pmatrix} -583/49 \\ -480/49 \\ 603/49 \end{pmatrix} \end{bmatrix}^{1/2} = \begin{bmatrix} \frac{1}{14 \cdot 49} (-583 - 960 + 1809) \end{bmatrix}^{1/2} \\ &= \sqrt{\frac{266}{2 \cdot 7^{3}}} = \sqrt{\frac{2 \cdot 7 \cdot 19}{2 \cdot 7^{3}}} = \sqrt{\frac{19}{49}} = \frac{\sqrt{19}}{7}. \end{split}$$

^{14. (}a) Write the basis-independent Hamiltonian for a free particle.

- (b) Write the basis-independent Hamiltonian for a simple harmonic oscillator.
- (c) Write the Hamiltonian for a free particle in position space.
- (d) Write the Hamiltonian for a simple harmonic oscillator in position space.
- (e) Write the Hamiltonian for an unknown potential in position space.
- (f) Write the Hamiltonian for a free particle in momentum space.
- (g) Write the Hamiltonian for a simple harmonic oscillator in momentum space.

The Hamiltonian operator is intrinsic to the Schrödinger equation. This problem is an intermediate step to writing the Schrödinger equation for systems under the influence of various potentials.

Postscript comments to problem 1 concerning the Schrodinger equation indicate that the classical Hamiltonian is H = T + V. The non-relativistic kinetic energy term is $T = p^2/2m$. The free particle is not influenced by any potential so the potential energy term for a free particle is V(x) = 0. The potential energy function for a simple harmonic oscillator is $V(x) = kx^2/2$. The dynamic variables of classical mechanics become quantum mechanical operators, $x \to \mathcal{X}$ and $p \to \mathcal{P}$ as $H \to \mathcal{H}$. A quantum mechanical Hamiltonian \mathcal{H} is expressed in terms of the basis-independent operators \mathcal{X} and \mathcal{P} is basis-independent.

In the position basis in one spatial dimension, $\mathcal{X} \to x$ and $\mathcal{P} \to -i\hbar \frac{d}{dx}$. In the momentum basis, $\mathcal{P} \to p$, $\mathcal{X} \to i\hbar \frac{d}{dp}$. Substitute the appropriate differential operators into the basis-independent Hamiltonian operators of parts (a) and (b) to attain the basis-dependent Hamiltonian operators for parts (c) through (g).

(a)
$$\mathcal{H} = \frac{\mathcal{P}^2}{2m}$$
.
(b) $\mathcal{H} = \frac{\mathcal{P}^2}{2m} + \frac{1}{2}k\mathcal{X}^2$.
(c) $\mathcal{H} = \frac{\mathcal{P}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$.
(d) $\mathcal{H} = \frac{\mathcal{P}^2}{2m} + \frac{1}{2}k\mathcal{X}^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2$.
(e) $\mathcal{H} = \frac{\mathcal{P}^2}{2m} + \mathcal{V}(\mathcal{X}) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$.
(f) $\mathcal{H} = \frac{\mathcal{P}^2}{2m} = \frac{p^2}{2m}$.
(g) $\mathcal{H} = \frac{\mathcal{P}^2}{2m} + \frac{1}{2}k\mathcal{X}^2 = \frac{p^2}{2m} - \frac{\hbar^2}{2}k\frac{d^2}{dp^2}$.

Postscript: We will explain why $\mathcal{X} \to x$ and $\mathcal{P} \to -i\hbar \frac{d}{dx}$ in position space in future problems. Accept that these and the momentum based representations are correct and use them. These representations are much more useful than the details that are necessary to derive them.

Notice that each potential is, or is assumed to be in part (e), a function of position only. This leads to the dramatic simplification known as the **time-independent Schrodinger equation**.

15. Expand
$$|\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\ 2\\ 3 \end{pmatrix}$$
 in the eigenbases of
(a) $\mathcal{A} = \begin{pmatrix} 3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5 \end{pmatrix}$ and (b) $\mathcal{L}_y = \begin{pmatrix} 0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0 \end{pmatrix}$.

(c) Check your expansions by calculating the probabilities of measuring each eigenvalue using the expansion coefficients.

A vector may be expressed in any eigenbasis that spans the appropriate space. The eigenvectors of Hermitian operators can therefore be used to represent any vector of the same dimension.

The first step is to attain the eigenvectors by solving the eigenvalue/eigenvector equation, and that has been previously completed for both of the given operators. Next, consider

$$|\psi\rangle = \mathcal{I} |\psi\rangle = \left(\sum_{i=1}^{n} |\alpha_{i}\rangle < \alpha_{i}|\right) |\psi\rangle = \sum_{i=1}^{n} |\alpha_{i}\rangle (<\alpha_{i} |\psi\rangle) = \sum_{i=1}^{n} c_{i} |\alpha_{i}\rangle,$$

where the c_i are complex numbers that are the inner product of each $\langle \alpha_i | \psi \rangle$. The process described by this equation is known as **expansion in an eigenbasis** and the c_i are called **expansion coefficients**.

Per postulates 3 and 4, a measurement will obtain the eigenvalues with the probabilities

$$P(\alpha_i) = \left| < \alpha_i \, | \, \psi > \right|^2 = \left| \, c_i \, \right|^2,$$

given that the eigenvectors and the state vector are normalized. The $|\alpha_i\rangle$ that form the basis vectors must remain normalized for $|c_i|^2$ to be correct probabilities. The eigenvectors of \mathcal{A} are unit vectors that are inherently of unit length. The eigenvectors of \mathcal{L}_y , however, contain normalization constants of 1/2 and $1/\sqrt{2}$ that cannot be absorbed into the expansion coefficients in part (b).

(a) The expansion can be done by inspection when the eigenvectors are unit vectors, nevertheless,

$$\begin{split} |\psi\rangle &= \sum_{i=1}^{3} |\alpha_{i}\rangle \langle \alpha_{i} |\psi\rangle \\ &= \begin{pmatrix} 1\\0\\0 \end{pmatrix} (1, 0, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} + \begin{pmatrix} 0\\1\\0 \end{pmatrix} (0, 1, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} + \begin{pmatrix} 0\\0\\1 \end{pmatrix} (0, 0, 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} \\ &= \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\0\\0 \end{pmatrix} (1+0+0) + \frac{1}{\sqrt{14}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} (0+2+0) + \frac{1}{\sqrt{14}} \begin{pmatrix} 0\\0\\1 \end{pmatrix} (0+0+3) \\ &= \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\0\\0 \end{pmatrix} + \frac{2}{\sqrt{14}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} + \frac{3}{\sqrt{14}} \begin{pmatrix} 0\\0\\1 \end{pmatrix} , \end{split}$$

where the last expression is the expansion of the state vector in the eigenbasis of \mathcal{A} . (b) The expansion of the state vector in the eigenbasis of \mathcal{L}_y is

$$\begin{split} |\psi\rangle &= \sum_{i=1}^{3} |\alpha_{i}\rangle < \alpha_{i} |\psi\rangle \\ &= \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}i}\right)\right] \frac{1}{2} \left(1, \sqrt{2}i, -1\right) \frac{1}{\sqrt{14}} \begin{pmatrix}1\\2\\3\end{pmatrix} + \left[\frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\1\end{pmatrix}\right] \frac{1}{\sqrt{2}} \left(1, 0, 1\right) \frac{1}{\sqrt{14}} \begin{pmatrix}1\\2\\3\end{pmatrix} \\ &+ \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}i}\right)\right] \frac{1}{2} \left(1, -\sqrt{2}i, -1\right) \frac{1}{\sqrt{14}} \begin{pmatrix}1\\2\\3\end{pmatrix} \\ &= \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}i}\right)\right] \frac{1}{2\sqrt{14}} \left(1 + 2\sqrt{2}i - 3\right) + \left[\frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\1\end{pmatrix}\right] \frac{1}{\sqrt{28}} \left(1 + 0 + 3\right) \\ &+ \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}i}\right)\right] \frac{1}{2\sqrt{14}} \left(1 - 2\sqrt{2}i - 3\right) \\ &= \frac{-1 + \sqrt{2}i}{\sqrt{14}} \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}i}\right)\right] + \frac{4}{\sqrt{28}} \left[\frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\1\end{pmatrix}\right] + \frac{-1 - \sqrt{2}i}{\sqrt{14}} \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}i}\right)\right]. \end{split}$$

(c) The six probabilities are

$$P(3) = \left| \frac{1}{\sqrt{14}} \right|^2 = \frac{1}{14}, \quad P(4) = \left| \frac{2}{\sqrt{14}} \right|^2 = \frac{4}{14}, \quad P(5) = \left| \frac{3}{\sqrt{14}} \right|^2 = \frac{9}{14} \quad \text{for } \mathcal{A}, \text{ and}$$

$$P(-\sqrt{2}) = \left| \frac{-1 + \sqrt{2}i}{\sqrt{14}} \right|^2 = \left(\frac{-1 - \sqrt{2}i}{\sqrt{14}} \right) \left(\frac{-1 + \sqrt{2}i}{\sqrt{14}} \right) = \frac{1 + 2}{14} = \frac{3}{14},$$

$$P(0) = \left| \frac{4}{\sqrt{28}} \right|^2 = \frac{16}{28} = \frac{8}{14},$$

$$P(\sqrt{2}) = \left| \frac{-1 - \sqrt{2}i}{\sqrt{14}} \right|^2 = \left(\frac{-1 + \sqrt{2}i}{\sqrt{14}} \right) \left(\frac{-1 - \sqrt{2}i}{\sqrt{14}} \right) = \frac{1 + 2}{14} = \frac{3}{14}, \quad \text{for } \mathcal{L}_y.$$

Postscript: Combining all the constants in the last line of part (b) yields a simpler expression, but the simpler expression hides the expansion coefficients $c_i = \langle \alpha_i | \psi \rangle$, and the capability to calculate probabilities from the simpler expression is compromised. The expansion coefficients are so closely related to probabilities that they are also known as **probability amplitudes**.

