Chapter 1

Who ever heard of rose petal soup? The soup de jour is rose petal soup. What kind of place is this? This is the place the client wanted the meet... so here we are... and no clam chowder on the menu... A faceless gray suit appeared in front of him. It sat down without a word and placed a slip of paper on the table. "Please enjoy your meal at the expense of our employer," were all the words from both men. He reached his hand to cover the paper and left it on the table under his palm while matching the gaze of his new non-acquaintance. The stare-down was worthy of two ten-year-old boys. The gray suit rose and disappeared. The uneasy feeling of unanticipated confrontation was balanced by the thought "Well..., I need the work." Lifting his hand unsteadily, he absorbed the large, bold words without picking the paper up. "What did de Broglie know?"

The Mathematics of Quantum Mechanics

Part 1, Objects and Arithmetic

The first part of chapter 1 introduces the objects and "arithmetic" of quantum mechanical calculation. You should finish this part with an appreciation of **scalars**, **vectors**, and **matrix operators**; and operations using these objects. You should understand the environment known as a **complex linear vector space** within which these objects and operations are described. You should leave part 1 with firm concepts of the process of **normalization**, and the properties of **linear independence** and **orthonormality**. The property of orthonormality is a practical necessity to any **basis** useful for quantum mechanical calculation. Finally, you should start part 2 with an understanding of **commutivity**. You should know which objects **commute** and which objects do not commute.

---

1. Given that $\alpha = 3 + 4i$, $\beta = 2 - 3i$, and $\gamma = 5e^{2+5i}$, find

(a) $\alpha + \beta$ and $\alpha - \beta$,

(b) the complex conjugates of α, β, and γ,

(c) $\alpha\,\beta$ and $\beta\,\alpha$, and compare the products. Then find

(d) $\alpha\,\alpha^*$, $\beta\,\beta^*$, and $\gamma\,\gamma^*$,

(e) and, $\alpha \div \beta$ and $\beta \div \alpha$.

A brief review of operations with complex numbers and complex conjugates is appropriate because the components and elements in the vectors and operators used to describe quantum mechanical phenomena are intrinsically complex. Complex numbers are added or subtracted by adding or subtracting real parts and imaginary parts. A complex conjugate is formed by multiplying the imaginary part by -1. Usually, the symbol "*" denotes a complex conjugate, *e.g.*, α^* denotes the complex conjugate of α. Multiplication of complex numbers is a form of binomial multiplication. Part (c) is a numerical example of **commutivity**. If two objects **commute**, their product is the same regardless of the order of multiplication, *i.e.*, if $a \cdot b = b \cdot a$, then a and b are said to commute. You will show that the product of complex conjugates is a real number in the next problem. This fact is used to divide complex numbers. "Division" of complex numbers is a form of rationalization. Multiplying the quotient desired by 1 in the form of the complex conjugate of the divisor divided by itself, *e.g.* $\dfrac{\alpha}{\beta} = \dfrac{\alpha}{\beta}\dfrac{\beta^*}{\beta^*}$, yields a scalar of the form $a + bi$, though the real and imaginary parts are often fractional.

(a)
$$\alpha + \beta = 3 + 4i + 2 - 3i = (3+2) + (4-3)i = 5 + i\,,$$
$$\alpha - \beta = 3 + 4i - (2 - 3i) = (3-2) + (4+3)i = 1 + 7i\,.$$

(b) $\alpha = 3+4i, \quad \beta = 2-3i, \quad \gamma = 5e^{2+5i} \quad \Rightarrow \quad \alpha^* = 3-4i, \quad \beta^* = 2+3i, \quad \text{and} \quad \gamma^* = 5e^{2-5i}.$

(c) $$\alpha\,\beta = (3+4i)(2-3i) = 3{\cdot}2 + 3(-3i) + 4i{\cdot}2 + 4i(-3i) = 6 - 9i + 8i + 12 = 18 - i\,.$$
$$\beta\,\alpha = (2-3i)(3+4i) = 2\cdot 3 + 2\cdot 4i - 3i\cdot 3 - 3i\cdot 4i = 6 + 8i - 9i + 12 = 18 - i\,.$$

These complex numbers commute. All complex numbers commute as you will show in problem 2.

