Tapestries from India, oriental vases, paintings by Dali, is that a Fabrege egg? Cases and stands made of ebony that matched the woodwork of the door frames, all intricately carved and highly polished. Is this a house or a gallery? He did not recognize most of the other guests; he did recognize a couple of ball players, a judge, and that beautiful lady who was a news reporter on channel 9. The guest list appeared to be affluent business people, successful entertainers, elected officials, appointed dignitaries...and a private eye who felt like a pair of unmatched socks in the silver drawer. Where can I get a beer? A petite maid in a black and white uniform appeared holding a tray of wine glasses and asked "Champagne?" Will have to do until I can find a beer... "Thank you." She added while turning toward adjacent guests, "The real values are those of Hermite."

The Mathematics of Quantum Mechanics, Part 2

Operators dominate part 2. When you finish part 2, you should understand the meaning of a **linear operator**, the **identity operator**, **inverse operators**, **Hermitian operators**, **unitary operators**, the **commutator**, a **diagonal** operator, and the **determinant** of an operator. You need to understand the meaning of **eigenvalues** and **eigenvectors**. You need to know how to solve the **eigenvalue/eigenvector** problem. You want to know how to **diagonalize** an operator. You want to know how to **simultaneously diagonalize** two operators. You want to understand the meaning of **degenerate operators**. The reason to understand simultaneous diagonalization is degenerate operators.

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1. Show that $\mathcal{L}_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ is linear.

A state vector, or wave function, will describe a quantum mechanical system. An observable quantity, or simply an **observable** is described by an operator. Examples of observable quantities are energy, momentum, and position. *All* the information about the *possible* outcomes we *could* observe is contained in the operator, without reference to a specific state vector or system.

Quantum mechanics appears to be a linear theory. In particular, we will informally state that quantum mechanics is dominated by **linear operators**. Specifically, *a linear operator commutes with scalars, and will distribute over vectors*. Symbolically this means

$$\Omega \Big[\alpha \,|\, v> \Big] = \alpha \, \Omega \,|\, v> , \tag{1}$$

$$\Omega \Big[\alpha \,|\, v_1> + \beta \,|\, v_2> \Big] = \alpha \, \Omega \,|\, v_1> + \beta \, \Omega \,|\, v_2> , \tag{2}$$

where the last equation is focal because it includes the operation described in the first equation.

Show that $\mathcal{L}_y$ satisfies these two equations. Since $\mathcal{L}_y$ is a three dimensional operator, pick generic three dimensional vectors like

$$|\, v_1> = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \text{and} \qquad |\, v_2> = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix},$$

and do the appropriate multiplications and additions to show that equations (1) and (2) are true. It may be wise to start on both ends and meet in the middle, which is how most proofs are actually completed.

$$\mathcal{L}_y\Big[\,\alpha\,|\,v\!>\Big] = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \left[\alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix} = \begin{pmatrix} -i\alpha b \\ i\alpha a - i\alpha c \\ i\alpha b \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(-ib) \\ \alpha(ia) + \alpha(-ic) \\ \alpha(ib) \end{pmatrix} = \alpha \begin{pmatrix} -ib \\ ia - ic \\ ib \end{pmatrix} = \alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha\,\mathcal{L}_y\,|\,v\!>,$$

so $\mathcal{L}_y$ satisfies equation (1). Seeing a three dimensional matrix in a vector to arrive at the next to last step is somewhat subtle, and is why we advised you to start at both ends. Equation (2) is

$$\mathcal{L}_y\Big[\,\alpha\,|\,v_1\!>+\,\beta\,|\,v_2\!>\Big] = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \left[\alpha \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \left[\begin{pmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{pmatrix} + \begin{pmatrix} \beta a_2 \\ \beta b_2 \\ \beta c_2 \end{pmatrix}\right] = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{pmatrix} + \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \beta a_2 \\ \beta b_2 \\ \beta c_2 \end{pmatrix}$$

and this is apparently two applications of the reduction for equation (1). We will continue for the purpose of completeness, but realize that the rest of this reduction duplicates the reduction of equation (1) twice. Proceeding,

$$\mathcal{L}_y\Big[\,\alpha\,|\,v_1\!>+\,\beta\,|\,v_2\!>\Big] = \begin{pmatrix} -i\alpha b_1 \\ i\alpha a_1 - i\alpha c_1 \\ i\alpha b_1 \end{pmatrix} + \begin{pmatrix} -i\beta b_2 \\ i\beta a_2 - i\beta c_2 \\ i\beta b_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(-ib_1) \\ \alpha(ia_1) + \alpha(-ic_1) \\ \alpha(ib_1) \end{pmatrix} + \begin{pmatrix} \beta(-ib_2) \\ \beta(ia_2) + \beta(-ic_2) \\ \beta(ib_2) \end{pmatrix} = \alpha \begin{pmatrix} -ib_1 \\ ia_1 - ic_1 \\ ib_1 \end{pmatrix} + \beta \begin{pmatrix} -ib_2 \\ ia_2 - ic_2 \\ ib_2 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \alpha\,\mathcal{L}_y\,|\,v_1\!>+\,\beta\,\mathcal{L}_y\,|\,v_2\!>\ .$$

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2. Show by explicit multiplication in $\mathbf{C}^3$ that an identity operator multiplying

(a) a ket,

(b) a bra,

(c) a matrix operator from the left, and

(d) a matrix operator from the right result in the original ket, bra, or matrix operator.

The identity operator is the analogy of "1" in the real number system. One times a number is the original number. This problem introduces the identity operator but also intends to persuade you that any legitimate multiplication by the identity operator results in the original object. In $\mathbf{C}^3$ the identity operator is

$$\mathcal{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An identity operator is a matrix with 1's on the principal diagonal and zeros elsewhere. Assume arbitrary objects like

$$|\,v\!> = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

You will find that $\mathcal{I}\,|\,v\!> = |\,v\!>$, $<\!v\,|\,\mathcal{I} = <\!v\,|$, $\mathcal{I}\,\mathcal{A} = \mathcal{A}$, and $\mathcal{A}\mathcal{I} = \mathcal{A}$.

(a) $$\mathcal{I}\,|\,v\!> = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1+0+0 \\ 0+b_2+0 \\ 0+0+b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = |\,v\!> .$$

(b) $$<\!v\,|\,\mathcal{I} = \big(b_1,\ b_2,\ b_3\big) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \big(b_1{+}0{+}0,\ 0{+}b_2{+}0,\ 0{+}0{+}b_3\big) = \big(b_1,\ b_2,\ b_3\big) = <\!v\,|\,.$$

(c) $$\mathcal{I}\,\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathcal{A}\,.$$

(d) $$\mathcal{A}\mathcal{I} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+0+0 & 0+a_{12}+0 & 0+0+a_{13} \\ a_{21}+0+0 & 0+a_{22}+0 & 0+0+a_{23} \\ a_{31}+0+0 & 0+a_{32}+0 & 0+0+a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathcal{A}\,.$$

Postscript: A scalar times an operator, from either side, is another operator. Symbolically,

$$\alpha\,\mathcal{I} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix},$$

the product is an operator with the scalar on the principal diagonal and zeros elsewhere. Operating on a vector or operator with this yields the same result as multiplication of the vector or operator by the scalar.

The scalar times the identity operator is what is intended when an author sets a scalar equal to an operator. For instance,

$$\mathcal{A}\mathcal{B} = \alpha \quad \text{means} \quad \mathcal{A}\mathcal{B} = \alpha\mathcal{I}\,.$$

---

3. Calculate the determinants of

$$\text{(a)} \qquad \mathcal{A} = \begin{pmatrix} 1-2i & 2+3i \\ 3-i & 4+2i \end{pmatrix}, \qquad \text{and} \qquad \text{(b)} \qquad \mathcal{C} = \begin{pmatrix} 2 & -4 & 7 \\ 8 & -3 & -5 \\ -4 & 9 & 1 \end{pmatrix}.$$

An integral with limits is a scalar, (though it may take considerable effort to calculate that scalar). Similarly, a **determinant** is a scalar. $\det \mathcal{A}$ is a scalar associated with a matrix operator. A determinant is a scalar associated with a square matrix. Specifically,

a determinant is a function of a matrix that is the sum of the products of the elements of any row or column and their respective cofactors.

$$\text{Symbolically} \qquad \mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}, \qquad \text{and}$$

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11}\big(a_{22}a_{33} - a_{32}a_{23}\big) - a_{21}\big(a_{12}a_{33} - a_{32}a_{13}\big) + a_{31}\big(a_{12}a_{23} - a_{22}a_{13}\big) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}. \end{aligned}$$

This can be extended to arbitrary or infinite dimension. For evaluation of determinants beyond dimension 3, refer to Arfken[1], Boas[2], or your favorite linear algebra text.

$$\text{(a)} \qquad \mathcal{A} = \begin{pmatrix} 1-2i & 2+3i \\ 3-i & 4+2i \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{A} = \begin{vmatrix} 1-2i & 2+3i \\ 3-i & 4+2i \end{vmatrix} = (1-2i)(4+2i) - (3-i)(2+3i)$$

$$= 4 + 2i - 8i + 4 - (6 + 9i - 2i + 3) = 8 - 6i - 6 - 9i + 2i - 3 = -1 - 13i\,,$$

which is a scalar in $\mathbf{C}^2$.

$$\text{(b)} \qquad \mathcal{C} = \begin{pmatrix} 2 & -4 & 7 \\ 8 & -3 & -5 \\ -4 & 9 & 1 \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{C} = \begin{vmatrix} 2 & -4 & 7 \\ 8 & -3 & -5 \\ -4 & 9 & 1 \end{vmatrix}$$

[1] Arfken *Mathematical Methods for Physicists*, Academic Press, 19–, chap 4.
[2] Boas *Mathematical Methods in the Physical Sciences*, John Wiley & Sons, 1983, pp. 87–94.

$$= (2)(-3)(1) - (2)(9)(-5) + (8)(9)(7) - (8)(-4)(1) + (-4)(-4)(-5) - (-4)(-3)(7)$$
$$= -6 + 90 + 504 + 32 - 80 - 84 = 456 \qquad \text{which is a scalar in } \mathbf{R}^3.$$

Postscript: A determinant is a scalar associated only with a square matrix. A determinant is not defined for a matrix that does not have an equal number of rows and columns.

A **singular** matrix is one for which $\det \mathcal{A} = 0$. A singular operator does not have an **inverse** operator. We will address inverse operators directly.

Evaluating determinants is a skill that is central to solving the eigenvalue/eigenvector problem.

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4. For an arbitrary matrix operator in $\mathbf{C}^2$,

(a) Show that $\det \mathcal{A}^T = \det \mathcal{A}$.

(b) Show that $\det\big(\mathcal{A}\,\mathcal{A}\big) = \det \mathcal{A}\,\det \mathcal{A}$.

(c) Show that $\det\big(\mathcal{B}^\dagger\big) = \big(\det \mathcal{B}^T\big)^*$ using explicitly complex matrix elements.

This problem is designed to increase your familiarity with determinants and some of their properties. Use a two-dimensional matrix like $\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. The condition $\in \mathbf{C}^2$ means that α, β, γ, and δ are arbitrary complex numbers. For parts (a) and (b), just calculate the necessary operators and their determinants. In part (c), explicitly complex elements means $\mathcal{B} = \begin{pmatrix} \alpha + ia & \beta + ib \\ \gamma + ic & \delta + id \end{pmatrix}$, for instance. Part (c) is straightforward using matrix elements with real and imaginary parts; it may be less accessible using implicitly complex operator elements.

(a) $\det \mathcal{A} = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma$ and $\det \mathcal{A}^T = \det \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \alpha\delta - \gamma\beta = \alpha\delta - \beta\gamma$

$$\text{therefore,} \quad \det \mathcal{A}^T = \det \mathcal{A}.$$

(b)
$$\mathcal{A}\,\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \alpha\gamma + \gamma\delta & \beta\gamma + \delta^2 \end{pmatrix}$$

$$\begin{aligned} \det\big(\mathcal{A}\,\mathcal{A}\big) &= (\alpha^2 + \beta\gamma)(\beta\gamma + \delta^2) - (\alpha\beta + \beta\delta)(\alpha\gamma + \gamma\delta) \\ &= \cancel{\alpha^2\beta\gamma} + \alpha^2\delta^2 + \beta^2\gamma^2 + \cancel{\beta\gamma\delta^2} - \cancel{\alpha^2\beta\gamma} - \alpha\beta\gamma\delta - \alpha\beta\gamma\delta - \cancel{\beta\gamma\delta^2} \\ &= \alpha^2\delta^2 + \beta^2\gamma^2 - 2\alpha\beta\gamma\delta. \end{aligned}$$

$$\begin{aligned} \text{Then } \det \mathcal{A}\,\det \mathcal{A} &= (\alpha\delta - \beta\gamma)(\alpha\delta - \beta\gamma) \\ &= \alpha^2\delta^2 - \alpha\beta\gamma\delta - \alpha\beta\gamma\delta + \beta^2\gamma^2 \\ &= \alpha^2\delta^2 + \beta^2\gamma^2 - 2\alpha\beta\gamma\delta, \end{aligned}$$

therefore, $\det\big(\mathcal{A}\,\mathcal{A}\big) = \det \mathcal{A}\det \mathcal{A}$. In general, $\det\,(\mathcal{A}\,\mathcal{B}) = \det \mathcal{A}\,\det \mathcal{B}$.