Using expansion coefficients to calculate probabilities is dominantly the easiest method for some of the problems that we will encounter in future chapters. The other reason to introduce the technique of expansion in an eigenbasis is that it is essential to the **time evolution** of the **stationary states** that are the solutions to the time-independent Schrödinger equation. 16. Discuss how the time-independent Schrödinger equation, $\mathcal{H} | E_i \rangle = E_i | E_i \rangle$, follows from the time-dependent Schrödinger equation stated in postulate 6.

A quantum mechanical Hamiltonian is either time-dependent, $\mathcal{H} = \mathcal{H}(t)$, or is time-independent, $\mathcal{H} \neq \mathcal{H}(t)$. A time-dependent matrix operator would have elements that are functions of time. A Schrodinger equation in which the Hamiltonian is independent of time is known as a **time-independent Schrodinger equation**. This is a qualified misnomer because the state vector is a function of time so the Schrodinger equation is not completely time-independent. When $\mathcal{H} \neq \mathcal{H}(t)$, the time-dependence becomes simple and probabilities, expectation values, and uncertainties are time-independent. The quantum mechanical Hamiltonians for the free particle and the simple harmonic oscillator are time-independent, for instance.

Total energy is represented $\mathcal{E} \to i\hbar \frac{d}{dt}$. The differential operator form is seen in the timedependent Schrodinger equation.

The Hamiltonian is the total energy operator. Energy is an observable quantity so the Hamiltonian is necessarily a Hermitian operator per postulate 2. Any state vector can be expanded in terms of the eigenvectors of the Hermitian Hamiltonian. The eigenvectors of the Hamiltonian are the energy eigenvectors, represented $|E_i\rangle$, and the eigenvalues of the Hamiltonian are the energy eigenvalues, denoted E_i . Thus,

$$\mathcal{H} | \psi \rangle = i\hbar \frac{d}{dt} | \psi \rangle \longrightarrow \mathcal{H} | \psi \rangle = \mathcal{E} | \psi \rangle \longrightarrow \mathcal{H} | E_i \rangle = \mathcal{E} | E_i \rangle$$

and assuming that $\mathcal{H} \neq \mathcal{H}(t)$, the last equation is simply an eigenvalue/eigenvector equation so that \mathcal{E} can be nothing other than the energy eigenvalues, or $\mathcal{H} | E_i > = E_i | E_i > .$

Postscript: The time-dependent Schrodinger equation must be used when $\mathcal{H} = \mathcal{H}(t)$. There are some exceptions, but the time-dependent Schrodinger equation is usually difficult or impossible to solve analytically. The usual approach to a Schrodinger equation with a weakly time-dependent Hamiltonian is to find a time-independent solution and then model the time dependence as a perturbation. A numerical solution is often the only recourse if the Hamiltonian is strongly time dependent, .

We will derive the representations introduced in problem 14 in later problems. The representation $\mathcal{E} \to i\hbar \frac{d}{dt}$ is different. This representation is beyond our scope.

17. Show that $|E_i(t)\rangle = e^{-iE_it/\hbar} |E_i\rangle$ for a system described by a time-independent Hamiltonian and a time-dependent state vector.

This problem is an introduction to **time evolution**. It is concurrently an introduction to **station**ary states. Stationary states are the energy eigenstates $|E_i\rangle$ of a time-independent Hamiltonian and the solutions to the time-independent Schrödinger equation. Notice that the stationary states evolve in time proportionally to the exponential of the energy eigenvalue. Use the given conditions of time-dependence of the state vector and time-independence of the Hamiltonian in the time-dependent Schrödinger equation to reason that

$$i\hbar \frac{d}{dt} | E_i(t) \rangle = E_i | E_i(t) \rangle$$
(1)

for individual eigenstates of \mathcal{H} . There are many fundamentals used to arrive at this equation. A time-dependent state vector can be denoted $|\psi(t)\rangle$. Is it a superposition of eigenstates? Why the energy eigenstates? Why are the energy eigenstates functions of time? Why can the equation be written for individual eigenstates? Why are the eigenvalues constants, E_i and not $E_i(t)$? The answers to these questions lie in properties of the eigenvalue/eigenvector equation, the technique of expanding a state vector, and postulate 1. Equation (1) is a variables separable differential equation, so separate the variables and integrate both sides from 0 to t. The last step is to substitute the conventional notation $|E_i\rangle$ for $|E_i(0)\rangle$.

The time-dependent Schrodinger equation is $i\hbar \frac{d}{dt} | \psi(t) \rangle = \mathcal{H} | \psi(t) \rangle$, where the timedependence of the state vector is indicated explicitly. The state vector can be expanded into a linear combination of the eigenvectors (postulate 1) of the total energy operator \mathcal{H} . Denote the energy eigenstates $|E_i(t)\rangle$ because the eigenbasis is that of the total energy operator \mathcal{H} . The eigenstates must be functions of time if the state vector is a function of time (postulate 1). A state vector can be an individual eigenstate, therefore the time-dependent Schrodinger equation applies to each eigenstate individually (postulate 1). If the Hamiltonian is time-independent, the eigenvalues are time-independent (eigenvalues are determined solely by the operator). Therefore,

$$i\hbar \frac{d}{dt} | E_i(t) \rangle = E_i | E_i(t) \rangle, \qquad (1)$$

where the eigenvalues are constants and not functions of time, again, because the Hamiltonian is independent of time. This is a variables separable differential equation that can be arranged

$$\frac{d | E_i(t) \rangle}{| E_i(t) \rangle} = \frac{E_i}{i\hbar} dt \quad \Rightarrow \quad \int_0^t \frac{d | E_i(t') \rangle}{| E_i(t') \rangle} = \frac{E_i}{i\hbar} \int_0^t dt'$$

where the independent variable is primed to differentiate it from the upper limit of integration. Multiplying numerator and denominator of the right side by i, the last equation implies

$$\begin{aligned} &\ln\left(\left|E_{i}\left(t'\right)\right\rangle\right)\Big|_{0}^{t} = \frac{-iE_{i}}{\hbar}t'\Big|_{0}^{t} \\ \Rightarrow &\ln\left|E_{i}\left(t\right)\right\rangle - \ln\left|E_{i}\left(0\right)\right\rangle = -iE_{i}t/\hbar \\ \Rightarrow &\ln\left(\frac{\left|E_{i}\left(t\right)\right\rangle}{\left|E_{i}\left(0\right)\right\rangle}\right) = -iE_{i}t/\hbar \\ \Rightarrow &\frac{\left|E_{i}\left(t\right)\right\rangle}{\left|E_{i}\left(0\right)\right\rangle} = e^{-iE_{i}t/\hbar} \\ \Rightarrow &\left|E_{i}\left(t\right)\right\rangle = e^{-iE_{i}t/\hbar} \left|E_{i}\left(0\right)\right\rangle = e^{-iE_{i}t/\hbar} \left|E_{i}\right\rangle \end{aligned}$$

after substituting $|E_i\rangle$ for $|E_i(0)\rangle$ as is conventional.

Postscript: The time evolution of an energy eigenstate is described by the product of $e^{-iE_it/\hbar}$ and that energy eigenstate. The state vector is a superposition of all the eigenstates, so

$$|\psi\rangle = \sum_{i} c_{i} |E_{i}\rangle \Rightarrow |\psi(t)\rangle = \sum_{i} c_{i} |E_{i}\rangle e^{-iE_{i}t/\hbar}$$

where $\mathcal{H} \neq \mathcal{H}(t)$. The energy eigenstates $|E_i\rangle$ are known as stationary states. The probability of a measurement is unaffected when $\mathcal{H} \neq \mathcal{H}(t)$. The probabilities are "stationary" as time advances. Expectation values and uncertainties are "stationary" since probabilities are unaffected. Stationary states are the result of time being separable from other observable quantities. The prerequisite for time being separable from other observables is a time-independent Hamiltonian.