(d) $$\alpha\,\alpha^* = (3+4i)(3-4i) = 3{\cdot}3 + 3(-4i) + 4i{\cdot}3 + 4i(-4i) = 9 - 12i + 12i + 16 = 25\,.$$
$$\beta\,\beta^* = (2-3i)(2+3i) = 2\cdot 2 + 2\cdot 3i - 3i\cdot 2 - 3i\cdot 3i = 4 + 6i - 6i + 9 = 13$$
$$\gamma\,\gamma^* = 5e^{2+5i}\cdot 5e^{2-5i} = 25e^{4+0i} = 25e^4\,.$$

These products of complex conjugates are real numbers. The products of complex conjugates are always real numbers which is another general result you will show in problem 2.

(e) $$\frac{\alpha}{\beta} = \frac{3+4i}{2-3i} = \frac{3+4i}{2-3i}\,\frac{2+3i}{2+3i} = \frac{3\cdot 2 + 3\cdot 3i + 4i\cdot 2 + 4i\cdot 3i}{13} = \frac{6 - 12 + 9i + 8i}{13} = -\frac{6}{13} + \frac{17}{13}i\,.$$
$$\frac{\beta}{\alpha} = \frac{2-3i}{3+4i} = \frac{2-3i}{3+4i}\,\frac{3-4i}{3-4i} = \frac{2\cdot 3 - 2\cdot 4i - 3i\cdot 3 + 3i\cdot 4i}{25} = \frac{6 - 12 - 8i - 9i}{25} = -\frac{6}{25} - \frac{17}{25}i\,.$$

Postscript: A **scalar** is defined as a tensor of zero order, and is invariant under rotation. An informal description is that any individual number is a scalar. A scalar may be real, imaginary, or complex. The first problem should reinforce the fact that the sum, difference, product, and quotient of two complex scalars is another scalar.

Observable and measurable quantities such as mass, charge, or energy are described using real scalars. Since the components and elements of vectors and operators used to describe quantum mechanical phenomena are intrinsically complex, calculations that predict measurable quantities must possess a mechanism that yields real numbers. This mechanism is embedded in the fact that the product of complex conjugates is a real number. Part (d) of the first problem provides three numerical examples of this fact.

---

2. Show in general that

(a) complex numbers commute under the operation of multiplication,

(b) the product of complex conjugates is the sum of the squares of the real and imaginary parts,

(c) and the product of complex conjugates is a real number.

Vectors and operators do not commute in general. Scalars do commute in general. This problem should fix this fact and two other general properties of complex numbers under the operation of multiplication. Assume general forms of complex numbers such as $a+bi$ and $c+di$ where a, b, c, and d are real numbers, then do the indicated multiplications.

(a) $$(a+bi)(c+di) = a\cdot c + a\cdot di + bi\cdot c + bi\cdot di = ac - bd + (ad+cb)i\,, \quad \text{and}$$

$$(c+di)(a+bi) = c\cdot a + c\cdot bi + di\cdot a + di\cdot bi = ac - bd + (ad+cb)i\,,$$

therefore $(a+bi)(c+di) = (c+di)(a+bi)$ and complex scalars commute in general. (b) $(a+bi)(a-bi) = a\cdot a + a(-bi) + bi\cdot a + bi(-bi) = a^2 - a\cancel{b}i + a\cancel{b}i + b^2 = a^2 + b^2$.

(c) Since a and b are real numbers, a^2 and b^2 are real numbers, and their sum is a real number.