(c)
$$\mathcal{B} = \begin{pmatrix} \alpha + ia & \beta + ib \\ \gamma + ic & \delta + id \end{pmatrix} \quad \Rightarrow \quad \mathcal{B}^T = \begin{pmatrix} \alpha + ia & \gamma + ic \\ \beta + ib & \delta + id \end{pmatrix}$$
$$\begin{aligned} \text{so} \quad \det \mathcal{B}^T &= (\alpha + ia)(\delta + id) - (\gamma + ic)(\beta + ib) \\ &= \alpha\delta + i\alpha d + ia\delta - ad - \beta\gamma - ib\gamma - i\beta c + bc \\ &= \alpha\delta - ad - \beta\gamma + bc + i(\alpha d + a\delta - b\gamma - \beta c) \\ \Rightarrow \quad (\det \mathcal{B}^T)^* &= \alpha\delta - ad - \beta\gamma + bc - i(\alpha d + a\delta - b\gamma - \beta c)\,. \end{aligned}$$

$$\begin{aligned} \det \mathcal{B}^\dagger &= \det \begin{pmatrix} \alpha - ia & \gamma - ic \\ \beta - ib & \delta - id \end{pmatrix} \\ &= (\alpha - ia)(\delta - id) - (\gamma - ic)(\beta - ib) \\ &= \alpha\delta - i\alpha d - ia\delta - ad - \beta\gamma + ib\gamma + i\beta c + bc \\ &= \alpha\delta - ad - \beta\gamma + bc - i(\alpha d + a\delta - b\gamma - \beta c)\,, \ \text{so} \ \det \mathcal{B}^\dagger = (\det \mathcal{B}^T)^*\,. \end{aligned}$$

Postscript: A determinant is a scalar so has the properties of scalar. For instance, determinants commute because scalars commute.

You have demonstrated that the determinant of an operator and its transpose are the same, that the determinant of a product is the same as the product of the determinants, and that the determinant of an adjoint is the conjugate of the determinant of the transpose. Parts (a) and (c) together imply that $\det\left(\mathcal{B}^\dagger\right) = \left(\det \mathcal{B}\right)^*$, the determinant of the adjoint is the same as the conjugate of the determinant. These are not central results, but they can be useful (see problem 13). Note that these results can be extended to operators of dimension larger than 2×2.

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5. Find the inverse of $\mathcal{D} = \begin{pmatrix} 1 & -2 \\ 4 & -2 \end{pmatrix}$, and then find $\mathcal{D}\,\mathcal{D}^{-1}$ and $\mathcal{D}^{-1}\,\mathcal{D}$ to verify your result.

The product of an operator and its inverse is the identity operator. Symbolically, the **inverse operator** of $\mathcal{A}$ is denoted $\mathcal{A}^{-1}$, and
$$\mathcal{A}\,\mathcal{A}^{-1} = \mathcal{A}^{-1}\,\mathcal{A} = \mathcal{I}\,.$$
Multiplication by an inverse is analogous to division. Calculating an inverse is an exercise in solving simultaneous equations. For example, if $\mathcal{A} = \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix}$, to find $\mathcal{A}^{-1}$,
$$\begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} 5a + 4c = 1 \\ 2a + 2c = 0 \\ \text{and} \\ 5b + 4d = 0 \\ 2b + 2d = 1 \end{array} \quad \begin{array}{c} \Rightarrow \quad \begin{array}{c} 5a + 4c = 1 \\ -4a - 4c = 0 \end{array} \quad \Rightarrow \quad a = 1 \\ \\ \Rightarrow \quad \begin{array}{c} 5b + 4d = 0 \\ -4b - 4d = -2 \end{array} \quad \Rightarrow \quad b = -2 \end{array}$$
and substituting these into the second and fourth equations,
$$\begin{array}{l} a = 1 \quad \Rightarrow \quad 2(1) + 2c = 0 \quad \Rightarrow \quad c = -1\,, \\ b = -2 \quad \Rightarrow \quad 2(-2) + 2d = 1 \quad \Rightarrow \quad d = 5/2\,, \end{array} \quad \Rightarrow \quad \mathcal{A}^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 5/2 \end{pmatrix}.$$

To verify this result,

$$\mathcal{A}\,\mathcal{A}^{-1} = \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 5/2 \end{pmatrix} = \begin{pmatrix} 5-4 & -10+10 \\ 2-2 & -4+5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}\,,$$

$$\mathcal{A}^{-1}\,\mathcal{A} = \begin{pmatrix} 1 & -2 \\ -1 & 5/2 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 5-4 & 4-4 \\ -5+5 & -4+5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}\,.$$

$$\begin{pmatrix} 1 & -2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} a-2c=1 \\ 4a-2c=0 \\ \text{and} \\ b-2d=0 \\ 4b-2d=1 \end{array} \begin{array}{c} \Rightarrow \\ \\ \\ \\ \Rightarrow \end{array} \begin{array}{c} a-2c=1 \\ -4a+2c=0 \\ \\ -b+2d=0 \\ 4b-2d=1 \end{array} \begin{array}{c} \Rightarrow \quad a=-\dfrac{1}{3} \\ \\ \\ \Rightarrow \quad b=\dfrac{1}{3} \end{array}$$

and substituting these into the first and third equations,

$$\begin{array}{l} a=-\dfrac{1}{3} \quad \Rightarrow \quad -\dfrac{1}{3}-2c=1 \quad \Rightarrow \quad c=-\dfrac{2}{3}\,, \\ b=\dfrac{1}{3} \quad \Rightarrow \quad \dfrac{1}{3}-2d=0 \quad \Rightarrow \quad d=\dfrac{1}{6}\,, \end{array} \quad \Rightarrow \quad \mathcal{D}^{-1} = \begin{pmatrix} -1/3 & 1/3 \\ -2/3 & 1/6 \end{pmatrix}.$$

To verify this result,

$$\mathcal{D}\,\mathcal{D}^{-1} = \begin{pmatrix} 1 & -2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} -1/3 & 1/3 \\ -2/3 & 1/6 \end{pmatrix} = \begin{pmatrix} -1/3+4/3 & 1/3-1/3 \\ -4/3+4/3 & 4/3-1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}\,,$$

$$\mathcal{D}^{-1}\,\mathcal{D} = \begin{pmatrix} -1/3 & 1/3 \\ -2/3 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1/3+4/3 & 2/3-2/3 \\ -2/3+2/3 & 4/3-1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}\,.$$

Postscript: The fact that you want to retain from this problem is $\mathcal{A}\,\mathcal{A}^{-1} = \mathcal{A}^{-1}\,\mathcal{A} = \mathcal{I}$. We will use this fact frequently. The numerical examples of this problem are intended to be illustrative, but they are not important. We will never explicitly calculate the inverse of a matrix operator.

If an operator is singular, *i.e.*, if $\det \mathcal{A} = 0$, the operator does not have an inverse. The products $\mathcal{A}\,\mathcal{A}^{-1}$ and $\mathcal{A}^{-1}\,\mathcal{A}$ are, therefore, undefined. This condition is equivalent to prohibiting the division by zero in the real number system. We will employ inverse operators without explicitly determining that the operator is non-singular. In other words, we will consistently assume that an inverse of an operator exists. We make this assumption because the classes of operators common to quantum mechanics are typically non-singular.

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6. Show that two dimensional matrix operators are associative.

Associativity is an often overlooked but frequently invoked property of matrix operators which says they can be grouped in any manner as long as the order does not change. Symbolically,

$$\mathcal{A}\left(\mathcal{B}\,\mathcal{C}\right) = \left(\mathcal{A}\,\mathcal{B}\right)\mathcal{C}\,.$$

By the way, $\mathcal{A}\,(\mathcal{B}\,\mathcal{C}\,)$ has the same meaning as $\mathcal{A}\,\mathcal{B}\,\mathcal{C}\,.$ Likely the simplest and clearest method of demonstrating associativity is using explicit forms of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, do the matrix multiplications and show that they are the same. An explicit form of $\mathcal{A}$ means

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

for instance. This problem is limited to two dimensions for reasons of simplicity. Your proof could be extended to any including infinite dimension.

We want to show

$$\mathcal{A}\,(\mathcal{B}\,\mathcal{C}\,) = (\,\mathcal{A}\,\mathcal{B}\,)\,\mathcal{C}\,. \tag{1}$$

The left side of the equation (1) is

$$\begin{aligned} \mathcal{A}\,(\mathcal{B}\,\mathcal{C}\,) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left[\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right] \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22} \end{pmatrix}$$

The right side of equation (1) is

$$\begin{aligned} (\,\mathcal{A}\,\mathcal{B}\,)\,\mathcal{C} &= \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right] \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22} \end{pmatrix}$$

when the second and third terms of each element are interchanged. This is the same as the first expansion. Since the two expansions equal the same thing, they must themselves be equal, or

$$\mathcal{A}\,(\mathcal{B}\,\mathcal{C}\,) = (\,\mathcal{A}\,\mathcal{B}\,)\,\mathcal{C}\,, \quad \text{and two dimensional matrix operators are associative.}$$

---

7. Show that $\left(\mathcal{A}\,\mathcal{B}\right)^{-1} = \mathcal{B}^{-1}\,\mathcal{A}^{-1}\,.$

This problem establishes the fact that the inverse of the product of two operators is the reverse order of the product of the inverses. It also introduces a technique known as **insertion of the identity** that is vital to many of the calculations of quantum mechanics. The identity operator is so named because anything upon which it operates remains identical. Use of the identity operator is analogous to multiplication by 1 in the real number system. The real utility of the identity operator (or any identity element including the number 1 in the real number system) is that it has an infinite number of different forms, $\mathcal{A}\,\mathcal{A}^{-1}$ and $\mathcal{B}\,\mathcal{B}^{-1}$ to name just two forms that are employed in this problem.

$$
\begin{aligned}
\mathcal{I} &= \mathcal{A}\,\mathcal{A}^{-1} && \text{statement of the identity}\\
\mathcal{I} &= \mathcal{A}\,\mathcal{I}\,\mathcal{A}^{-1} && \text{insertion of the identity}\\
\mathcal{I} &= \mathcal{A}\,\mathcal{B}\,\mathcal{B}^{-1}\,\mathcal{A}^{-1} && \text{use of the form of the identity } \mathcal{I} = \mathcal{B}\,\mathcal{B}^{-1}\\
\mathcal{I} &= \big(\mathcal{A}\,\mathcal{B}\big)\,\mathcal{B}^{-1}\,\mathcal{A}^{-1} && \text{associative property}\\
\big(\mathcal{A}\,\mathcal{B}\big)^{-1}\mathcal{I} &= \big(\mathcal{A}\,\mathcal{B}\big)^{-1}\big(\mathcal{A}\,\mathcal{B}\big)\,\mathcal{B}^{-1}\,\mathcal{A}^{-1} && \text{operate on both sides with } \big(\mathcal{A}\,\mathcal{B}\big)^{-1}\\
\big(\mathcal{A}\,\mathcal{B}\big)^{-1}\mathcal{I} &= \mathcal{I}\,\mathcal{B}^{-1}\,\mathcal{A}^{-1} && \text{use of the form of the identity } \mathcal{I} = \big(\mathcal{A}\,\mathcal{B}\big)^{-1}\big(\mathcal{A}\,\mathcal{B}\big)\\
\big(\mathcal{A}\,\mathcal{B}\big)^{-1} &= \mathcal{B}^{-1}\,\mathcal{A}^{-1} && \text{operation by the identity}
\end{aligned}
$$

Postscript: A short comment on lines 5 and 6 may be helpful. Remember that the product of two operators is another operator, so $\big(\mathcal{A}\,\mathcal{B}\big)$ is an operator, and its inverse is denoted $\big(\mathcal{A}\,\mathcal{B}\big)^{-1}$.