This problem explicitly uses time as an independent variable within ket vectors. Time is the only quantity that can be used this way. Time is not an observable quantity in the same sense as position and momentum. Time is intrinsic to all spaces. The notation $|\psi(t)\rangle$ says only that time moves forward (or backward) in every space. $|\psi(t)\rangle$ may be represented in any space by forming the inner product with an appropriate bra, for instance, $\langle x | \psi(t) \rangle = \psi(x,t)$ in position space and $\langle p | \psi(t) \rangle = \hat{\psi}(p,t)$ in momentum space.

18. (a) Find the time-dependent state vector for

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix}$$
 where $\mathcal{H} = \begin{pmatrix} 3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5 \end{pmatrix}$.

(b) Calculate the probability of each possible result of a measurement of energy as the state vector evolves in time.

Part (a) requires you to apply the result of problem 17. Remember that $|\psi(t)\rangle$ is a superposition of all time-evolving eigenstates, in this case

$$|\psi(t)\rangle = \sum_{i=1}^{3} c_i |E_i\rangle e^{-iE_it/\hbar}$$

Having expanded this state vector in the eigenbasis of this operator previously, there is little to do for part (a) except to write the answer. Part (b) is a numerical example illustrating the fact that the time evolution of stationary states does not affect calculations of probabilities.

(a) The energy eigenvalues corresponding to the energy eigenvectors are $E_i = 3, 4, \text{ and } 5$, so

$$|\psi(t)\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\0\\0 \end{pmatrix} e^{-i3t/\hbar} + \frac{2}{\sqrt{14}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} e^{-i4t/\hbar} + \frac{3}{\sqrt{14}} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i5t/\hbar} = \frac{1}{\sqrt{14}} \begin{pmatrix} e^{-i3t/\hbar}\\2e^{-i4t/\hbar}\\3e^{-i5t/\hbar} \end{pmatrix},$$

which is the time-dependent state vector exhibiting time evolution.

(b) The probabilities are

$$\begin{split} P\left(E=3\right) &= \left| \left(1,\ 0,\ 0\right) \frac{1}{\sqrt{14}} \left(\begin{array}{c} e^{-i3t/\hbar} \\ 2\ e^{-i4t/\hbar} \\ 3\ e^{-i5t/\hbar} \end{array} \right) \right|^2 = \frac{1}{14} \left| \ e^{-i3t/\hbar} \right|^2 \\ &= \frac{1}{14} \left(e^{-i3t/\hbar} \right) \left(e^{+i3t/\hbar} \right) = \frac{1}{14} e^0 = \frac{1}{14} \,. \end{split}$$

$$\begin{split} P\left(E=4\right) &= \left| \left(0,\ 1,\ 0\right) \frac{1}{\sqrt{14}} \left(\begin{array}{c} e^{-i3t/\hbar} \\ 2\ e^{-i4t/\hbar} \\ 3\ e^{-i5t/\hbar} \end{array} \right) \right|^2 = \frac{1}{14} \left| \ 2\ e^{-i4t/\hbar} \right|^2 \\ &= \frac{1}{14} \left(2\ e^{-i4t/\hbar} \right) \left(2\ e^{+i4t/\hbar} \right) = \frac{4}{14} e^0 = \frac{4}{14} \,. \end{split}$$

$$\begin{split} P\left(E=5\right) &= \left| \left(0,\ 0,\ 1\right) \frac{1}{\sqrt{14}} \left(\begin{array}{c} e^{-i3t/\hbar} \\ 2\ e^{-i4t/\hbar} \\ 3\ e^{-i5t/\hbar} \end{array} \right) \right|^2 = \frac{1}{14} \left| \ 3\ e^{-i5t/\hbar} \right|^2 \\ &= \frac{1}{14} \left(3\ e^{-i5t/\hbar} \right) \left(3\ e^{+i5t/\hbar} \right) = \frac{9}{14} e^0 = \frac{9}{14} \,. \end{split}$$

The probabilities are independent of time and they are the same probabilities that were attained when time was not considered.

19. Consider a system described by the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \text{ which at } t = 0 \text{ is in the state } |\psi(0)\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

- (a) If the energy is measured, what results can be obtained, and with what probabilities will these results be obtained?
- (b) Calculate the expectation value $\langle \mathcal{H} \rangle = \langle \psi(0) | \mathcal{H} | \psi(0) \rangle$. Then show that your expectation value agrees with your calculations from part (a) using $\langle \mathcal{H} \rangle = \sum_{i} P(E_i) E_i$.
- (c) Expand the initial state vector $|\psi(0)\rangle$ in the energy eigenbasis to calculate the time dependent state vector $|\psi(t)\rangle$.
- (d) If the energy is measured at time t, what results can be obtained, and with what probabilities will these results be obtained? Compare your answers with the t = 0 case of part (a). Explain why these probabilities are independent of time even though the state vector is time dependent.
- (e) Suppose that you measure the energy of the system at t = 0 and you find E = 7. What is the state vector of the system immediately after your measurement? Now let the system evolve without any additional measurements until t = 10. What is the state vector $|\psi(10)\rangle$ at t = 10? What energies will you measure if you repeat the energy measurement at t = 10?

This problem is intended to provide insight into the meaning and applications of the postulates of quantum mechanics. The first questions for a measurement of any system are what are the possibilities and what are their respective probabilities? Postulate 3 addresses the possibilities and postulate 4 determines the respective probabilities for part (a). Notice that the given state vector is normalized. Using your probabilities and eigenvalues from part (a), you must find that $\langle \psi(0) | \mathcal{H} | \psi(0) \rangle = \sum_{i} P(E_i) E_i$ for part (b). Use the procedures of problem 15 to expand the state vector in the energy eigenbasis. Use the procedures of problem 17 illustrated in problem 18 for part (c). Postulate 3 addresses the possibilities and postulate 4 determines the respective probabilities for part (d), without regard to time dependence. You have made an error if your answers do not agree with part (a). Remember to complex conjugate the components when you form bras! When you find E = 7 for part (e), the state vector changes in accordance with postulate 5, so $|\psi(0)\rangle \rightarrow |E = 7\rangle$. There is one possible result with a probability of 1, and the other two possible results have probability zero.

(a) The only possible results of the measurement of energy are the energy eigenvalues which are 3, 5, and 7. The probabilities are

$$\begin{split} P\left(E_{i}=3\right) &= \left| \left(1, \ 0, \ 0\right) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\3\\2 \end{pmatrix} \right|^{2} = \left| \frac{1}{\sqrt{14}} \left(1\right) \right|^{2} = \frac{1}{14} \,, \\ P\left(E_{i}=5\right) &= \left| \left(0, \ 1, \ 0\right) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\3\\2 \end{pmatrix} \right|^{2} = \left| \frac{1}{\sqrt{14}} \left(3\right) \right|^{2} = \frac{9}{14} \,, \\ P\left(E_{i}=7\right) &= \left| \left(0, \ 0, \ 1\right) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\3\\2 \end{pmatrix} \right|^{2} &= \left| \frac{1}{\sqrt{14}} \left(2\right) \right|^{2} = \frac{4}{14} \,. \end{split}$$

Notice that the sum of the probabilities is 1.