---

3. If $\alpha = 3+4i$ and $|\,v> = \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix}$, find $\alpha\,|\,v>$.

The first postulate of quantum mechanics states that a system is represented by a **state vector**. Operations with vectors are therefore fundamental to quantum mechanical calculations. Three types of multiplication are possible with vectors. This problem addresses **scalar multiplication** of a vector. Vector-vector, operator-vector, and vector-operator multiplication are the other types of vector multiplication that we will encounter in part 1. To multiply a vector by a scalar, multiply each component of the vector by the scalar. A scalar times a vector is another vector. Symbolically,

$$\alpha\,|\,v> = \alpha \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\,\beta \\ \alpha\,\gamma \\ \alpha\,\delta \end{pmatrix}.$$

$$\begin{aligned} \alpha\,|\,v> &= (3+4i)\begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} = \begin{pmatrix} (3+4i)(2+i) \\ (3+4i)(1-3i) \\ (3+4i)(3-2i) \end{pmatrix} \\ &= \begin{pmatrix} 3\cdot 2 + 3\cdot i + 4i\cdot 2 + 4i\cdot i \\ 3\cdot 1 + 3(-3i) + 4i\cdot 1 + 4i(-3i) \\ 3\cdot 3 + 3(-2i) + 4i\cdot 3 + 4i(-2i) \end{pmatrix} = \begin{pmatrix} 6+3i+8i-4 \\ 3-9i+4i+12 \\ 9-6i+12i+8 \end{pmatrix} = \begin{pmatrix} 2+11i \\ 15-5i \\ 17+6i \end{pmatrix}. \end{aligned}$$

---

4. If $\gamma = 2i$, and $\mathcal{B} = \begin{pmatrix} -2-2i & 3-i & 1+i \\ -3+2i & -2+i & 3-i \\ -2i & -3-3i & 4 \end{pmatrix}$, find $\gamma\,\mathcal{B}$.

An **operator** or a **matrix operator** contains the same information as a differential operator. Dirac demonstrated the equivalence of the Schrodinger differential equation and Heisenberg matrix formulations of quantum mechanics. Both types of operators have advantages in given circumstances, though you will see more matrix operators than differential operators in part 1 because matrix operators provide clarity to the **eigenvalue/eigenvector** problem and favor the economy contained in **Dirac notation**.

To multiply a matrix operator by a scalar, multiply each element of the operator by the scalar. A scalar times an operator is another operator.

$$\alpha\,\mathcal{A} = \alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \alpha\, a_{11} & \alpha\, a_{12} & \alpha\, a_{13} \\ \alpha\, a_{21} & \alpha\, a_{22} & \alpha\, a_{23} \\ \alpha\, a_{31} & \alpha\, a_{32} & \alpha\, a_{33} \end{pmatrix}.$$

$$\gamma\,\mathcal{B} = 2i \begin{pmatrix} -2-2i & 3-i & 1+i \\ -3+2i & -2+i & 3-i \\ -2i & -3-3i & 4 \end{pmatrix} = \begin{pmatrix} 2i(-2-2i) & 2i(3-i) & 2i(1+i) \\ 2i(-3+2i) & 2i(-2+i) & 2i(3-i) \\ 2i(-2i) & 2i(-3-3i) & 2i(4) \end{pmatrix}$$

$$= \begin{pmatrix} -4i+4 & 6i+2 & 2i-2 \\ -6i-4 & -4i-2 & 6i+2 \\ 4 & -6i+6 & 8i \end{pmatrix} = \begin{pmatrix} 4-4i & 2+6i & -2+2i \\ -4-6i & -2-4i & 2+6i \\ 4 & 6-6i & 8i \end{pmatrix}.$$

---

5. Find (a) $|\,v\!> + \,|\,w\!>$ and (b) $|\,v\!> - \,|\,w\!>$ given that

$$|\,v\!> = \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} \quad \text{and} \quad |\,w\!> = \begin{pmatrix} -1-2i \\ -2-2i \\ 1+i \end{pmatrix}.$$

Add or subtract two vectors by adding or subtracting corresponding components. Symbolically,

$$|\,a\!> = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad |\,b\!> = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \Rightarrow \quad |\,a\!> \pm \,|\,b\!> = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}.$$