---

8. Show that the commutator of

$$\mathcal{A} = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix} \quad \text{is zero.}$$

The order of the operators matters in matrix multiplication. Generally,

$$\mathcal{A}\,\mathcal{B} \neq \mathcal{B}\,\mathcal{A}\,, \qquad \text{or} \qquad \mathcal{A}\,\mathcal{B} - \mathcal{B}\,\mathcal{A} \neq 0\,.$$

The left side of the last equation is commonly denoted

$$\big[\,\mathcal{A},\,\mathcal{B}\,\big] = \mathcal{A}\,\mathcal{B} - \mathcal{B}\,\mathcal{A}\,.$$

The object $[\,\mathcal{A},\,\mathcal{B}\,]$ is called a **commutator**. Most of the physics of quantum mechanics can be generated using commutators. If $[\,\mathcal{A},\,\mathcal{B}\,] = 0\,,$ the operators $\mathcal{A}$ and $\mathcal{B}$ are said to commute. Operators that represent observable quantities and commute become particularly focal.

Do the indicated multiplcations and you will find that the difference of the products is "zero." It is actually the zero operator. If all of the elements are zero, the operator is said to be zero.

$$
\begin{aligned}
\big[\,\mathcal{A},\,\mathcal{B}\,\big] = \mathcal{A}\,\mathcal{B} - \mathcal{B}\,\mathcal{A} &= \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}\begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix} - \begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}\\
&= \begin{pmatrix} (1+i)(-1-i)+0 & 0+0 \\ 0+0 & 0+(1-i)(-1+i) \end{pmatrix} - \begin{pmatrix} (-1-i)(1+i)+0 & 0+0 \\ 0+0 & 0+(-1+i)(1-i) \end{pmatrix}\\
&= \begin{pmatrix} -1-i-i+1 & 0 \\ 0 & -1+i+i+1 \end{pmatrix} - \begin{pmatrix} -1-i-i+1 & 0 \\ 0 & -1+i+i+1 \end{pmatrix}\\
&= \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix} - \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix} = \begin{pmatrix} -2i+2i & 0 \\ 0 & 2i-2i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Postscript: A commutator is itself an operator. The product of two operators is another operator, and the difference of two operators is an operator, so the difference of the products of two operators is an operator.

The operators in this problem are **diagonal operators**. A diagonal operator is a matrix that has non-zero elements only on the principal diagonal. The identity matrix is another example of a diagonal operator. All diagonal operators commute.

---

9. Is the operator $\mathcal{L}_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ Hermitian?

A **Hermitian operator** is an operator that is identical to its adjoint, *i.e.*,

$$\mathcal{A} = \mathcal{A}^\dagger \, .$$

An adjoint is a transpose conjugate. Switch the rows and columns and conjugate all the elements.

The transpose of $\mathcal{L}_y$ is $\mathcal{L}_y^{\mathrm{T}} = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$, and the conjugate of the transpose, or transpose conjugate, is the adjoint, or $\mathcal{L}_y^{\mathrm{T}\,*} = \mathcal{L}_y^\dagger = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \mathcal{L}_y$, therefore, $\mathcal{L}_y$ is Hermitian.

Postscript: All observable quantities, such as energy, momentum, or position; are represented by Hermitian operators. *All observable quantities are represented by Hermitian operators.*

The operator $\mathcal{L}_y$ in this problem is one of the three component angular momentum operators.

An **anti-Hermitian operator** is an operator that is identical to the negative of its adjoint, *i.e.*,

$$\mathcal{A} = -\mathcal{A}^\dagger \, .$$

---

10. Is the operator $\mathcal{R} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ unitary?

A **unitary operator** is an operator whose product with its adjoint is the identity, *i.e.*,

$$\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger \mathcal{U} = \mathcal{I} \, .$$

Multiplication by a unitary operator preserves the norm, or in geometric terms, the length of a vector. A unitary operator is analogous to an identity operator with a rotation. Geometrically, operation by the identity operator changes neither the direction nor the length. Operation by a unitary operator does not change the length but does change the direction of a vector.

The adjoint of $\mathcal{R}$ is the tranpose conjugate, so

$$\mathcal{R}^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad\Rightarrow\quad \mathcal{R}^{\mathrm{T}\,*} = \left(\frac{1}{\sqrt{2}}\right)^{*}\begin{pmatrix} 1^* & 1^* \\ -1^* & 1^* \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \mathcal{R}^{\dagger},$$

$$\Rightarrow\quad \mathcal{R}\,\mathcal{R}^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I},$$

therefore $\mathcal{R}$ is unitary.

Postscript: If $\mathcal{U}\mathcal{U}^{\dagger} = \mathcal{I}$, then $\mathcal{U}^{\dagger}\mathcal{U} = \mathcal{I}$. Similarly, $\mathcal{U}^{\dagger}\mathcal{U} = \mathcal{I} \;\Rightarrow\; \mathcal{U}\mathcal{U}^{\dagger} = \mathcal{I}$. The proof is short and straightforward using techniques similar to problem 7. See problem 32.

The operator $\mathcal{R}$ in this problem rotates a vector clockwise through an angle $\pi/4$.

---

11. Show that if $\mathcal{A}$ and $\mathcal{B}$ are two-dimensional operators, $\left(\mathcal{A}\,\mathcal{B}\right)^{\dagger} = \mathcal{B}^{\dagger}\,\mathcal{A}^{\dagger}$.

This problem proves a useful result. Problem 12 is a more elegant reduction of this fact for unitary operators using operator algebra. Use

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

for instance. Simply form the matrix products of the left and right side of $\left(\mathcal{A}\,\mathcal{B}\right)^{\dagger} = \mathcal{B}^{\dagger}\,\mathcal{A}^{\dagger}$, the adjoint of the product and the product of the adjoints in the reverse order, compare them, and they will be the same. Again, this proof can be easily extended to higher dimension.

$$\left(\mathcal{A}\,\mathcal{B}\right)^{\dagger} = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right]^{\dagger} = \left[\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}\right]^{\dagger}$$

$$= \begin{pmatrix} a_{11}^*b_{11}^* + a_{12}^*b_{21}^* & a_{21}^*b_{11}^* + a_{22}^*b_{21}^* \\ a_{11}^*b_{12}^* + a_{12}^*b_{22}^* & a_{21}^*b_{12}^* + a_{22}^*b_{22}^* \end{pmatrix}.$$

The product of the adjoints in the opposite order is

$$\mathcal{B}^{\dagger}\,\mathcal{A}^{\dagger} = \left[\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right]^{\dagger}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]^{\dagger} = \begin{pmatrix} b_{11}^* & b_{21}^* \\ b_{12}^* & b_{22}^* \end{pmatrix}\begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}^*b_{11}^* + a_{12}^*b_{21}^* & a_{21}^*b_{11}^* + a_{22}^*b_{21}^* \\ a_{11}^*b_{12}^* + a_{12}^*b_{22}^* & a_{21}^*b_{12}^* + a_{22}^*b_{22}^* \end{pmatrix}.$$

This is, in fact, the same as the first expansion, therefore $\left(\mathcal{A}\,\mathcal{B}\right)^{\dagger} = \mathcal{B}^{\dagger}\,\mathcal{A}^{\dagger}$.

12. (a) Show that the adjoint of a non-singular unitary operator is equal to its inverse.

(b) Show that the product of unitary operators is unitary.

(c) Use operator algebra to show that if $\mathcal{A}$ and $\mathcal{B}$ are unitary, $\left(\mathcal{A}\,\mathcal{B}\right)^{\dagger} = \mathcal{B}^{\dagger}\,\mathcal{A}^{\dagger}$.

This problem is intended to provide the opportunity to use operator algebra. All three parts are intended to be completed without the use of an explicit matrix. You can operate on both sides of an equation with the same operator if you operate from the same side. Operation must be from the left on both sides or from the right on both sides because operators do not generally commute. Operator algebra allows insertion of the operator $\mathcal{I}$ in any circumstance that is convenient.

Part (a) is an application of the definitions of unitary and inverse relations, $\mathcal{A}\,\mathcal{A}^{\dagger} = \mathcal{I}$, and $\mathcal{A}\,\mathcal{A}^{-1} = \mathcal{I}$, for instance. To show that the product of any two unitary operators is unitary, start with the product of two arbitrary unitary operators, say $\mathcal{A}\,\mathcal{B} = \mathcal{C}$. Operate on both sides with appropriate operators to show that $\mathcal{C}\,\mathcal{C}^{\dagger} = \mathcal{I}$. Remember that you have $\left(\mathcal{A}\,\mathcal{B}\right)^{\dagger} = \mathcal{B}^{\dagger}\mathcal{A}^{\dagger}$ from problem 11. Part (c) is a more elegant determination of this fact for unitary operators. Start with $\mathcal{A}\,\mathcal{A}^{\dagger} = \mathcal{I}$ and insert the identity between the two operators. It is likely $\mathcal{B}\,\mathcal{B}^{\dagger}$ will be a convenient form for this problem. You will likely need the result of part (b) to complete part (c).

(a) If $\mathcal{A}$ is unitary, $\mathcal{A}\,\mathcal{A}^{\dagger} = \mathcal{I}$ and $\mathcal{A}\,\mathcal{A}^{-1} = \mathcal{I}$ for all non-singular operators, so

$$
\begin{aligned}
\mathcal{A}\,\mathcal{A}^{\dagger} &= \mathcal{A}\,\mathcal{A}^{-1} && \text{because both are equal to } \mathcal{I} \\
\mathcal{A}^{-1}\,\mathcal{A}\,\mathcal{A}^{\dagger} &= \mathcal{A}^{-1}\,\mathcal{A}\,\mathcal{A}^{-1} && \text{operate from the left with } \mathcal{A}^{-1} \\
\mathcal{I}\,\mathcal{A}^{\dagger} &= \mathcal{I}\,\mathcal{A}^{-1} && \text{because } \mathcal{A}^{-1}\,\mathcal{A} = \mathcal{I} \\
\mathcal{A}^{\dagger} &= \mathcal{A}^{-1} && \text{which is the desired result.}
\end{aligned}
$$

(b) If $\mathcal{A}$ and $\mathcal{B}$ are unitary, then $\mathcal{A}\,\mathcal{A}^{\dagger} = \mathcal{B}\,\mathcal{B}^{\dagger} = \mathcal{I}$. Consider their product

$$
\begin{aligned}
\mathcal{A}\;\mathcal{B} &= \mathcal{C} && \text{the product of two operators is an operator} \\
\mathcal{A}\mathcal{B}\,\mathcal{B}^{\dagger} &= \mathcal{C}\,\mathcal{B}^{\dagger} && \text{operate from the right with } \mathcal{B}^{\dagger} \\
\mathcal{A}\mathcal{I} = \mathcal{A} &= \mathcal{C}\,\mathcal{B}^{\dagger} && \text{because } \mathcal{B}\,\mathcal{B}^{\dagger} = \mathcal{I} \\
\mathcal{A}\,\mathcal{A}^{\dagger} &= \mathcal{C}\,\mathcal{B}^{\dagger}\,\mathcal{A}^{\dagger} && \text{operate from the right with } \mathcal{A}^{\dagger} \\
\mathcal{I} &= \mathcal{C}\,\mathcal{B}^{\dagger}\mathcal{A}^{\dagger} && \text{because } \mathcal{A}\,\mathcal{A}^{\dagger} = \mathcal{I} \\
\mathcal{I} &= \mathcal{C}\left(\mathcal{A}\mathcal{B}\right)^{\dagger} && \text{use of } \left(\mathcal{A}\mathcal{B}\right)^{\dagger} = \mathcal{B}^{\dagger}\mathcal{A}^{\dagger} \\
\mathcal{I} &= \mathcal{C}\,\mathcal{C}^{\dagger} && \text{definition of } \mathcal{C}
\end{aligned}
$$

or $\mathcal{C}\,\mathcal{C}^{\dagger} = \mathcal{I}$, and therefore, the product of unitary operators is unitary.