(b) The t = 0 expectation value of the energy is

$$\langle \mathcal{H} \rangle = \langle \psi(0) | \mathcal{H} | \psi(0) \rangle = \frac{1}{\sqrt{14}} (1, 3, 2) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$
$$= \frac{1}{14} (1, 3, 2) \begin{pmatrix} 3 \\ 15 \\ 14 \end{pmatrix} = \frac{1}{14} (3 + 45 + 28) = \frac{76}{14} = \frac{38}{7} \approx 5.43, \text{ and}$$
$$\sum_{i} P(E_i) E_i = \frac{1}{14} 3 + \frac{9}{14} 5 + \frac{4}{14} 7 = \frac{3 + 45 + 28}{14} = \frac{76}{14} = \frac{38}{7} \approx 5.43.$$

(c) Expanding the state vector in the energy eigenbasis,

$$\begin{aligned} |\psi(0)\rangle &= \sum_{i} |E_{i}\rangle \langle E_{i} |\psi(0)\rangle \\ &= \begin{pmatrix} 1\\0\\0 \end{pmatrix} (1, 0, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\3\\2 \end{pmatrix} + \begin{pmatrix} 0\\1\\0 \end{pmatrix} (0, 1, 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\3\\2 \end{pmatrix} + \begin{pmatrix} 0\\0\\1 \end{pmatrix} (0, 0, 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\3\\2 \end{pmatrix} \\ &= \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\0\\0 \end{pmatrix} + \frac{3}{\sqrt{14}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} + \frac{2}{\sqrt{14}} \begin{pmatrix} 0\\0\\1 \end{pmatrix}. \end{aligned}$$

With the time zero expansion, we can easily write the complete time-dependent state vector

$$\begin{aligned} |\psi(t)\rangle &= \sum_{i} |E_{i}\rangle \langle E_{i} |\psi(0)\rangle e^{-iE_{i}t/\hbar} \\ &= \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\0\\0 \end{pmatrix} e^{-i3t/\hbar} + \frac{3}{\sqrt{14}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} e^{-i5t/\hbar} + \frac{2}{\sqrt{14}} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i7t/\hbar} = \frac{1}{\sqrt{14}} \begin{pmatrix} e^{-i3t/\hbar}\\3e^{-i5t/\hbar}\\2e^{-i7t/\hbar} \end{pmatrix} \end{aligned}$$

(d) At any time, the only possible results of a measurement of energy are the eigenenergies of the system, which are 3, 5, and 7. The "time-dependent" probabilities are

$$P(3) = \left| (1, 0, 0) e^{+i3t/\hbar} \frac{1}{\sqrt{14}} \begin{pmatrix} e^{-i3t/\hbar} \\ 3 e^{-i5t/\hbar} \\ 2 e^{-i7t/\hbar} \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (1+0+0) e^0 \right|^2 = \frac{1}{14},$$

$$P(5) = \left| (0, 1, 0) e^{+i5t/\hbar} \frac{1}{\sqrt{14}} \begin{pmatrix} e^{-i3t/\hbar} \\ 3 e^{-i5t/\hbar} \\ 2 e^{-i7t/\hbar} \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (0+3+0) e^0 \right|^2 = \frac{9}{14},$$

$$P(7) = \left| (0, 0, 1) e^{+i7t/\hbar} \frac{1}{\sqrt{14}} \begin{pmatrix} e^{-i3t/\hbar} \\ 3 e^{-i5t/\hbar} \\ 2 e^{-i7t/\hbar} \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{14}} (0+0+2) e^0 \right|^2 = \frac{4}{14}.$$

These are exactly the same probabilities obtained in part (a). There is no time dependence in the probabilities because the eigenvectors of the Hamiltonian have only a time-dependent phase that "cancels" in the sense that $e^0 = 1$ in this calculation. Time dependency will "cancel" in one way or another in all cases that the Hamiltonian is independent of time, *i.e.*, $\mathcal{H} \neq \mathcal{H}(t)$. The probabilities are "stationary" in time, which is the meaning of the term "stationary states."

(e) An energy measurement with the result E = 7 forces the system into the energy eigenstate

$$|\psi\left(t>0\right)\rangle = \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i7t/\hbar} \quad \Rightarrow \quad |\psi\left(t=10\right)\rangle = \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i7(10)/\hbar},$$

so the probability of measuring E = 7 at t = 10 is

$$\left| \begin{pmatrix} 0, \ 0, \ 1 \end{pmatrix} e^{i70/\hbar} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i70/\hbar} \right|^2 = (0+0+1) e^{(i70-i70)/\hbar} = 1 \cdot e^0 = 1,$$

and the probability of measuring any other eigenenergy is zero since

$$\left| (1, 0, 0) e^{i30/\hbar} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i70/\hbar} \right|^2 = \left| (0, 1, 0) e^{i50/\hbar} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i70/\hbar} \right|^2 = 0$$

20. Consider a system described by the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \text{ which at } t = 0 \text{ is in the state } |\psi(0)\rangle = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- (a) Is \mathcal{H} Hermitian?
- (b) Solve the eigenvalue/eigenvector problem to attain the eigenvalues and eigenvectors of \mathcal{H} .
- (c) If the energy is measured, what results can be obtained, and with what probabilities will these results be obtained?
- (d) Calculate the expectation value of the Hamiltonian using both

$$\langle \mathcal{H} \rangle = \langle \psi(0) | \mathcal{H} | \psi(0) \rangle$$
 and $\langle \mathcal{H} \rangle = \sum_{i} P(E_i) E_i.$

- (e) Calculate the time dependent state vector $|\psi(t)\rangle$.
- (f) If the energy is measured at time t, what results can be obtained, and with what probabilities will these results be obtained? Compare your answer with the t = 0 case in part (c).
- (g) Diagonalize \mathcal{H} using a unitary transformation. Transform $|\psi(0)\rangle$ and both eigenvectors to be consistent with this unitary transformation.
- (h) Calculate $|\psi(t)\rangle$ and both t > 0 probabilities in the basis in which \mathcal{H} is diagonal.
- (i) Suppose that you measure the energy of the system at t = 0 and you find E = -2. Find the state vector of the system immediately after your measurement and at time t = 10 in both the basis in which \mathcal{H} is diagonal and in the basis of part (b). What energies will you measure if you repeat the energy measurement at t = 10?

This problem emphasizes the postulates and their applications using a Hamiltonian that is not diagonal. It is in two dimensions to minimize the calculations though you will likely find the calculations to be substantial. Parts (c) through (f) may be more interesting than similar calculations using diagonal matrices because the off-diagonal elements contribute cross terms. Review chapter 1 techniques to diagonalize the Hamiltonian for part (g) if required. Understanding diagonalization is the initial step to understanding simultaneous diagonalization, and simultaneous diagonalization is the foundation underlying the essential concept of a complete set of commuting observables.

Expand the state vector in the energy eigenbasis using $|\psi(0)\rangle = \sum_{j} |E_{j}\rangle \langle E_{j}|\psi(0)\rangle$

where $|E_j\rangle$ are the normalized eigenvectors of the Hamiltonian matrix for part (e). The only time dependence is that of the energy eigenvectors $|E_j(t)\rangle = \exp(-iE_jt/\hbar)|E_j(0)\rangle$, per previous problems. The probabilities for part (f) are identical to those from part (c). Transform \mathcal{H} using \mathcal{UHU}^{\dagger} for part (g). Form \mathcal{U} from the eigenvectors of \mathcal{H} as done in part 2 of chapter 1. Transform the state vector $\mathcal{U}^{\dagger} | \psi(0) \rangle$ to establish it in the same basis as \mathcal{UHU}^{\dagger} . Of course, probabilities for part (h) are identical to those found in parts (c) and (f). You should find that calculations done in the basis in which the Hamiltonian is diagonal to be shorter and easier than the non-diagonal basis.

(a)
$$\mathcal{H}^{\dagger} = \left[\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \right]^{\mathrm{T}*} = \left[\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \right]^* = \begin{pmatrix} 1^* & 3^* \\ 3^* & 1^* \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \mathcal{H},$$

therefore, \mathcal{H} is Hermitian. Postulate 2 says that this is important to quantum mechanics.

(b) To attain the eigenvalues, det $\begin{pmatrix} 1-\alpha & 3\\ 3 & 1-\alpha \end{pmatrix} = ((1-\alpha)^2 - 9) = 0$

$$\Rightarrow 1 - 2\alpha + \alpha^2 - 9 = 0 \Rightarrow \alpha^2 - 2\alpha - 8 = 0 \Rightarrow (\alpha - 4)(\alpha + 2) = 0$$

 $\Rightarrow \quad \alpha = -2, \ 4, \ \text{ are the eigenvalues of the Hamiltonian operator. For } \alpha = -2 \,,$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{aligned} a & + & 3b & = & -2a \\ 3a & + & b & = & -2b \end{aligned} \Rightarrow \begin{aligned} b &= -a \\ a &= -b \end{aligned}$$
$$\Rightarrow \quad |-2\rangle = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \quad |-2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is the normalized eigenvector. The eigenvector corresponding to the eigenvalue 4 is

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{array}{c} a & + & 3b & = & 4a \\ 3a & + & b & = & 4b \end{array} \implies \begin{array}{c} b = a \\ a = b \\ a = b \\ \end{array}$$
$$\Rightarrow \quad |4\rangle = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \quad |4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{is the normalized eigenvector.}$$

(c) We can now address the t = 0 probabilities which are

$$P(-2) = \left| \frac{1}{\sqrt{2}} (1, -1) \frac{1}{\sqrt{13}} {\binom{2}{3}} \right|^2 = \left| \frac{1}{\sqrt{26}} (2-3) \right|^2 = \left| \frac{-1}{\sqrt{26}} \right|^2 = \frac{1}{26},$$
$$P(4) = \left| \frac{1}{\sqrt{2}} (1, 1) \frac{1}{\sqrt{13}} {\binom{2}{3}} \right|^2 = \left| \frac{1}{\sqrt{26}} (2+3) \right|^2 = \left| \frac{5}{\sqrt{26}} \right|^2 = \frac{25}{26}.$$

The probabilities sum to 1, as they must.