(c) Given that $\mathcal{A}$ and $\mathcal{B}$ are unitary,

$$\begin{aligned}
\mathcal{A}\mathcal{A}^\dagger &= \mathcal{I} && \text{definition of unitary} \\
\mathcal{A}\mathcal{I}\mathcal{A}^\dagger &= \mathcal{I} && \text{insertion of the identity} \\
\mathcal{A}\mathcal{B}\mathcal{B}^\dagger\mathcal{A}^\dagger &= \mathcal{I} && \text{use of the form } \mathcal{I} = \mathcal{B}\mathcal{B}^\dagger \\
(\mathcal{A}\mathcal{B})\,\mathcal{B}^\dagger\mathcal{A}^\dagger &= \mathcal{I} && \text{associative property} \\
(\mathcal{A}\mathcal{B})^\dagger\,(\mathcal{A}\mathcal{B})\,\mathcal{B}^\dagger\mathcal{A}^\dagger &= (\mathcal{A}\mathcal{B})^\dagger\,\mathcal{I} && \text{operation from the left with } (\mathcal{A}\mathcal{B})^\dagger \\
\mathcal{I}\,\mathcal{B}^\dagger\mathcal{A}^\dagger &= (\mathcal{A}\mathcal{B})^\dagger && \text{because } (\mathcal{A}\mathcal{B})^\dagger(\mathcal{A}\mathcal{B}) = \mathcal{I} \text{ from part (b)} \\
\mathcal{B}^\dagger\mathcal{A}^\dagger &= \big(\mathcal{A}\mathcal{B}\big)^\dagger && \text{or} \quad \big(\mathcal{A}\mathcal{B}\big)^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger\,.
\end{aligned}$$

---

13. Show that the determinant of any two-dimensional unitary matrix has unit modulus.

This problem establishes a property of unitary matrices just as problem 4 established three properties of determinants. This problem is fairly short and straightforward if you use the three results of problem 4 appropriately.

Show that $\mathcal{U}\mathcal{U}^\dagger = I$ implies $\det\big(\mathcal{U}\mathcal{U}^\dagger\big) = 1$. Then show that $\det\big(\mathcal{U}\mathcal{U}^\dagger\big) = \det\mathcal{U}\,\det\mathcal{U}^*$. Realize that $\det\mathcal{U}\,\det\mathcal{U}^*$ is the product of a complex number and its conjugate. Finally, remember that the modulus of a complex number is the square root of the product of the number and its complex conjugate.

For a unitary matrix,

$$\mathcal{U}\mathcal{U}^\dagger = \mathcal{I} \quad\Rightarrow\quad \det\big(\mathcal{U}\mathcal{U}^\dagger\big) = \det\mathcal{I} = 1\,,$$

so using previously developed results, we have

$$\begin{aligned}
\det\big(\mathcal{U}\mathcal{U}^\dagger\big) &= \det\mathcal{U}\,\det\mathcal{U}^\dagger && \text{from part (b) of problem 4} \\
&= \det\mathcal{U}\,\det\mathcal{U}^{T*} = \det\mathcal{U}\,\big(\det\mathcal{U}^T\big)^* && \text{from part (c) of problem 4} \\
&= \det\mathcal{U}\,\det\mathcal{U}^* = 1\,, && \text{from part (a) of problem 4}
\end{aligned}$$

The determinant is generally a complex scalar, so let $\det\mathcal{U} = a + ib \;\Rightarrow\; \det\mathcal{U}^* = a - ib$

$$\Rightarrow\quad \det\mathcal{U}\,\det\mathcal{U}^* = (a+ib)(a-ib) = a^2 + b^2 = 1 = |\det\mathcal{U}|^2\,,$$

$\Rightarrow\;\; \det\mathcal{U} = \sqrt{a^2+b^2} = 1$, therefore the determinant of a unitary matrix is of unit modulus.

---

14. Given the matrix operator $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$,

(a) show that $|\,2> = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

(b) and that $\,|\,3\!>\,=\begin{pmatrix}1\\2\end{pmatrix}$.

(c) Interpret these facts geometrically.

This problem is a numerical introduction to **eigenvalues** and **eigenvectors** and the **eigenvalue/eigenvector** equation. It also attempts to familiarize you with some popular notation.

The product of an operator and a vector is another vector, $\mathcal{A}\,|\,v\,> \;=\; |\,w\,>$. For every operator, there are products of the operator and special vectors such that the new vector is a product of a scalar and the original vector, *i.e.*,

$$\mathcal{A}\,|\,v_i\!> = \alpha_i\,|\,v_i\!> \;.$$

This eigenvalue/eigenvector equation actually describes a family of equations, thus subscripts are commonly used. There are as many matching scalars and vectors as the dimension of the space. The scalars are known as eigenvalues and the vectors are known as eigenvectors. It is popular to place the eigenvalue between the $|$ and $>$ that indicate the ket to identify the corresponding eigenvector. Thus, part (a) asks you to show that

$$\begin{pmatrix}1&1\\-2&4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix},$$

which is simply a numerical statement of the eigenvalue/eigenvector equation. Consider "length" and "direction" of the vectors for part (c).

(a) $\begin{pmatrix}1&1\\-2&4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1+1\\-2+4\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix}$, so for the operator $\mathcal{A}=\begin{pmatrix}1&1\\-2&4\end{pmatrix}$,

2 is an eigenvalue, and $|\,2\!>\,=\begin{pmatrix}1\\1\end{pmatrix}$ is the corresponding eigenvector.

(b) $\begin{pmatrix}1&1\\-2&4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1+2\\-2+8\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix}=3\begin{pmatrix}1\\2\end{pmatrix}$, so for the operator $\mathcal{A}=\begin{pmatrix}1&1\\-2&4\end{pmatrix}$,

3 is an eigenvalue, and $|\,3\!>\,=\begin{pmatrix}1\\2\end{pmatrix}$ is the corresponding eigenvector.

In a 2 dimensional space, we expect exactly two eigenvalues and two eigenvectors.

(c) In part (a), the effect of the operation is a vector twice as long and in exactly the same direction as the original vector. In part (b), the effect of the operation is a vector three times as long as the original vector and in exactly the same direction. An operator acting on an eigenvector results in changing the length of the vector without rotating it. The length of the eigenvector is changed by a factor equal to its eigenvalue.

Postscript: Physicists use numerous other methods of denoting eigenvectors other than by placing the eigenvalue between the $|$ and $>$ that indicate the eigenvector or **eigenket**. For instance, $|\,v_1\!>$

or $|\,1>$ might identify the first eigenvector and $|\,v_2>$ or $|\,2>$ might denote the second eigenvector. Quantum numbers are often placed between the $|$ and $>$ that indicate the eigenket.

Also, be aware that the eigenvalue/eigenvector equation always denotes a family of equations. Subscripts are not always used. There are as many eigenvalues and eigenvectors as the dimension of the operator under consideration.

---

15. (a) Interpret the operation of $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ on the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ geometrically.

(b) Show that the result of the operation of $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ on the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ can be expressed as a linear combination of the eigenvectors given in problem 14.

An operator operating on an eigenvector results in an elongation or contraction of the vector. An operator operating on a vector that is not an eigenvector also includes some sort of rotation. There is generally an elongation or contraction unless the operator is unitary, but if the vector is not an eigenvector, there will always be something equivalent to a rotation.

Part (a) is intended that you recognize these facts. For instance, you should recognize that

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = a\begin{pmatrix} -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 6 \end{pmatrix},$$

and interpret the result in a Cartesian coordinate system. Part (b) intends for you to solve

$$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \; b\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a)

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

which cannot be arranged into a scalar times the original vector. The new vector can be arranged

$$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 6\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

which is a scalar times the original vector plus another vector. Given Cartesian coordinates, the first vector represents an elongation by a factor of six in the original direction, and the second vector represents 6 steps in the x direction which is a clockwise rotation around the origin.

(b)

$$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 0 = a+b \\ 6 = a+2b \end{matrix}.$$

The top equation says $a = -b$, and using this in the bottom equation yields $b = 6$ so $a = -6$, and

$$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a linear combination of $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ in terms of the eigenvectors of the operator.

Postscript: Operation on an eigenvector is analogous to an elongation or contraction without a rotation. Rotation is included as part of any operation on a vector that is not an eigenvector.

The linearly independent eigenvectors of this operator constitute a basis in $\mathbf{R}^2$. We will have frequent occasion to use the eigenvectors of an operator as a basis.

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16. Find the eigenvalues and normalized eigenvectors of $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

The eigenvalue/eigenvector problem is one of the preeminent calculations in quantum mechanics.

Here is a roadmap to solve an eigenvalue/eigenvector problem.

1) Set $\det\left(\mathcal{A} - \alpha\mathcal{I}\right) = 0$. This is known as the **characteristic equation**.
2) Solve the characteristic equation. The solutions are the eigenvalues.
3) Use the eigenvalue/eigenvector equation to solve for the eigenvectors.
4) Normalize the eigenvectors.

There are some conventions associated with the **eigenvalue/eigenvector** problem. We will work from the eigenvalue of least magnitude to the eigenvalue of greatest magnitude. The simultaneous equations that result from the eigenvalue/eigenvector equation are generally indeterminate. In this circumstance, we choose the first non-zero element of the eigenket to be positive and real, most often 1. This is a convention that is popular but not universal[3].

$$\det\left(\mathcal{A} - \alpha\mathcal{I}\right) = \det\left[\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}\right] = \det\begin{pmatrix} 1-\alpha & 1 & 2 \\ 0 & 2-\alpha & 0 \\ 0 & 1 & 3-\alpha \end{pmatrix}$$

$$= (1-\alpha)(2-\alpha)(3-\alpha) + 0 + 0 - 0 - 0 - 0 = (1-\alpha)(2-\alpha)(3-\alpha)$$

is the **characteristic polynomial**. The characteristic equation is formed by setting the characteristic polynomial equal to zero, or

$$(1-\alpha)(2-\alpha)(3-\alpha) = 0$$

which has the solutions $\alpha_i = 1,\ 2,$ and 3. These are the eigenvalues. For the eigenvalue of least magnitude, $\alpha_1 = 1$, the eigenvalue/eigenvector equation is

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{ccccccc} a & + & b & + & 2c & = & a \\ & & 2b & & & = & b \\ & & b & + & 3c & = & c \end{array}$$

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[3] Shankar, *Principles of Quantum Mechanics* (Plenum Press, New York, 1994), 2nd ed., p. 34.

which is three equations in three unknowns. The middle equation implies $b = 0$. Using this in the bottom equation implies $c = 0$ also. Using both of these in the top equation yields $a = a$, which has an infinite number of solutions. This system is indeterminate. Following the convention of choosing the top element of the eigenket positive and real, we choose $a = 1$, so the eigenket is

$$|\,1\!> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for the eigenvalue } \alpha_1 = 1.$$

This choice also has the advantage that the eigenvector is already normalized. For $\alpha_1 = 2$,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{ccccccc} a & + & b & + & 2c & = & 2a \\ & & 2b & & & = & 2b \\ & & b & + & 3c & = & 2c \end{array}$$

The middle equation implies b is anything we want, and bottom equation indicates $b = -c$. Substituting $b = -c$ in the top equation yields $a = c$. We choose $a = 1$, which determines $c = 1$ and $b = -1$, so the eigenket is

$$|\,2\!> = A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{with a normalization constant attached. Normalizing}$$

$$\big(1,\ -1,\ 1\big) A^* A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = |\,A\,|^2\big(1+1+1\big) = 3\,|\,A\,|^2 = 1 \ \Rightarrow\ A = \frac{1}{\sqrt{3}} \ \Rightarrow\ |\,2\!> = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

for the eigenvalue $\alpha_2 = 2$. For the third eigenvalue

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{ccccccc} a & + & b & + & 2c & = & 3a \\ & & 2b & & & = & 3b \\ & & b & + & 3c & = & 3c \end{array}$$

The middle equation implies $b = 0$. Then the bottom equation implies $c =$ anything. We would like $a = 1$, which we can have if $c = 1$. The third eigenket is $|\,3\!> = A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Normalizing,

$$\big(1,\ 0,\ 1\big) A^* A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = |\,A\,|^2\big(1+0+1\big) = 2\,|\,A\,|^2 = 1 \ \Rightarrow\ A = \frac{1}{\sqrt{2}} \ \Rightarrow\ |\,3\!> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

---

17. What are the eigenvalues and eigenvectors for $\mathcal{L}_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$?

Use the procedures introduced in problem 16. The operator in this problem contains imaginary elements but is about the same degree of difficulty. There is another key difference that we will address in the postscript.

$$\det\left(\mathcal{L}_y - \alpha\mathcal{I}\right) = \det\begin{pmatrix} -\alpha & -i & 0 \\ i & -\alpha & -i \\ 0 & i & -\alpha \end{pmatrix} = (-\alpha)^3 - (-\alpha)(i)(-i) - (-i)(i)(-\alpha) = -\alpha^3 + \alpha + \alpha = 0$$

$$\Rightarrow \quad \alpha^3 - 2\alpha = 0 \quad \Rightarrow \quad \alpha(\alpha^2 - 2) = \alpha(\alpha - \sqrt{2}\,)(\alpha + \sqrt{2}\,) = 0 \quad \Rightarrow \quad \alpha = -\sqrt{2}\,,\ 0,\ \sqrt{2}$$

so $-\sqrt{2}\,,\ 0,$ and $\sqrt{2}$ are the eigenvalues. Calculating the eigenvectors starting with the smallest eigenvalue,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(-\sqrt{2}\,\right)\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} & -bi & & = & -\sqrt{2}\,a \\ ai & - & ci & = & -\sqrt{2}\,b \\ & bi & & = & -\sqrt{2}\,c \end{matrix}$$

Adding the top and bottom equation, we get $-\sqrt{2}\,a - \sqrt{2}\,c = 0 \quad \Rightarrow \quad a = -c$. Using this in the middle equation, we get $ai + ai = -\sqrt{2}\,b \quad \Rightarrow \quad -\dfrac{2ai}{\sqrt{2}} = b \quad \Rightarrow \quad b = -\sqrt{2}\,ai$. If we choose $a = 1$, then $b = -\sqrt{2}\,i$, and $c = -1$, so the eigenket is

$$|-\sqrt{2}> = A\begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix} \quad \Rightarrow <-\sqrt{2}\,|-\sqrt{2}> = \left(1,\ \sqrt{2}\,i,\ -1\right) A^* A \begin{pmatrix} 1 \\ -\sqrt{2}\,i \\ -1 \end{pmatrix} = \left|\, A \,\right|^2 \left(1+2+1\right)$$

$$= 4\left|\, A\,\right|^2 = 1 \quad \Rightarrow \quad |-\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2}\,i \\ -1 \end{pmatrix} \quad \text{for the eigenvalue} \quad -\sqrt{2}\,.$$

To find the eigenvector corresponding to the next smallest eigenvalue 0,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} & -bi & & = & 0 \\ ai & - & ci & = & 0 \\ & bi & & = & 0 \end{matrix}$$

The top and bottom equation say $b = 0$, and the middle equation says $a = c$, so by convention, $a = 1 \quad \Rightarrow \quad c = 1$, and the eigenket is

$$|\,0> = A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad <0\,|\,0> = \left(1,\ 0,\ 1\right) A^* A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \left|\,A\,\right|^2\left(1+0+1\right) = 2\left|\,A\,\right|^2 = 1$$

$$\Rightarrow \quad |\,0> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for the eigenvalue} \ \ 0\,.$$

For the eigenvector corresponding to the largest eigenvalue,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} & -bi & & = & \sqrt{2}\,a \\ ai & - & ci & = & \sqrt{2}\,b \\ & bi & & = & \sqrt{2}\,c \end{matrix}$$

Adding the top and bottom equation, we get $\sqrt{2}\,a + \sqrt{2}\,c = 0 \quad \Rightarrow \quad a = -c\,.$ Using this in the middle equation, we get

$$ai + ai = \sqrt{2}\,b \quad \Rightarrow \quad \frac{2ai}{\sqrt{2}} = b \quad \Rightarrow \quad b = \sqrt{2}\,ai\,.$$

If we choose $a = 1\,,$ then $b = \sqrt{2}\,i,$ and $c = -1\,,$ the eigenket is

$$|\,\sqrt{2}> = A\begin{pmatrix} 1 \\ \sqrt{2}\,i \\ -1 \end{pmatrix} \quad \Rightarrow \quad <\sqrt{2}\,|\,\sqrt{2}> = \big(\,1,\ -\sqrt{2}\,i,\ -1\big)\,A^*A\begin{pmatrix} 1 \\ \sqrt{2}\,i \\ -1 \end{pmatrix} = |\,A\,|^2\big(1+2+1\big)$$

$$= 4\,|\,A\,|^2 = 1 \quad \Rightarrow \quad |\,\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix} \quad \text{for the eigenvalue } \sqrt{2}\,.$$

Postscript: Notice that $\mathcal{L}_y$ in this problem is Hermitian, and that operator $\mathcal{A}$ of problem 16 is not Hermitian. Hermitian operators have two particularly desirable properties.

(1) The eigenvalues of Hermitian operators are real numbers, and

(2) the eigenvectors of Hermitian operators are orthogonal,

thus can be made orthonormal by the process of normalization. Property (1) is a necessity because any physical observable, for instance mass, charge, energy, momentum, or position, is quantified using a real number. Measurements in the real world require real numbers as outcomes. Property (2) is valuable because orthonormality is a necessity for a quantum mechanical basis. Use the procedures of problem 15 from part 1 to show that the eigenvectors of $\mathcal{L}_y$ are orthogonal, (or orthonormal if you include the normalization constants), if you want. Hermitian operators are central to quantum mechanical calculation because Hermitian operators represent all physically observable quantities.

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18. What are the eigenvalues and eigenvectors for $\mathcal{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$?

The operator $\mathcal{D}$ is a diagonal operator. All non-zero elements are on the principal diagonal and all elements that are not on the principal diagonal are zero. Diagonal operators have some valuable properties. This problem intends to expose you to these properties. Solve the eigenvalue/eigenvector equation in the traditional way. You will find that the eigenvalues are the elements on the principal diagonal and the eigenvectors are the unit vectors that correspond to the position of the eigenvalue. And, since they are unit vectors, the eigenvectors are orthonormal unit vectors that do not require normalization.

$$\det\Big(\mathcal{D} - \alpha\mathcal{I}\Big) = \det\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2-\alpha & 0 & 0 \\ 0 & 3-\alpha & 0 \\ 0 & 0 & 4-\alpha \end{pmatrix} = (2-\alpha)(3-\alpha)(4-\alpha)+0+0-0-0-0 = (2-\alpha)(3-\alpha)(4-\alpha)$$

$$\Rightarrow \quad (2-\alpha)(3-\alpha)(4-\alpha) = 0$$

is the characteristic equation, so the eigenvalues are $\alpha_i = 2,\ 3,$ and 4. The eigenvectors are

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 2a & = & 2a \\ 3b & = & 2b \\ 4c & = & 2c \end{array}$$

The middle and bottom equations tell us $b = c = 0$. We choose $a = 1$, so the eigenket is

$$|\,2> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for the eigenvalue} \quad \alpha_1 = 2\,.$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 2a & = & 3a \\ 3b & = & 3b \\ 4c & = & 3c \end{array}$$

The top and bottom equations tell us $a = c = 0$. We choose $b = 1$, so the eigenket is

$$|\,3> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{for the eigenvalue} \quad \alpha_2 = 3\,.$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 4 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 2a & = & 4a \\ 3b & = & 4b \\ 4c & = & 4c \end{array}$$

The top and middle equations tell us $a = b = 0$. We choose $c = 1$, so the eigenket is

$$|\,4> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for the eigenvalue} \quad \alpha_3 = 4\,.$$

Postscript: The eigenvalues and eigenvectors of a diagonal operator are found by inspection. Also, orthonormality of the eigenvectors is inherent. For instance,

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \Rightarrow \quad |-3> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\,1> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |\,5> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

without calculation. We would clearly prefer to work with diagonal operators if we had the opportunity. In fact, all Hermitian operators can be transformed into diagonal operators using a **unitary transformation**.

---

19. Diagonalize $\mathcal{L}_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$.

Unitary operators were introduced because they are central to the process of **diagonalization**. All Hermitian operators can be transformed into diagonal operators. Having the eigenvalues as the elements on the diagonal and the unit vectors as eigenvectors can be a significant convenience. The process of diagonalization can be accomplished using a **unitary transformation**. For an appropriate choice of $\mathcal{U}$, the transformation

$$\mathcal{U}^\dagger \mathcal{A} \mathcal{U} = \mathcal{A}'$$

yields an operator $\mathcal{A}'$ that is diagonal. The appropriate unitary operator is constructed from the eigenvectors of the operator to be diagonalized. There is, however, more than one way to arrange the eigenvectors to attain a unitary operator that will diagonalize the original operator. The convention that we will generally use is to keep the eigenvectors in columns, and place them in a matrix from left to right in the order of the vector corresponding to the lowest eigenvalue to the vector corresponding to the highest eigenvalue. For instance, given that the eigenvectors of $\mathcal{L}_y$ are

$$|-\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2}\,i \\ -1 \end{pmatrix}, \qquad |\,0> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad |\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2}\,i \\ -1 \end{pmatrix},$$

the unitary operator of interest is

$$\mathcal{U} = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -i\sqrt{2}/2 & 0 & i\sqrt{2}/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{pmatrix}.$$

$$\mathcal{U} = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -i\sqrt{2}/2 & 0 & i\sqrt{2}/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{pmatrix} \quad \Rightarrow \quad \mathcal{U}^\dagger = \begin{pmatrix} 1/2 & i\sqrt{2}/2 & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -i\sqrt{2}/2 & -1/2 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{U}^\dagger \mathcal{A} \mathcal{U} = \begin{pmatrix} 1/2 & i\sqrt{2}/2 & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -i\sqrt{2}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -i\sqrt{2}/2 & 0 & i\sqrt{2}/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & i\sqrt{2}/2 & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -i\sqrt{2}/2 & -1/2 \end{pmatrix} \begin{pmatrix} -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ i & 0 & i \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2}/4 - \sqrt{2}/2 - \sqrt{2}/4 & 0 & \sqrt{2}/4 - \sqrt{2}/2 + \sqrt{2}/4 \\ -1/2 + 1/2 & 0 & 1/2 - 1/2 \\ -\sqrt{2}/4 + \sqrt{2}/2 - \sqrt{2}/4 & 0 & \sqrt{2}/4 + \sqrt{2}/2 + \sqrt{2}/4 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

where the eigenvalues are on the principal diagonal in order of the smallest at the upper left to the largest at the lower right. The eigenvectors of this transformed operator are now the unit vectors, *i.e.*,

$$|-\sqrt{2}> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |0> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad |\sqrt{2}> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

A unitary transformation is a **change of basis**. It is another expression of the same operator in terms of a different set of linearly independent vectors. In a sense, our unitary transformation "rotates" the eigenvectors to align with the unit vectors. The transformed operator contains the same information as the original operator.

Of particular note is that *every Hermitian operator may be diagonalized by a unitary change of basis*, which is a result we will use, but will rely on other texts, such as Shankar[4], to support.

Our convention for forming a unitary operator is to keep the eigenvectors in columns, and place them in a matrix from left to right in the order of the eigenvectors that correspond to ascending eigenvalues. This convention aids organization and provides eigenvalues on the main diagonal in ascending order. There are other ways to form a unitary matrix that will diagonalize an operator. We will demonstrate an option in chapter 2.

---

20. Show that the eigenvalues of a matrix are invariant under a unitary transformation.

The postulates of quantum mechanics indicate that the eigenvalues of any operator are the only possible result of any measurement. Eigenvalues, therefore, are of preeminent importance to quantum mechanical measurement. This problem is intended to amplify and further legitimize the unitary transformation of problem 19. "Rotating" the eigenvectors to align with unit vectors does not affect the eigenvalues.

If you can show that $\Omega' = \mathcal{U}^\dagger \Omega \mathcal{U}$ and Ω have the same characteristic equation, then you are essentially done since the respective eigenvalues are the roots of identical characteristic equations. Start by letting $\Omega' = \mathcal{U}^\dagger \Omega \mathcal{U}$, a statement that says only that an operator subject to a unitary transformation is another operator. Work to justify $\Omega' - \omega \mathcal{I} = \mathcal{U}^\dagger (\Omega - \omega \mathcal{I}) \mathcal{U}$. We will insert an identity of the form $\mathcal{I} = \mathcal{U}^\dagger \mathcal{U}$ and use some of the operator algebra developed previously to attain this intermediate result. Remember that a determinant is a scalar, and that scalars commute with anything. Part (b) of problem 4 is pertinent. Identity operators also commute with anything. Conclude that $\det\big(\Omega' - \omega \mathcal{I}\big) = \det\big(\Omega - \omega \mathcal{I}\big)$ and reiterate that respective eigenvalues are the roots of identical characteristic equations.