(d) The expectation value is

$$\langle \mathcal{H} \rangle = \langle \psi(0) | \mathcal{H} | \psi(0) \rangle = \frac{1}{\sqrt{13}} \begin{pmatrix} 2, & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 2, & 3 \end{pmatrix} \begin{pmatrix} 2+9 \\ 6+3 \end{pmatrix}$$
$$= \frac{1}{13} \begin{pmatrix} 2, & 3 \end{pmatrix} \begin{pmatrix} 11 \\ 9 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 22+27 \end{pmatrix} = \frac{49}{13} \approx 3.77 \,.$$
$$\langle \mathcal{H} \rangle = \sum_{i} P(E_{i}) E_{i} = \frac{1}{26} (-2) + \frac{25}{26} 4 = -\frac{2}{26} + \frac{100}{26} = \frac{98}{26} = \frac{49}{13} \approx 3.77 \,.$$

The value of using a second procedure is that it is a check for all previous calculations.

(e) The expansion of the t = 0 state vector in the energy basis is

$$\begin{split} |\psi(0)\rangle &= \sum_{i} |E_{i}\rangle \langle E_{i} |\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-1 \end{pmatrix} \frac{1}{\sqrt{2}} (1, -1) \frac{1}{\sqrt{13}} \begin{pmatrix} 2\\3 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\1 \end{pmatrix} \frac{1}{\sqrt{2}} (1, 1) \frac{1}{\sqrt{13}} \begin{pmatrix} 2\\3 \end{pmatrix} \\ &= \frac{1}{2\sqrt{13}} \begin{pmatrix} 1\\-1 \end{pmatrix} (2-3) + \frac{1}{2\sqrt{13}} \begin{pmatrix} 1\\1 \end{pmatrix} (2+3) = -\frac{1}{2\sqrt{13}} \begin{pmatrix} 1\\-1 \end{pmatrix} + \frac{5}{2\sqrt{13}} \begin{pmatrix} 1\\1 \end{pmatrix} , \\ \Rightarrow |\psi(t)\rangle &= -\frac{1}{2\sqrt{13}} \begin{pmatrix} 1\\-1 \end{pmatrix} e^{i2t/\hbar} + \frac{5}{2\sqrt{13}} \begin{pmatrix} 1\\1 \end{pmatrix} e^{-i4t/\hbar} = \frac{1}{2\sqrt{13}} \begin{pmatrix} -e^{i2t/\hbar} + 5e^{-i4t/\hbar} \\ e^{i2t/\hbar} + 5e^{-i4t/\hbar} \end{pmatrix}. \end{split}$$

(f) The complex components are conjugated to form time-dependent bras so the probabilities are

$$\begin{split} P\left(-2\right) &= \left|\frac{1}{\sqrt{2}}\left(1,\ -1\right)e^{-i2t/\hbar}\frac{1}{2\sqrt{13}}\left(\frac{-e^{i2t/\hbar}+5\,e^{-i4t/\hbar}}{e^{i2t/\hbar}+5\,e^{-i4t/\hbar}}\right)\right|^2 \\ &= \left|\frac{1}{2\sqrt{26}}\left(-e^{(-i2t+i2t)/\hbar}+5\,e^{(-i2t-i4t)/\hbar}-e^{(-i2t+i2t)/\hbar}-5\,e^{(-i2t-i4t)/\hbar}\right)\right|^2 \\ &= \left|\frac{1}{2\sqrt{26}}\left(-1+5\not\!e^{-i6t/\hbar}\!-1-5\not\!e^{-i6t/\hbar}\right)\right|^2 = \left|-\frac{1}{\sqrt{26}}\right|^2 = \frac{1}{26}\,. \end{split}$$

$$\begin{split} P\left(4\right) &= \left|\frac{1}{\sqrt{2}}\left(1,\ 1\right)e^{i4t/\hbar}\frac{1}{2\sqrt{13}}\left(\frac{-e^{i2t/\hbar}+5\,e^{-i4t/\hbar}}{e^{i2t/\hbar}+5\,e^{-i4t/\hbar}}\right)\right|^2 \\ &= \left|\frac{1}{2\sqrt{26}}\left(-e^{(i4t+i2t)/\hbar}+5\,e^{(i4t-i4t)/\hbar}+e^{(i4t+i2t)/\hbar}+5\,e^{(i4t-i4t)/\hbar}\right)\right|^2 \\ &= \left|\frac{1}{2\sqrt{26}}\left(-e^{(i6t/\hbar}+5+e^{i6t/\hbar}+5)\right|^2 = \left|\frac{5}{\sqrt{26}}\right|^2 = \frac{25}{26}\,. \end{split}$$

(g) Placing the eigenvector corresponding to the smaller eigenvector on the left and the eigenvector corresponding to larger eigenvalue on the right yields the unitary transformation matrix,

$$\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1\\ -1 & 1 \end{pmatrix} \Rightarrow \mathcal{U}^{\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1\\ 1 & 1 \end{pmatrix}$$
$$\Rightarrow \mathcal{U}^{\dagger} \mathcal{H} \mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1\\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3\\ 3 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1\\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1\\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 4\\ 2 & 4 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} -2 - 2 & 4 - 4\\ -2 + 2 & 4 + 4 \end{pmatrix} = \begin{pmatrix} -2 & 0\\ 0 & 4 \end{pmatrix}.$$

A consistent unitary transformation for the this is found using

$$\langle i | \mathcal{H} | i \rangle = \langle i | \mathcal{I} | \mathcal{H} | \mathcal{I} | i \rangle = \langle i | \mathcal{U} \left(\mathcal{U}^{\dagger} | \mathcal{H} | \mathcal{U} \right) \left(\mathcal{U}^{\dagger} | i \rangle \right),$$

so since we transform operators as $\mathcal{U}^{\dagger}\mathcal{H}\mathcal{U}$, the kets transform as $\mathcal{U}^{\dagger}|i>$. The state vector is

$$\mathcal{U}^{\dagger} | \psi(0) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 2-3 \\ 2+3 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

The eigenvectors are easily found by inspection but notice that they also transform correctly,

$$\mathcal{U}^{\dagger} | -2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+1 \\ 1-1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{U}^{\dagger} | 4 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-1 \\ 1+1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(h) The t = 0 expansion is

$$\begin{split} |\psi\left(0\right)\rangle &= \begin{pmatrix} 1\\0 \end{pmatrix} \begin{pmatrix} 1, \ 0 \end{pmatrix} \frac{1}{\sqrt{26}} \begin{pmatrix} -1\\5 \end{pmatrix} + \begin{pmatrix} 0\\1 \end{pmatrix} \begin{pmatrix} 0, \ 1 \end{pmatrix} \frac{1}{\sqrt{26}} \begin{pmatrix} -1\\5 \end{pmatrix} \\ &= \frac{1}{\sqrt{26}} \begin{pmatrix} 1\\0 \end{pmatrix} \begin{pmatrix} -1+0 \end{pmatrix} + \frac{1}{\sqrt{26}} \begin{pmatrix} 0\\1 \end{pmatrix} \begin{pmatrix} 0+5 \end{pmatrix} = -\frac{1}{\sqrt{26}} \begin{pmatrix} 1\\0 \end{pmatrix} + \frac{5}{\sqrt{26}} \begin{pmatrix} 0\\1 \end{pmatrix} , \\ \Rightarrow & |\psi\left(t\right)\rangle &= -\frac{1}{\sqrt{26}} \begin{pmatrix} 1\\0 \end{pmatrix} e^{i2t/\hbar} + \frac{5}{\sqrt{26}} \begin{pmatrix} 0\\1 \end{pmatrix} e^{-i4t/\hbar} = \frac{1}{\sqrt{26}} \begin{pmatrix} -e^{i2t/\hbar}\\5 e^{-i4t/\hbar} \end{pmatrix} . \end{split}$$

Probabilities are

$$P(-2) = \left| (1, 0) \frac{1}{\sqrt{26}} \left(\begin{array}{c} -e^{i2t/\hbar} \\ 5 e^{-i4t/\hbar} \end{array} \right) \right|^2 = \left| \frac{1}{\sqrt{26}} \left(e^{-i2t/\hbar} \right) \right|^2 = \frac{1}{26} \left(e^{+i2t/\hbar} \right) \left(e^{-i2t/\hbar} \right) = \frac{1}{26} ,$$

$$P(4) = \left| (0, 1) \frac{1}{\sqrt{26}} \left(\begin{array}{c} -e^{i2t/\hbar} \\ 5 e^{-i4t/\hbar} \end{array} \right) \right|^2 = \left| \frac{1}{\sqrt{26}} \left(5 e^{-i4t/\hbar} \right) \right|^2 = \frac{1}{26} \left(5 e^{+i4t/\hbar} \right) \left(5 e^{-i4t/\hbar} \right) = \frac{25}{26} .$$