Consider $\Omega' = \mathcal{U}^\dagger \Omega \mathcal{U}$ where $\mathcal{U}$ is unitary. The characteristic equation is formed by setting the determinant $\det(\Omega' - \omega I) = \det(\mathcal{U}^\dagger \Omega \mathcal{U} - \omega I)$ equal to zero. Consider first

$$\begin{aligned} \Omega' - \omega \mathcal{I} &= \mathcal{U}^\dagger \Omega \mathcal{U} - \omega \mathcal{I} = \mathcal{U}^\dagger \Omega \mathcal{U} - \omega \mathcal{I} \mathcal{I} \\ &= \mathcal{U}^\dagger \Omega \mathcal{U} - \omega \mathcal{I} \mathcal{U}^\dagger \mathcal{U} = \mathcal{U}^\dagger \Omega \mathcal{U} - \mathcal{U}^\dagger \omega \mathcal{I} \mathcal{U} = \mathcal{U}^\dagger (\Omega - \omega \mathcal{I}) \mathcal{U}. \end{aligned}$$

[4] Shankar, *Principles of Quantum Mechanics* (Plenum Press, New York, 1994), 2nd ed., p. 40.

Since these matrices are equal, their determinants are equal. So

$$\begin{aligned}
\det\left(\Omega' - \mathcal{I}\,\omega\right) &= \det\left[\mathcal{U}^{\dagger}\left(\Omega - \mathcal{I}\,\omega\right)\mathcal{U}\,\right] = \det\left(\mathcal{U}^{\dagger}\right)\det\left(\Omega - \mathcal{I}\,\omega\right)\det\left(\mathcal{U}\right) \\
&= \det\left(\mathcal{U}^{\dagger}\right)\det\left(\mathcal{U}\right)\det\left(\Omega - \mathcal{I}\,\omega\right) = \det\left(\mathcal{U}^{\dagger}\mathcal{U}\right)\det\left(\Omega - \mathcal{I}\,\omega\right) \\
&= \det\left(\mathcal{I}\right)\det\left(\Omega - \mathcal{I}\,\omega\right) = \left(1\right)\det\left(\Omega - \mathcal{I}\,\omega\right) = \det\left(\Omega - \mathcal{I}\,\omega\right) \\
\text{therefore}\quad \det\left(\mathcal{U}^{\dagger}\Omega\,\mathcal{U} - \mathcal{I}\,\omega\right) &= \det\left(\Omega - \mathcal{I}\,\omega\right).
\end{aligned}$$

Since the determinants of $\Omega - \mathcal{I}\,\omega$ and $\Omega' - \mathcal{I}\,\omega = \mathcal{U}^{\dagger}\Omega\,\mathcal{U} - \mathcal{I}\,\omega$ are equal, their characteristic equations are identical, and consequently, the eigenvalues of the operators Ω and $\Omega' = \mathcal{U}^{\dagger}\Omega\,\mathcal{U}$ are identical since the respective eigenvalues are the roots of identical characteristic equations. Therefore, the eigenvalues of an operator are not changed by a unitary change of basis.

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21. Consider the two operators $\mathcal{A} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

(a) Are they Hermitian?

(b) Do they commute?

(c) Find the eigenvalues and eigenvectors of $\mathcal{A}$.

(d) Find $\mathcal{U}^{\dagger}\mathcal{A}\,\mathcal{U}$ where $\mathcal{U}$ is constructed from the eigenvectors of $\mathcal{A}$. Is it diagonal and what are its eigenvectors?

(e) Find $\mathcal{U}^{\dagger}\mathcal{B}\,\mathcal{U}$ using the same $\mathcal{U}$ constructed for part (d). Is it diagonal and what are its eigenvectors?

This is a toy problem meant to introduce a concept. This problem illustrates feasible mathematical mechanics to **simultaneously diagonalize** two matrix operators. Physics will likely never ask you to do such a calculation. Quantum mechanics, however, demands that you grasp the underlying concept. Simultaneous diagonalization of more than one operator is the mechanism that leads to a **complete set of commuting observables** required to describe many realistic physical systems.

All physical observables are represented by Hermitian matrices. Thus, part (a) asks you to check that $\mathcal{A} = \mathcal{A}^{\dagger}$, for instance. As stated in the last problem, every Hermitian operator can be diagonalized by a unitary change of basis, therefore, both of these Hermitian matrices can be diagonalized. When can they be simultaneously diagonalized? The answer is when they commute! *Two Hermitian operators that commute will both be diagonalized by the same unitary transformation.* This means that they share a common **eigenbasis**. Given the appropriate rotation, the eigenvalues do not change but both operators will have the same eigenvectors! The common eigenbasis for Hermitian matrices will be unit vectors. Part (b) asks you to check that the commutator, $[\,\mathcal{A},\ \mathcal{B}\,] = \mathcal{A}\,\mathcal{B} - \mathcal{B}\,\mathcal{A} = 0$. You will find that these two operators commute.

You need to solve the eigenvalue/eigenvector problem for one of the two operators. The eigenvectors of either operator are satisfactory to form an appropriate unitary matrix. We arbitrarily choose to solve the eigenvalue/eigenvector problem for $\mathcal{A}$ in part (c) of this problem. The eigenvalues of $\mathcal{A}$ are 2, 3, and 4. Use the procedures of problem 20 to form your unitary matrix. You will find that both $\mathcal{U}^{\dagger}\mathcal{A}\,\mathcal{U}$ and $\mathcal{U}^{\dagger}\mathcal{B}\,\mathcal{U}$ are diagonal operators in parts (d) and (e).

(a) Both operators are Hermitian, because $\mathcal{A} = \mathcal{A}^\dagger$ and $\mathcal{B} = \mathcal{B}^\dagger$.

(b) The commutator is

$$\left[\mathcal{A}, \mathcal{B}\right] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+1 & 0+0+0 & -3+0+0 \\ 0+0+0 & 0-9+0 & 0+0+0 \\ 0+0-3 & 0+0+0 & 1+0+0 \end{pmatrix} - \begin{pmatrix} 0+0+1 & 0+0+0 & 0+0-3 \\ 0+0+0 & 0-9+0 & 0+0+0 \\ -3+0+0 & 0+0+0 & 1+0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -3 \\ 0 & -9 & 0 \\ -3 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -3 \\ 0 & -9 & 0 \\ -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \cdot \mathcal{I}\,, \quad \text{or just } 0\,,$$

therefore $\mathcal{A}$ and $\mathcal{B}$ commute. This means that there is a basis of common eigenvectors that simultaneously diagonalize both $\mathcal{A}$ and $\mathcal{B}$.

(c) The unitary transformation to this basis can be constructed from the eigenvectors of either operator, so we arbitrarily pick $\mathcal{A}$ and find its eigenvalues and eigenvectors.

$$\det\left(\mathcal{A} - \mathcal{I}\,\alpha\right) = \begin{pmatrix} 3-\alpha & 0 & -1 \\ 0 & 3-\alpha & 0 \\ -1 & 0 & 3-\alpha \end{pmatrix} = (3-\alpha)^3 + 0 + 0 - 0 - 0 - (3-\alpha) = 0$$

$$\Rightarrow \quad 27 - 27\alpha + 9\alpha^2 - \alpha^3 - 3 + \alpha = 0 \quad \Rightarrow \quad \alpha^3 - 9\alpha^2 + 26\alpha - 24 = 0$$

$$\Rightarrow \quad (\alpha-2)(\alpha-3)(\alpha-4) = 0 \quad \Rightarrow \quad \alpha = 2,\ 3,\ \text{and } 4 \text{ are the eigenvalues of } \mathcal{A}\,.$$

Next, find the eigenvectors.

$$\alpha = 2 \quad \Rightarrow \quad \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 3a - c &=& 2a \\ 3b &=& 2b \\ -a + 3c &=& 2c \end{array} \quad \Rightarrow \quad \begin{array}{rcl} a &=& c \\ b &=& 0 \\ a &=& c \end{array}$$

Choose $a = 1 \ \Rightarrow \ c = 1$, and $b = 0$, so $\;|\,2\!> = A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $<\!2\,|\,2\!> = 1$ means that

$$\left(1,\ 0,\ 1\right)A^* A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = |\,A\,|^2(1+0+1) = 2\,|\,A\,|^2 = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad |\,2\!> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\alpha = 3 \quad \Rightarrow \quad \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 3a - c &=& 3a \\ 3b &=& 3b \\ -a + 3c &=& 3c \end{array} \quad \Rightarrow \quad \begin{array}{rcl} c &=& 0 \\ b &=& 1 \\ a &=& 0 \end{array}$$

$$|\,3\!> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{which is a unit vector so is already normalized.}$$

$$\alpha = 4 \quad \Rightarrow \quad \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 4 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 3a - c &=& 4a \\ 3b &=& 4b \\ -a + 3c &=& 4c \end{array} \quad \Rightarrow \quad \begin{array}{rcl} -c &=& a \\ b &=& 0 \\ -a &=& c \end{array}$$

Choose $a = 1 \ \Rightarrow \ c = -1$, and $b = 0$, so $|\,4\!> \, = A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $<\!4\,|\,4\!> \, = 1$ means that

$$(1,\ 0, -1) A^* A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = |\,A\,|^2 (1+0+1) = 2\,|\,A\,|^2 = 1 \ \Rightarrow \ A = \frac{1}{\sqrt{2}} \ \Rightarrow \ |\,4\!> \, = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(d) Form $\mathcal{U}$ from the eigenvectors of $\mathcal{A}$ by placing them in a matrix from left to right in order of ascending eigenvalue.

$$\mathcal{U} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \quad \Rightarrow \quad \mathcal{U}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad \mathcal{U}^\dagger \mathcal{A} \mathcal{U} &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3/\sqrt{2} - 1/\sqrt{2} & 0 & 3/\sqrt{2} + 1/\sqrt{2} \\ 0 & 3 & 0 \\ -1/\sqrt{2} + 3/\sqrt{2} & 0 & -1/\sqrt{2} - 3/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2/\sqrt{2} & 0 & 4/\sqrt{2} \\ 0 & 3 & 0 \\ 2/\sqrt{2} & 0 & -4/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2/2 + 2/2 & 0 & 4/2 - 4/2 \\ 0 & 3 & 0 \\ 2/2 - 2/2 & 0 & 4/2 + 4/2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \end{aligned}$$

This operator is diagonal. The eigenvalues on the principal diagonal are as calculated in part (c), but the eigenvectors are now

$$|\,2\!> \, = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\,3\!> \, = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |\,4\!> \, = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(e) Applying the same transformation to $\mathcal{B}$,

$$\begin{aligned} \mathcal{U}^\dagger \mathcal{B} \mathcal{U} &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -3 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -1/2-1/2 & 0 & 1/2-1/2 \\ 0 & -3 & 0 \\ -1/2+1/2 & 0 & 1/2+1/2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a diagonal matrix. The eigenvalues and corresponding eigenvectors are

$$|-1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-3> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |1> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Postscript: Notice that after the unitary transformations are completed the common eigenvectors of both operators are the unit vectors.

Suppose that we did part (e) before part (d). Given that we solved the eigenvalue/eigenvector problem for $\mathcal{A}$ in part (c) so have the eigenvalues, the diagonal form of $\mathcal{A}$ can be constructed by inspection. Do you understand why?

The diagonal operator for $\mathcal{B}$ has eigenvalues on the principal diagonal but they are not in ascending order. That is because the unitary matrix was formed from the eigenvectors of $\mathcal{A}$. If the unitary matrix was formed from the eigenvectors of $\mathcal{B}$, the eigenvalues of $\mathcal{B}$ would be in ascending order and the eigenvalues of the transformed $\mathcal{A}$ would appear in the order 3, 2, 4.

The key thing to take from this problem is that any two Hermitian operators that commute have a common set of eigenvectors so can be simultaneously diagonalized.

---

22. Given that

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and that the state function is the superposition} \quad |\psi> = \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

what are the possible results of a measurement of $\mathcal{D}$?

This problem introduces substance that is the reason for the next two problems. It informally introduces two of the postulates of quantum mechanics. The primary reason for this lengthy mathematics preliminary is to make the postulates of quantum mechanics and the calculations that follow from them accessible.

The eigenvalues and corresponding eigenvectors of $\mathcal{D}$ are

$$|1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |3> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The first postulate of quantum mechanics indicates that any possible state vector is a linear combination of the eigenvectors, *i.e.*,

$$|\psi> = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where α, β, and γ are scalars. This type of linear combination is known as a **superposition**. A state vector is a linear combination or superposition of the eigenvectors. The third postulate of quantum mechanics indicates that the only possible results of a measurement are the eigenvalues of the operator that represents the observable. In general, the possible results of a measurement of $\mathcal{D}$ are 1, 2, and 3. The third postulate also indicates, however, that the state vector must contain the eigenvector in order to measure the corresponding eigenvalue. The given state vector contains only two of the three eigenvectors, so only two of the three eigenvalues are possible results of a measurement.