(i) According to postulate 5, if you measure an energy of E = -2, the t = 0 state vector becomes

$$|\psi'(0)\rangle = \begin{pmatrix} 1\\ 0 \end{pmatrix} \Rightarrow |\psi'(t)\rangle = \begin{pmatrix} 1\\ 0 \end{pmatrix} e^{i2t/\hbar},$$

in the basis in which \mathcal{H} is diagonal. The state vector is

$$|\psi'(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-1 \end{pmatrix} \quad \Rightarrow \quad |\psi'(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-1 \end{pmatrix} e^{i2t/\hbar}.$$

in the original non-diagonal basis. Using operators, bras, and kets consistent with the appropriate basis, identical results necessarily follow. For instance,

$$\begin{split} P\left(-2\right) &= \left| \begin{pmatrix} 1, \ 0 \end{pmatrix} e^{-i2t/\hbar} \begin{pmatrix} 1\\0 \end{pmatrix} e^{i2t/\hbar} \right|^2 = \left| \begin{pmatrix} 1+0 \end{pmatrix} e^0 \right|^2 = 1, \\ P\left(4\right) &= \left| \begin{pmatrix} 0, \ 1 \end{pmatrix} e^{i4t/\hbar} \begin{pmatrix} 1\\0 \end{pmatrix} e^{i2t/\hbar} \right|^2 = \left| \begin{pmatrix} 0+0 \end{pmatrix} e^{i6t/\hbar} \right|^2 = 0, \\ P\left(-2\right) &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1, \ -1 \end{pmatrix} e^{-i2t/\hbar} \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-1 \end{pmatrix} e^{i2t/\hbar} \right|^2 = \left| \frac{1}{2} \begin{pmatrix} 1+1 \end{pmatrix} e^0 \right|^2 = 1, \\ P\left(4\right) &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1, \ 1 \end{pmatrix} e^{i4t/\hbar} \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-1 \end{pmatrix} e^{i2t/\hbar} \right|^2 = \left| \frac{1}{2} \begin{pmatrix} 1-1 \end{pmatrix} e^{i6t/\hbar} \right|^2 = 0. \end{split}$$

21. A system is described by a Hamiltonian \mathcal{H} and by a second observable Λ where

$$\mathcal{H} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ for which } |\psi(0)\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Show that \mathcal{H} and Λ commute, *i.e.*, show that $[\mathcal{H}, \Lambda] = 0$.

- (b) If the energy is measured, what results can be obtained and with what probabilities will these results be obtained? If Λ is measured, what results can be obtained and with what probabilities will these results be obtained?
- (c) Calculate the expectation values of the Hamiltonian $\langle \mathcal{H} \rangle$ and the Lambda operator $\langle \Lambda \rangle$ using the initial state vector. Then show that your expectation values agree with your part (b) probabilities and eigenvalues by using the general expression $\langle \Omega \rangle = \sum_{i} P(\omega_i) \omega_i$.
- (d) Calculate the time dependent state vector $|\psi(t)\rangle$ in the energy eigenbasis.
- (e) Transform to the basis that simultaneously diagonalizes \mathcal{H} and Λ . Calculate the new form of the initial state vector $|\psi(0)\rangle$ in this diagonal basis.
- (f) Repeat parts (b) and (c) in the diagonal basis. Which basis do you prefer? Why?
- (g) Calculate the time evolution of the state vector in the diagonal basis. Calculate the possibilities and probabilities of measuring the energy, E_i , and the "lambda-ness," λ_i , at time t.
- (h) Describe the state vector immediately after each measurement and the result of each measurement if you do a gedanken experiment by alternating \mathcal{H} and Λ measurements starting with an \mathcal{H} measurement, *i.e.*, you measure \mathcal{H} , Λ , \mathcal{H} , Λ , \mathcal{H} ,..., for the three possible cases:
 - 1) You find 2 when you first measure \mathcal{H} .
 - 2) You find 3 when you first measure \mathcal{H} .
 - 3) You find -1 when you first measure \mathcal{H} .
- (i) Explain how the two-measurement process removes the degeneracy in Λ .

This problem features two Hermitian operators that commute. Operators that commute have a common set of eigenvectors. The eigenvectors of Hermitian operators are orthogonal so can be made orthonormal. Λ is degenerate so that a measurement of the eigenvalue 0 does not uniquely determine the state vector of the system. Since Λ commutes with \mathcal{H} , together they form a complete set of commuting observables for the system. Part (i) should reinforce that the meaning of a complete set of commuting observables is that the state vector of the system can be uniquely determined by making two measurements in the instance that one of the operators is degenerate.

This problem unifies significant amounts of the mathematics of chapter 1 and quantitative interpretations of the postulates. The results of problem 38 of part 2 of chapter 1 are easily adapted to some of the questions for this problem because it uses the same two operators. Work to understand the concept of a complete set of commuting observables in the light of two, small-dimensional operators, one of which is degenerate. A complete set of commuting observables is often necessary to describe realistic systems in infinite-dimensional or continuous space—the concept is the same encountered here.

Operators that commute have a common eigenbasis. It is essentially impossible to find a common eigenbasis by solving the eigenvalue/eigenvector problem for a degenerate operator. Solve the eigenvalue/eigenvector problem for the non-degenerate operator \mathcal{H} to find the common eigenbasis

$$|E = -1 > = \frac{1}{\sqrt{6}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix}, \qquad |E = 2 > = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix}, \qquad |E = 3 > = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix}, \quad \text{and}$$

$$|\lambda_1 = 0\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix}, \qquad |\lambda_2 = 0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix}, \qquad |\lambda = 2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix}.$$

Parts (b) and (c) are straightforward calculations using postulates 3, 4, and appropriate expressions for expectation values. Part (d) asks for the energy eigenbasis so expand the state vector in the eigenvectors of the Hamiltonian. You should find that the unitary operator for part (e) is

$$\mathcal{U} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \Rightarrow \mathcal{U}^{\dagger} \mathcal{H} \mathcal{U} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \text{ and } \mathcal{U}^{\dagger} \Lambda \mathcal{U} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) The operators Λ and \mathcal{H} commute, because

$$\begin{bmatrix} \Lambda, \mathcal{H} \end{bmatrix} = \Lambda \mathcal{H} - \mathcal{H} \Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} = 0.$$

(b) Calculate the eigenvalues of \mathcal{H} ,

$$\det (\mathcal{H} - \beta \mathcal{I}) = \det \begin{pmatrix} 2 - \beta & 1 & 1 \\ 1 & -\beta & -1 \\ 1 & -1 & 2 - \beta \end{pmatrix}$$

= $(2 - \beta)^2 (-\beta) + (-1) + (-1) - (2 - \beta) - (2 - \beta) - (-\beta) = 0$
 $\Rightarrow (4 - 4\beta + \beta^2)(-\beta) - 1 - 1 - 2 + \beta - 2 + \beta + \beta = 0$
 $\Rightarrow -\beta^3 + 4\beta^2 - \beta - 6 = 0 \Rightarrow \beta^3 - 4\beta^2 + \beta + 6 = 0$
 $\Rightarrow (\beta - 3)(\beta - 2)(\beta + 1) = 0 \Rightarrow \beta = 3, 2, -1$ are the eigenvalues of \mathcal{H} .

To find the eigenvector associated with the eigenvalue $\beta = 3$,

$$\beta = 3 \implies \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies 2a+b+c = 3a \qquad -a+b+c = 0$$
$$\Rightarrow a-c = 3b \implies a-3b-c = 0$$
$$a-b+2c = 3c \qquad a-b-c = 0$$

Add the top and middle equations to attain $b = 0 \Rightarrow c = a$. Following our convention, let $a = 1 \Rightarrow c = 1$ then normalize,

$$(1, 0, 1) A^* A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = |A|^2 (1+0+1) = |A|^2 (2) = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad |3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta = 2 \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} 2a+b+c &= 2a & b &= -c \\ a-c &= 2b & \Rightarrow & a &= b \\ a-b+2c &= 2c & a &= b \end{array}$$

using the top equation to attain the middle equation. Let $a = 1 \Rightarrow b = 1$ and c = -1 and

$$(1, 1, -1) A^* A \begin{pmatrix} 1\\1\\-1 \end{pmatrix} = |A|^2 (1+1+1) = |A|^2 (3) = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad |2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\1\\-1 \end{pmatrix}.$$

$$\beta = -1 \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -1 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} 2a+b+c &= -a & 3a+b+c &= 0 \\ a-c &= -b \Rightarrow a+b-c &= 0 \\ a-b+2c &= -c & a-b+3c &= 0 \end{aligned}$$

where adding the middle and bottom equations yields $a = -c \Rightarrow b = -2a$. Choose $a = 1 \Rightarrow c = -1$ and b = -2 then normalize,

$$(1, -2, -1) A^* A \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = |A|^2 (1+4+1) = |A|^2 (6) = 1 \implies A = \frac{1}{\sqrt{6}} \implies |-1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Applying similar procedures to the operator Λ ,

$$\det (\Lambda - \lambda \mathcal{I}) = \det \begin{pmatrix} 1 - \lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 (-\lambda) - -\lambda = 0$$

$$\Rightarrow \quad (1 - 2\lambda + \lambda^2)(\lambda) - \lambda = 0 \quad \Rightarrow \quad \lambda - 2\lambda^2 + \lambda^3 - \lambda = 0$$

$$\Rightarrow \quad \lambda^2 (\lambda - 2) = 0 \quad \Rightarrow \quad \lambda = 0, \ 0, \ 2 \text{ are the eigenvalues of } \Lambda \text{. For } \lambda = 2,$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \begin{array}{c} a+c &= 2a & a=c \\ \Rightarrow & 0 &= 2b \\ a+c &= 2c & a=c \end{array} \qquad \begin{array}{c} a = c \\ \Rightarrow & |\lambda = 2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

in a normalization procedure identical to that of |E = 3 >. The other two eigenvalues are $\lambda = 0$. We now desire two eigenvectors that are orthonormal to $|\lambda = 2 >$ and each other. We know that eigenvectors exist that meet these conditions because Λ is Hermitian. There is, however, only one eigenvector equation,

$$\lambda = 0 \implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies a + c = 0 \qquad a = -c \\ a + c = 0 \qquad a = -c$$

The middle component is arbitrary—it can be anything. You know that Λ and \mathcal{H} share a common eigenbasis because they commute, so choose

$$|\lambda = 0\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix}$$
 and $|\lambda = 0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix}$.

These choices satisfy the requirement of the eigenvector equation, the top component is opposite the bottom component, and these choices are orthogonal and normalized, or are already orthonormal.

The possibilities of a measurement are the eigenvalues per postulate 3. The probabilities of any given measurement are $\langle \omega_i | \psi(0) \rangle$ per postulate 4. If the energy is measured, the possible results are the eigenvalues of the Hamiltonian, -1, 2, or 3. The corresponding probabilities are

$$P(-1) = \left| \frac{1}{\sqrt{6}} (1, -2, -1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{18}} (1 - 2 - 1) \right|^2 = \left| \frac{-2}{\sqrt{18}} \right|^2 = \frac{4}{18} = \frac{2}{9},$$
$$P(2) = \left| \frac{1}{\sqrt{3}} (1, 1, -1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right|^2 = \left| \frac{1}{3} (1 + 1 - 1) \right|^2 = \left| \frac{1}{3} \right|^2 = \frac{1}{9},$$
$$P(3) = \left| \frac{1}{\sqrt{2}} (1, 0, 1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{6}} (1 + 0 + 1) \right|^2 = \left| \frac{2}{\sqrt{6}} \right|^2 = \frac{4}{6} = \frac{2}{3}.$$

The possible results of a measurement of Λ are its eigenvalues, 0, 0, or 2. In calculations that are identical to those for $P(E_i)$, the probabilities are $|\langle \lambda_i | \psi(0) \rangle|^2$,

$$P(\lambda_1 = 0) = \frac{2}{9}, \quad P(\lambda_2 = 0) = \frac{1}{9}, \quad \Rightarrow \quad P(\lambda = 0) = \frac{1}{3}, \quad \text{and} \quad P(\lambda = 2) = \frac{2}{3}.$$

(c) Using $\langle \Omega \rangle = \langle \psi(0) | \Omega | \psi(0) \rangle$, the t = 0 expectation values of \mathcal{H} and Λ are

$$<\mathcal{H}> = \frac{1}{\sqrt{3}} \begin{pmatrix} 1, \ 1, \ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1, \ 1, \ 1 \end{pmatrix} \begin{pmatrix} 4\\ 0\\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4+0+2 \end{pmatrix} = \frac{6}{3} = 2 ,$$

$$<\Lambda> = \frac{1}{\sqrt{3}} \begin{pmatrix} 1, \ 1, \ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1, \ 1, \ 1 \end{pmatrix} \begin{pmatrix} 2\\ 0\\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2+0+2 \end{pmatrix} = \frac{4}{3} .$$

The sums of the products of the probabilities and the corresponding eigenvalues are

$$<\mathcal{H}>=\sum_{i} P(E_{i}) E_{i} = \frac{2}{9} (-1) + \frac{1}{9} (2) + \frac{6}{9} (3) = \frac{-2+2+18}{9} = 2,$$

and $<\Lambda>=\sum_{i} P(\lambda_{i}) \lambda_{i} = \frac{2}{9} (0) + \frac{1}{9} (0) + \frac{2}{3} (2) = \frac{4}{3}.$

(d) Expanding the initial state in terms of the eigenvectors of \mathcal{H} ,

$$\begin{split} |\psi(0)\rangle &= \sum_{j} |j\rangle \langle j|\psi(0)\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix} \frac{1}{\sqrt{6}} (1, -2, -1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix} \\ &+ \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix} \frac{1}{\sqrt{3}} (1, 1, -1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (1, 0, 1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix} \\ &= \frac{1}{6\sqrt{3}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix} (-2) + \frac{1}{3\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix} (1) + \frac{1}{2\sqrt{3}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} (2) \\ &= -\frac{1}{3\sqrt{3}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix} + \frac{1}{3\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} , \end{split}$$

and then multiplying by the appropriate time-dependent phase factors,

$$\begin{split} |\psi(t)\rangle &= \sum_{j} |j\rangle \langle j | \psi(0) \rangle e^{-iE_{j}t/\hbar} \\ &= -\frac{1}{3\sqrt{3}} \begin{pmatrix} 1\\ -2\\ -1 \end{pmatrix} e^{+it/\hbar} + \frac{1}{3\sqrt{3}} \begin{pmatrix} 1\\ 1\\ -1 \end{pmatrix} e^{-i2t/\hbar} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} e^{-i3t/\hbar} \\ &= \frac{1}{3\sqrt{3}} \begin{pmatrix} -e^{it/\hbar} + e^{-i2t/\hbar} + 3e^{-i3t/\hbar} \\ 2e^{it/\hbar} + e^{-i2t/\hbar} \\ e^{it/\hbar} - e^{-i2t/\hbar} + 3e^{-i3t/\hbar} \end{pmatrix} \end{split}$$

in its most compact form. The probabilities at t > 0 are the same as at t = 0 but we do not ask as this question is not asked here because the algebra with the cross terms can be numbing. We hope the point was made in the previous two dimensional problem. A more efficient approach is transforming to a diagonal basis.

(e) Per part (a), $[\Lambda, \mathcal{H}] = [\mathcal{H}, \Lambda] = 0$. The unitary matrix \mathcal{U} formed from the eigenvectors of \mathcal{H} by placing them from left to right in order of ascending eigenvalue is

$$\mathcal{U} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \quad \Rightarrow \quad \mathcal{U}^{\dagger} = \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

The diagonal operators are

$$\begin{split} \mathcal{U}^{\dagger}\mathcal{H}\mathcal{U} &= \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{6} & 2/\sqrt{3} & 3/\sqrt{2} \\ 2/\sqrt{6} & 2/\sqrt{3} & 0 \\ 1/\sqrt{6} & -2/\sqrt{3} & 3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \\ \mathcal{U}^{\dagger}\Lambda\mathcal{U} &= \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \\ &\text{If we transform the operators } \begin{pmatrix} \mathcal{U}^{\dagger} | \mathcal{H} | \mathcal{U} \end{pmatrix}, \text{ we must transform kets } \begin{pmatrix} \mathcal{U}^{\dagger} | \psi \rangle \end{pmatrix}, \text{ so} \end{split}$$

$$\begin{aligned} \mathcal{U}^{\dagger} \mid \psi(0) \rangle &= \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} (1-2-1)/\sqrt{6} \\ (1+1-1)/\sqrt{3} \\ (1+0+1)/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{3} \\ 2/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{18}} \begin{pmatrix} -2 \\ \sqrt{2} \\ 2\sqrt{3} \end{pmatrix}. \end{aligned}$$