Possible outcomes of a measurement are 2, the eigenvalue corresponding to the second eigenvector, and 3, the eigenvalue corresponding to the third eigenvector. A measurement with a result of 1 is impossible, because the state vector does not contain the eigenvector that corresponds to the eigenvalue 1.

Postscript: The condition that the state vector does not contain the first eigenvector is equivalent to $\alpha = 0$. We will examine the meaning of the coefficients α, β, and γ in chapter 2.

We have examined only operators that have the property that there is one eigenvector for each eigenvalue to this point. Consider the operator $\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This has eigenvalues of 1, 1, and 2. An eigenvalue is repeated. The eigenvectors are

$$|\,1\!> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\,1\!> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |\,2\!> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\mathcal{A}$ is Hermitian so could represent a measurable quantity. Only 1 or 2 are possible outcomes of a measurement of $\mathcal{A}$ because those are its only eigenvalues. But two eigenvectors are associated with the eigenvalue 1. If the outcome of a measurement is 1, to which eigenvector does this correspond? It is impossible to tell. This is an example of **degeneracy**. An operator that has an eigenvalue corresponding to two or more eigenvectors is known as degenerate.

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23. Is $\mathcal{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ degenerate?

Solve the characteristic equation. If $\mathcal{B}$ has any repeated eigenvalues, then that eigenvalue will correspond to more than one eigenvector and $\mathcal{B}$ is degenerate.

$$\det\left[\mathcal{B} - \alpha\,\mathcal{I}\right] = \det\begin{pmatrix} 1-\alpha & 0 & 1 \\ 0 & 2-\alpha & 0 \\ 1 & 0 & 1-\alpha \end{pmatrix} = (1-\alpha)^2(2-\alpha) - (2-\alpha)$$

$$= (1-2\alpha+\alpha^2)(2-\alpha) - (2-\alpha) = (1-2\alpha+\alpha^2-1)(2-\alpha) = (-2\alpha+\alpha^2)(2-\alpha) \;\;\Rightarrow\;\; -\alpha\,(2-\alpha)^2 = 0$$

(because $(-2\alpha+\alpha^2) = -\alpha\,(2-\alpha)\,)$, is the characteristic equation. The eigenvalues are 0, 2, and 2. The eigenvalue 2 corresponds to more than one eigenstate, so the operator $\mathcal{B}$ is degenerate.

Postscript: Notice that we have referred to an eigenvector as an **eigenstate**. The two terms are used interchangeably. Since the superposition of eigenvectors is known as the "state vector," the term eigenstate frequently preferred within the realm of physics.

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24. Simultaneously diagonalize $\mathcal{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

The procedures to this problem parallel problem 22 except that operator $\mathcal{A}$ is degenerate. Both operators are Hermitian. The next question is do they commute? Check to see if these operators commute and they will... because this is a non-problem if they do not commute. Operators that do not commute do not have a common eigenbasis so cannot be diagonalized simultaneously. Solve the eigenvalue/eigenvector problem for operator $\mathcal{B}$. The choice was arbitrary in problem 22. It is not arbitrary in this problem because $\mathcal{A}$ is degenerate. Solve the eigenvalue/eigenvector problem for $\mathcal{B}$ because it is the non-degenerate operator. Procedures for forming the unitary matrix and doing the unitary transformations are the same as problem 22.

$$\begin{aligned}\left[\mathcal{A},\mathcal{B}\right] &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

These operators commute so they can be simultaneously diagonalized. $\mathcal{A}$ is degenerate from problem 23, so solve the eigenvalue/eigenvector problem for $\mathcal{B}$ and hope that it is not degenerate.

$$\det\begin{pmatrix} -\alpha & 1 & 0 \\ 1 & -\alpha & 1 \\ 0 & 1 & -\alpha \end{pmatrix} = (-\alpha)^3 - (-\alpha) - (-\alpha) = -\alpha^3 + 2\alpha \quad\Rightarrow\quad -\alpha^3 + 2\alpha = 0$$

$$\Rightarrow\quad \alpha^3 - 2\alpha = 0 \quad\Rightarrow\quad \alpha\,(\alpha^2-2) = 0 \quad\Rightarrow\quad \alpha\,(\alpha+\sqrt{2}\,)(\alpha-\sqrt{2}\,) = 0$$

so the eigenvalues are $\alpha = -\sqrt{2},\ 0,\ \sqrt{2}$. $\mathcal{B}$ is not degenerate. Therefore, $\mathcal{A}$ and $\mathcal{B}$ can be diagonalized simultaneously by forming the unitary matrix from the eigenvectors of $\mathcal{B}$. We need to find the eigenvectors.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(-\sqrt{2}\right)\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad\Rightarrow\quad \begin{aligned} b &= -\sqrt{2}\,a \\ a+c &= -\sqrt{2}\,b \\ b &= -\sqrt{2}\,c \end{aligned}$$

The top and bottom equations tell us $a = c$. Using this in the middle equation,

$$a + a = -\sqrt{2}\,b \quad\Rightarrow\quad b = -\frac{2}{\sqrt{2}}\,a \quad\Rightarrow\quad b = -\sqrt{2}\,a\,,$$

so choosing $a = 1$ determines b and c, or $|-\sqrt{2}> = A \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$. Normalizing,

$$\big(1,\ -\sqrt{2},\ 1\big) A^* A \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = 4\,\big|\,A\,\big|^2 = 1 \quad\Rightarrow\quad A = \frac{1}{2} \;\Rightarrow\; |-\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad\Rightarrow\quad \begin{array}{rcl} b & = & 0 \\ a+c & = & 0 \\ b & = & 0 \end{array}$$

$\Rightarrow$ $b = 0$ and $a = -c$, so choosing $a = 1$, $|\,0> = A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Normalizing,

$$\big(1,\ 0,\ -1\big) A^* A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \big|\,A\,\big|^2\big(1+0+1\big) = 2\,\big|\,A\,\big|^2 = 1 \;\Rightarrow\; A = \frac{1}{\sqrt{2}} \quad\Rightarrow\quad |\,0> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{2} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad\Rightarrow\quad \begin{array}{rcl} b & = & \sqrt{2}\,a \\ a+c & = & \sqrt{2}\,b\,. \\ b & = & \sqrt{2}\,c \end{array}$$

The top and bottom equations tell us $a = c$. Using this in the middle equation,

$$a + a = \sqrt{2}\,b \quad\Rightarrow\quad b = \frac{2}{\sqrt{2}}\,a \quad\Rightarrow\quad b = \sqrt{2}\,a\,,$$

so choosing $a = 1$ determines b and c, or $|\sqrt{2}> = A \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$. Normalizing,

$$\big(1,\ \sqrt{2},\ 1\big) A^* A \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \big|\,A\,\big|^2\big(1+2+1\big) = 4\,\big|\,A\,\big|^2 = 1 \;\Rightarrow\; A = \frac{1}{2} \quad\Rightarrow\quad |\sqrt{2}> = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Now that we have the eigenvectors, we can form the appropriate unitary operator,

$$\mathcal{U} = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix} \quad\Rightarrow\quad \mathcal{U}^\dagger = \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix}$$

$$\Rightarrow\quad \mathcal{U}^\dagger \mathcal{A}\,\mathcal{U} = \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix}\begin{pmatrix} 1/2+1/2 & 1/\sqrt{2}-1/\sqrt{2} & 1/2+1/2 \\ -2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 1/2+1/2 & 1/\sqrt{2}-1/\sqrt{2} & 1/2+1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2+2/2+1/2 & 0 & 1/2-2/2+1/2 \\ 1/\sqrt{2}-1/\sqrt{2} & 0 & 1/\sqrt{2}-1/\sqrt{2} \\ 1/2-2/2+1/2 & 0 & 1/2+2/2+1/2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We do not need to do the diagonalization for $\mathcal{B}$ because of our method of constructing the unitary matrix. Place the eigenvalues on the diagonal in ascending order, and the eigenvectors are the unit vectors by inspection, meaning

$$\mathcal{U}^\dagger \mathcal{B} \mathcal{U} = \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Postscript: The eigenvectors for both $\mathcal{A}$ and $\mathcal{B}$ are the unit vectors where the eigenvalues correspond to the unit eigenvector appropriate to that position. Look at what has developed. If we measure that $\mathcal{A}$ has the eigenvalue 2 and $\mathcal{B}$ has the eigenvalue $-\sqrt{2}$, we can ascertain the eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. If we measure that $\mathcal{A}$ has the eigenvalue 2 and $\mathcal{B}$ has the eigenvalue $\sqrt{2}$, the eigenvector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We can now tell which eigenvector corresponds to the eigenvalue of 2 for $\mathcal{A}$ by measuring both $\mathcal{A}$ and $\mathcal{B}$. We have removed the degeneracy, however, we must specify two eigenvalues to identify one eigenstate. That is one of the consequence of degeneracy.

The concept is much more important than the mathematical mechanics. This is a toy problem intended for illustration and exercise. Physical systems are commonly degenerate. The process used to remove the degeneracy in problem 24 is the same process used to remove degeneracy in realistic physical systems. We have developed what is known as a **complete set of commuting observables** for the operators of the last problem.

A complete set of commuting observables is used to remove a degeneracy. The first step in developing a complete set of commuting observables is to find a non-degenerate operator with which the degenerate operator commutes. The eigenvectors of the non-degenerate operator can be used to form a unitary matrix for unitary transformations that will simultaneously diagonalize both operators.

In the instance that $\mathcal{A}$ and $\mathcal{B}$ are both degenerate, it is necessary to find an operator $\mathcal{C}$ with which both $\mathcal{A}$ and $\mathcal{B}$ commute to develop a complete set of commuting observables. In this circumstance, three eigenvalues would be required to specify one eigenstate.

It is not a mathematical necessity to use the eigenvectors of the non-degenerate operator to form the unitary matrix, though it is a practical necessity. The indeterminacy in the eigenvector/eigenvalue problem of the degenerate operator make it difficult to attain a unitary matrix that leads to transformations that diagonalize operators other than itself.

---

25. Show that for any Hermitian operator Ω,

$$\det \Omega = \text{the product of the eigenvalues of } \Omega = \prod_{i=1}^{n} \omega_i \,.$$

Remember that any Hermitian operator can be diagonalized. That the determinant of a matrix is the same as the product of the eigenvalues is a fact that is fairly accessible if the matrix is diagonal.

You need to show that the determinant of an operator is invariant under a unitary transformation. You already know that the eigenvalues are not changed by a unitary change of basis. A unitary transformation diagonalizes Ω. So if you can show that the statement is true in the diagonal basis, then you are essentially done. The fact that the determinant of a matrix is the product of the eigenvalues is a simple consistency check for eigenvalue/eigenvector problems.

If Ω is Hermitian, there exists a unitary operator $\mathcal{U}$ such that $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal. That the eigenvalues of $\mathcal{U}^\dagger \Omega \mathcal{U}$ and Ω are the same has been demonstrated previously. Further, the determinants of $\mathcal{U}^\dagger \Omega \mathcal{U}$ and Ω are identical because

$$\det \mathcal{U}^\dagger \Omega \mathcal{U} = \det \mathcal{U}^\dagger \det \Omega \det \mathcal{U} = \det \mathcal{U}^\dagger \det \mathcal{U} \det \Omega = \det \left(\mathcal{U}^\dagger \mathcal{U} \right) \det \Omega = \left(1 \right) \det \Omega = \det \Omega \,.$$

Let $\Lambda = \mathcal{U}^\dagger \Omega \mathcal{U}$. Λ is diagonal because $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal. To find the eigenvalues,

$$\det \left(\mathcal{U}^\dagger \Omega \mathcal{U} - \mathcal{I}\omega \right) = \det \left(\Lambda - \omega I \right) = 0$$

$$\Rightarrow \quad \det \begin{pmatrix} \Lambda_{11} - \omega & 0 & 0 & \cdots & 0 \\ 0 & \Lambda_{22} - \omega & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_{33} - \omega & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_{nn} - \omega \end{pmatrix} = 0 \,.$$

This has the characteristic equation $\left(\Lambda_{11} - \omega \right) \left(\Lambda_{22} - \omega \right) \left(\Lambda_{33} - \omega \right) \cdots \left(\Lambda_{nn} - \omega \right) = 0$, which has n roots $\omega_i = \Lambda_{ii}$, $i = 1, 2, \ldots, n$. The elements on the principal diagonal of a diagonal matrix are the eigenvalues. Therefore, the form of $\det \Lambda$ or $\det U^\dagger \, \Omega \, U$ in the diagonal basis is

$$\det \Omega = \det \left(\mathcal{U}^\dagger \Omega \mathcal{U} \right) = \det \begin{pmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ 0 & 0 & \omega_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_n \end{pmatrix} = (\omega_1)(\omega_2)(\omega_3) \cdots (\omega_n) = \prod_{i=1}^{n} \omega_i \,. \quad \text{Q.E.D.}$$

Supplementary Problems

26. Show that

$$\mathcal{A} = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{is linear.}$$

Like problem 1, show that $\mathcal{A}$ commutes with scalars and that it distributes over vectors.