(f) The eigenvalues are on the main diagonals so are trivial to attain for both operators. The eigenvectors are also found by inspection. The probabilities of measuring the eigenvalues of \mathcal{H} are

$$P(-1) = \left| (1, 0, 0) \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\\sqrt{2}\\2\sqrt{3} \end{pmatrix} \right|^2 = \left| \frac{-2}{\sqrt{18}} \right|^2 = \frac{4}{18} = \frac{2}{9},$$
$$P(2) = \left| (0, 1, 0) \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\\sqrt{2}\\2\sqrt{3} \end{pmatrix} \right|^2 = \left| \frac{\sqrt{2}}{\sqrt{18}} \right|^2 = \frac{2}{18} = \frac{1}{9},$$
$$P(3) = \left| (0, 0, 1) \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\\sqrt{2}\\2\sqrt{3} \end{pmatrix} \right|^2 = \left| \frac{2\sqrt{3}}{\sqrt{18}} \right|^2 = \frac{12}{18} = \frac{2}{3}.$$

Identical calculations for Λ yield $P(\lambda_1 = 0) = \frac{2}{9}$, $P(\lambda_2 = 0) = \frac{1}{9}$, and $P(2) = \frac{2}{3}$. In agreement with previous results, expectation values are

$$<\mathcal{H}> = \left(-2, \sqrt{2}, 2\sqrt{3}\right) \frac{1}{\sqrt{18}} \begin{pmatrix} -1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3 \end{pmatrix} \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\ \sqrt{2}\\ 2\sqrt{3} \end{pmatrix}$$
$$= \frac{1}{18} \left(-2, \sqrt{2}, 2\sqrt{3}\right) \begin{pmatrix} 2\\ 2\sqrt{2}\\ 6\sqrt{3} \end{pmatrix} = \frac{1}{18} \left(-4 + 4 + 36\right) = \frac{36}{18} = 2,$$
$$<\Lambda> = \left(-2, \sqrt{2}, 2\sqrt{3}\right) \frac{1}{\sqrt{18}} \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2 \end{pmatrix} \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\ \sqrt{2}\\ 2\sqrt{3} \end{pmatrix}$$
$$= \frac{1}{18} \left(-2, \sqrt{2}, 2\sqrt{3}\right) \begin{pmatrix} 0\\ 0\\ 4\sqrt{3} \end{pmatrix} = \frac{1}{18} \left(0 + 0 + 24\right) = \frac{24}{18} = \frac{4}{3}.$$

The calculations are much simpler in the diagonal basis. The eigenvalue/eigenvector problem for Λ is made unnecessary by the process of diagonalization. The absence of cross terms is a significant algebraic advantage. Finally, a basis of unit vectors offer conceptual advantages over a basis of eigenvectors having multiple non-zero components. (Do the probability calculations for t > 0 using $|\psi(0)\rangle$ from part (d) if you are in need of further argument).

(g) The time dependence of the state vector in the diagonal basis is

$$\begin{split} |\psi(t)\rangle &= \sum_{j} |j\rangle \langle j|\psi(0)\rangle e^{-iE_{j}t/\hbar} \\ &= \begin{pmatrix} 1\\0\\0 \end{pmatrix} (1, 0, 0) \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\\sqrt{2}\\2\sqrt{3} \end{pmatrix} e^{it/\hbar} + \begin{pmatrix} 0\\1\\0 \end{pmatrix} (0, 1, 0) \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\\sqrt{2}\\2\sqrt{3} \end{pmatrix} e^{-i2t/\hbar} \\ &+ \begin{pmatrix} 0\\0\\1 \end{pmatrix} (0, 0, 1) \frac{1}{\sqrt{18}} \begin{pmatrix} -2\\\sqrt{2}\\2\sqrt{3} \end{pmatrix} e^{-i3t/\hbar} \\ &= -\frac{2}{\sqrt{18}} \begin{pmatrix} 1\\0\\0 \end{pmatrix} e^{it/\hbar} + \frac{\sqrt{2}}{\sqrt{18}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} e^{-i2t/\hbar} + \frac{2\sqrt{3}}{\sqrt{18}} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i3t/\hbar} = \frac{1}{\sqrt{18}} \begin{pmatrix} -2e^{it/\hbar}\\\sqrt{2}e^{-i2t/\hbar}\\2\sqrt{3}e^{-i3t/\hbar} \end{pmatrix} \end{split}$$

Probability calculations are duplicative,

$$P(E = -1) = P(\lambda_1 = 0) = \left| (1, 0, 0) \left(-\frac{2}{\sqrt{18}} \right) \begin{pmatrix} 1\\0\\0 \end{pmatrix} e^{it/\hbar} \right|^2$$
$$= \left| -\frac{2}{\sqrt{18}} e^{it/\hbar} \right|^2 = \frac{4}{18} \left(e^{-it/\hbar} \right) \left(e^{+it/\hbar} \right) = \frac{2}{9},$$
$$P(E = 2) = P(\lambda_2 = 0) = \left| (0, 1, 0) \frac{\sqrt{2}}{\sqrt{18}} \begin{pmatrix} 0\\1\\0 \end{pmatrix} e^{-i2t/\hbar} \right|^2$$
$$= \left| \frac{\sqrt{2}}{\sqrt{18}} e^{-i2t/\hbar} \right|^2 = \frac{1}{9} \left(e^{+i2t/\hbar} \right) \left(e^{-i2t/\hbar} \right) = \frac{1}{9},$$
$$P(E = 3) = P(\lambda = 2) = \left| (0, 0, 1) \frac{2\sqrt{3}}{\sqrt{18}} \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^{-i3t/\hbar} \right|^2$$
$$= \left| \frac{2\sqrt{3}}{\sqrt{18}} e^{-i3t/\hbar} \right|^2 = \frac{12}{18} \left(e^{+i3t/\hbar} \right) \left(e^{-i3t/\hbar} \right) = \frac{2}{3}.$$

These are identical to the t = 0 probabilities, as expected.

(h) If \mathcal{H} is measured and E = 2 is obtained, the state vector is $|\psi(t > 0)\rangle = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$ in

the diagonal basis, per postulate 5. A measurement of Λ can yield only $\lambda = 0$ because the corresponding eigenvector is identical to the state vector and no other eigenvector has a second component that is non-zero. In other words, $P(\lambda = 0) = 1$, per postulate 4. Subsequent alternate measurements of \mathcal{H} and Λ can yield only E = 2 and $\lambda = 0$. If \mathcal{H} is measured and E = 3

measurements of \mathcal{H} and Λ can yield only $\mathcal{L} = \begin{pmatrix} 0\\0\\1 \end{pmatrix}$ in the diagonal basis. A measurement

of Λ can yield only $\lambda = 2$ per postulate 4. Subsequent alternate measurements of \mathcal{H} and Λ can yield only E = 3 and $\lambda = 2$. If \mathcal{H} is measured and E = -1 is obtained, the state vector is $|\psi(t > 0)\rangle = \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}$ in the diagonal basis. A measurement of Λ can yield only $\lambda = 0$.

Subsequent measurements of \mathcal{H} and Λ can yield only E = -1 and $\lambda = 0$.

(i) If we measure Λ at t = 0 and find $\lambda = 0$, then the state vector is one of the two eigenvectors of Λ corresponding to $\lambda = 0$. We do not know which. Postulates 1 and 4 indicate the best possible interpretation is that $|\psi'(t > 0)\rangle$ is in the linear combination of the two $\lambda = 0$ eigenvectors

$$|\psi'(t>0)\rangle = c_1 \begin{pmatrix} 1\\0\\0 \end{pmatrix} + c_2 \begin{pmatrix} 0\\1\\0 \end{pmatrix}.$$

Therefore, E = -1 or E = 2 are possible for a subsequent measurement of \mathcal{H} . If a measurement of \mathcal{H} yields E = -1, then the state vector is $|\psi''(t > 0)\rangle = \begin{pmatrix} 1\\0\\0 \end{pmatrix}$. If a measurement of

 \mathcal{H} yields E = 2, then the state vector is $|\psi''(t > 0) \rangle = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$. Measurement of both Λ

and \mathcal{H} determines the state vector uniquely. In other words, the degeneracy in Λ is removed by measuring both Λ and \mathcal{H} .

Postscript: Notice in part (g) that the square of the expansion coefficients are the probabilities. This is true in all orthonormal bases. Thus the expansion coefficients are also known as the probability amplitudes. The probabilities are more generally $|c_j|^2 = c_j^* c_j$ because the expansion coefficients are generally complex numbers.

Again, part (i) should reinforce that the meaning of a complete set of commuting observables is that the state vector of the system can be uniquely determined by making two measurements in the instance that one of the operators is degenerate. The concept of a complete set of commuting observables is employed in many realistic systems. It is much more difficult to comprehend when employed to describe a realistic system using a basis that is infinite-dimensional, instead of threedimensional, composed of special functions, instead of unit vectors... ensure that you understand the idea addressed in part (i).