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27. Use explicit calculations to show that

(a) $\det\big(\mathcal{A}\mathcal{A}\big) = \det\mathcal{A}\det\mathcal{A}$, and

(b) $\det\big(\mathcal{A}\mathcal{B}\big) = \det\mathcal{A}\det\mathcal{B}$,

$$\text{for} \quad \mathcal{A} = \begin{pmatrix} 2 & i & 1 \\ -i & 1 & -i \\ 1 & i & 2 \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

This is a numerical exercise intended to reinforce the fact that the determinant of the products is the same as the product of the determinants. Simply form the appropriate determinants and products for the right and left sides of the equations, and they will be the same.

---

28. Verify that the operator $\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ is unitary.

This problem is practice in forming adjoints and should help solidify the meaning of a unitary matrix. $\mathcal{R}$ represents a rotation of $\pi/2$ around the x-axis in three dimensions. Calculate $\mathcal{R}^\dagger$ and $\mathcal{R}\mathcal{R}^\dagger$. You will find that $\mathcal{R}\mathcal{R}^\dagger = \mathcal{I}$, so can conclude that this matrix is unitary.

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29. Show that $\big(\mathcal{A} + \mathcal{B}\big)^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger$ in two dimensions.

That the adjoint of the sum is the same as the sum of the adjoints is not obvious but the fact is used frequently. The proof in two dimensions is simplest. The result extends to arbitrary or infinite dimension. Use

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Calculate the adjoint of the sum and the sum of the adjoints. They will be the same.

30. (a) Show that any two diagonal matrices commute.

(b) Show that a diagonal matrix does not commute with an arbitrary matrix.

The fact that diagonal operators commute is both a useful tool and another reason to diagonalize operators when the opportunity exists. Part (b) warns you that there is an end to this good fortune because you cannot generally assume commutivity. For part (a), explicitly multiply two diagonal matrices of arbitrary dimension. Your result should be another diagonal matrix. The elements of the resulting diagonal matrix will be the product of two numbers. Is each of these products the same if you reverse the order of the factors? Why? For part (b), explicitly multiply a matrix with all non-zero elements by a diagonal matrix. You should show that the off-diagonal elements are different when you change the order of matrix multiplication.

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31. Show that $\mathcal{U}\,\mathcal{U}^\dagger = \mathcal{I} \quad \Rightarrow \quad \mathcal{U}^\dagger \mathcal{U} = \mathcal{I}\,.$

This problem is an application of the insertion of the identity. Remember that both $\mathcal{U}\,\mathcal{U}^\dagger = \mathcal{I}$ and $\mathcal{U}^{-1}\,\mathcal{U} = \mathcal{I}\,.$ Assume that $\mathcal{U}$ is non-singular.

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32. Given that Ω and Λ are Hermitian, show that

(a) $\Omega\,\Lambda$ is not Hermitian,

(b) $\Omega\Lambda + \Lambda\Omega$ is Hermitian,

(c) $\big[\,\Omega,\,\Lambda\,\big]$ is anti-Hermitian, and

(d) $i\big[\,\Omega,\,\Lambda\,\big]$ is Hermitian.

This problem is designed to deepen your understanding of the properties of Hermitian operators. It should also provide a vehicle for use of commutators and some previous results. All parts use the result of problem 11. As an example of how you want to proceed, here is the solution to part (d). Hermitian for part (d) means $\big(\,i\,\big[\,\Omega,\,\Lambda\,\big]\,\big)^\dagger = i\,\big[\,\Omega,\,\Lambda\,\big]$. To show that this is true,

$$
\begin{aligned}
\big(\,i\,\big[\,\Omega,\,\Lambda\,\big]\,\big)^\dagger \;&=\; (i)^*\,\big(\,\Omega\,\Lambda - \Lambda\,\Omega\,\big)^\dagger \quad = -i\,\Big(\big(\,\Omega\,\Lambda\,\big)^\dagger - \big(\,\Lambda\,\Omega\,\big)^\dagger\Big)\\
&= -i\,\big(\,\Lambda^\dagger\,\Omega^\dagger - \Omega^\dagger\,\Lambda^\dagger\,\big) \;= -i\,\big(\,\Lambda\,\Omega - \Omega\,\Lambda\,\big)\\
&= -i\,\big(\,-\,\Omega\,\Lambda + \Lambda\,\Omega\,\big) \quad = \;\; i\,\big(\,\Omega\,\Lambda - \Lambda\,\Omega\,\big)\\
&= \;\; i\,\big[\,\Omega,\,\Lambda\,\big], \quad \text{so it is Hermitian.}
\end{aligned}
$$

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33. Consider the matrix $\quad \Omega = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$

(a) Is it Hermitian?

(b) Find its eigenvalues and normalized eigenvectors.

(c) Verify that $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal, where $\mathcal{U}$ is the matrix of eigenvectors of Ω.

The eigenvalue/eigenvector problem is one of the preeminent problems of quantum mechanics, so you need to understand it thoroughly. Part (a) is simply to recognize Hermiticity. Show that $\Omega = \Omega^\dagger$. Solve the eigenvalue/eigenvector problem. You should find

$$|-2> = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \qquad |0> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad |4> = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Use the convention of keeping the column vectors in columns and placing them in a matrix from left to right in the ascending order of the eigenvalues to form the unitary matrix. The unitary matrix that diagonalizes Ω is

$$\mathcal{U} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix}$$

using our convention. $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal, with eigenvalues from left to right in ascending order.

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34. Consider the matrix

$$\Omega = \begin{pmatrix} 0 & 4i & 0 \\ -4i & 0 & -3i \\ 0 & 3i & 0 \end{pmatrix}.$$

(a) Is it Hermitian?

(b) Find its eigenvalues and normalized eigenvectors.

(c) Verify that $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal, where $\mathcal{U}$ is the unitary matrix constructed from the eigenvectors of Ω.

The intent and procedures for this problem are essentially identical to the previous one, however, this operator has elements that are imaginary. The operator Ω is Hermitian. The eigenvalues are integral but the eigenvectors are slightly more challenging than the last problem. Using the convention that the leading non-zero component should be 1, you should find

$$|-5> = \frac{4}{5\sqrt{2}} \begin{pmatrix} 1 \\ i5/4 \\ 3/4 \end{pmatrix}, \qquad |0> = \frac{3}{5} \begin{pmatrix} 1 \\ 0 \\ -4/3 \end{pmatrix}, \qquad \text{and} \qquad |5> = \frac{4}{5\sqrt{2}} \begin{pmatrix} 1 \\ -i5/4 \\ 3/4 \end{pmatrix}.$$

The eigenvectors above lead to the unitary matrix

$$\mathcal{U} = \begin{pmatrix} 4/5\sqrt{2} & 3/5 & 4/5\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 3/5\sqrt{2} & -4/5 & 3/5\sqrt{2} \end{pmatrix}$$

if formed using our convention. The transformation $\mathcal{U}^\dagger \Omega \mathcal{U}$ does diagonalize this matrix.

35. Consider the matrix

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 4 & -i \\ 0 & i & 0 \end{pmatrix}.$$

(a) Is it Hermitian?

(b) Find its eigenvalues and normalized eigenvectors.

(c) Verify that $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal, where $\mathcal{U}$ is the unitary matrix constructed from the eigenvectors of Ω.

This problem has both real and imaginary elements. Again, this operator is Hermitian. You should find

$$|-1> = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 \\ -1/2 \\ i/2 \end{pmatrix}, \quad |0> = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2i \end{pmatrix}, \quad \text{and} \quad |5> = \frac{2}{\sqrt{30}} \begin{pmatrix} 1 \\ 5/2 \\ i/2 \end{pmatrix},$$

for eigenvalues and eigenvectors. These lead to the unitary matrix

$$\mathcal{U} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 0 & 5/\sqrt{30} \\ i/\sqrt{6} & -2i/\sqrt{5} & i/\sqrt{30} \end{pmatrix}$$

if formed using our convention. We have done some arithmetic so that denominators are common in each column. If you have the foresight to do this, you save having to find common denominators in the following steps. Again, the transformation $\mathcal{U}^\dagger \Omega \mathcal{U}$ must diagonalize this matrix.

36. Consider the operator $\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$

(a) Show that Ω is unitary.

(b) Show that Ω's eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$.

(c) Find the corresponding eigenvectors, and show that they are orthogonal.

(d) Verify that $\mathcal{U}^\dagger \Omega \mathcal{U}$ is diagonal, where $\mathcal{U}$ is the unitary matrix constructed from the eigenvectors of Ω.

You should proceed forward undaunted even though this operator has components that are trigonometric functions. You will probably want to use the quadratic formula to solve the eigenvalue problem. You may also want to recall Euler's equation $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$.

37. Show that $\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\mathcal{B} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ cannot be simultaneously diagonalized.

If they do not commute, then they cannot be simultaneously diagonalized because there is no common eigenbasis!

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38. Consider the operators $\mathcal{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

(a) Show that they are Hermitian.

(b) Show that they commute.

(c) Diagonalize $\mathcal{A}$ and $\mathcal{B}$ using a unitary matrix formed from the eigenvectors of $\mathcal{B}$.

The operators are Hermitian. Part (b) means to check if $\left[\mathcal{A}, \mathcal{B}\right] = 0$, and it does. Part (c) means solve the eigenvalue/eigenvector problem for $\mathcal{B}$ and form $\mathcal{U}$ from the eigenvectors of that operator so as to diagonalize both operators using the transformations $\mathcal{U}^\dagger \mathcal{A} \mathcal{U}$ and $\mathcal{U}^\dagger \mathcal{B} \mathcal{U}$. You want to form the unitary operator from the eigenvectors of $\mathcal{B}$ because $\mathcal{A}$ is degenerate.

The most important idea here is that when one operator is degenerate, simultaneous diagonalization is essentially a necessary procedure. It is the procedure used in problems where degeneracy occurs to attain a complete set of commuting operators. In fact, where two or more **quantum numbers** are required to specify an eigenstate, you can be sure that simultaneous diagonalization or an equivalent process has been accomplished.

If a degenerate operator is encountered, you can lift the degeneracy using simultaneous diagonalization by finding another non-degenerate operator with which it commutes, and using the eigenvectors of the non-degenerate operator to form a unitary transformation that diagonalizes both operators. If you have two operators which are both degenerate, you have to find a third operator that commutes with both of these that is not degenerate. That is why it takes at least three quantum numbers to specify an eigenstate for the hydrogen atom, two of the quantum numbers specify eigenstates of degenerate operators. The concept is more important than the mathematical mechanics, but you are not likely to comprehend the concept without mastering the math.

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39. Consider the two operators $\mathcal{A} = \begin{pmatrix} 4 & -i & 0 \\ i & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Show that they are Hermitian.

(b) Show that they commute.

(c) What are $\mathcal{A}$ and $\mathcal{B}$ in the eigenbasis in which both are diagonal?

Use the same procedures as the previous problem. These operators have both real and imaginary elements which makes this problem more challenging than the last one. One of the operators is degenerate. If the characteristic equation yields duplicate eigenvalues for the operator you choose, then you want to solve the eigenvalue/eigenvector equation for the other operator and use the eigenvectors of the non-degenerate operators to form your unitary matrix.

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40. Consider the operators $\mathcal{A} = \begin{pmatrix} 3 & i & 0 \\ -i & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ $\mathcal{B} = \begin{pmatrix} -1 & 2i & 0 \\ -2i & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. What are $\mathcal{A}$ and $\mathcal{B}$ in the eigenbasis in which both are diagonal?

This should look a lot like the previous two problems. These operators are Hermitian and they commute. You may have perceived that these properties are important. If you have not, perceive so now! Physical quantities are represented by Hermitian operators. Operators that commute have a common eigenbasis so can be simultaneously diagonalized. Pick one of the two operators and solve the eigenvalue/eigenvector problem. If it is not degenerate, you can form a unitary matrix from its eigenvectors and do a unitary transformation for both operators.