(a)
$$\begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} + \begin{pmatrix} -1-2i \\ -2-2i \\ 1+i \end{pmatrix} = \begin{pmatrix} 2+i-1-2i \\ 1-3i-2-2i \\ 3-2i+1+i \end{pmatrix} = \begin{pmatrix} 2-1+i-2i \\ 1-2-3i-2i \\ 3+1-2i+i \end{pmatrix} = \begin{pmatrix} 1-i \\ -1-5i \\ 4-i \end{pmatrix}.$$

(b)
$$\begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} - \begin{pmatrix} -1-2i \\ -2-2i \\ 1+i \end{pmatrix} = \begin{pmatrix} 2+i+1+2i \\ 1-3i+2+2i \\ 3-2i-1-i \end{pmatrix} = \begin{pmatrix} 2+1+i+2i \\ 1+2-3i+2i \\ 3-1-2i-i \end{pmatrix} = \begin{pmatrix} 3+3i \\ 3-i \\ 2-3i \end{pmatrix}.$$

Postscript: The two vectors being added need to have the same number of components. Addition for vectors possessing a dissimilar number of components is not defined. For instance, addition of a scalar and a vector is undefined. The sum of two vectors is another vector.

---

6. Find (a) $\mathcal{A}+\mathcal{B}$ and (b) $\mathcal{A}-\mathcal{B}$ given that

$$\mathcal{A}=\begin{pmatrix} -4+2i & 2+i & -1-3i \\ 6 & 2+3i & -3i \\ -3-i & 2-3i & -5 \end{pmatrix}, \quad \text{and} \quad \mathcal{B}=\begin{pmatrix} -2-2i & 3-i & 1+i \\ -3+2i & -2+i & 3-i \\ -2i & -3-3i & 4 \end{pmatrix}$$

The sum/difference of two matrix operators is the sum/difference of the corresponding elements.

(a) $$\begin{pmatrix} -4-2+2i-2i & 2+3+i-i & -1+1-3i+i \\ 6-3+2i & 2-2+3i+i & 3-3i-i \\ -3-i-2i & 2-3-3i-3i & -5+4 \end{pmatrix} = \begin{pmatrix} -6 & 5 & -2i \\ 3+2i & 4i & 3-4i \\ -3-3i & -1-6i & -1 \end{pmatrix}.$$

(b) $$\begin{pmatrix} -4+2+2i+2i & 2-3+i+i & -1-1-3i-i \\ 6+3-2i & 2+2+3i-i & -3-3i+i \\ -3-i+2i & 2+3-3i+3i & -5-4 \end{pmatrix} = \begin{pmatrix} -2+4i & -1+2i & -2-4i \\ 9-2i & 4+2i & -3-2i \\ -3+i & 5 & -9 \end{pmatrix}.$$

Postscript: Operators useful for quantum mechanical calculation are square matrices, *i.e.*, they have the same number of elements in columns as in rows. Operator addition is undefined unless the operators have identical configurations. You cannot add a 2 X 2 matrix to a 3 X 3 matrix, for instance. The sum of two matrix operators is another matrix operator. Addition of a scalar and an operator, and addition of a vector and an operator are undefined.

---

7. What are (a) $<v\,|$, (b) $<w\,|$, (c) $<v\,| + <w\,|$, and (d) $<v\,| - <w\,|$,

$$\text{given that} \quad |\,v> = \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} \quad \text{and} \quad |\,w> = \begin{pmatrix} -1-2i \\ -2-2i \\ 1+i \end{pmatrix}.$$

Dirac called a column vector a **ket**, denoted $|\,u>$, for instance. He called a row vector a **bra** and denoted it $<u\,|$, for example. The rationale behind these labels is explained in the next problem. A bra is "formed" from a ket by forming a row vector of the complex conjugates of the components of the column vector. There is a bra $<u\,|$ whose components correspond to the complex conjugates for every ket $|\,u>$ and a ket $|\,u>$ whose components correspond to the complex conjugates for every bra $<u\,|$. For example, to the ket

$$|\,u> = \begin{pmatrix} 2+5i \\ 7-3i \\ -4 \\ 6i \end{pmatrix} \quad \text{there corresponds a bra} \quad <u\,| = \big(2-5i,\ 7+3i,\ -4,\ -6i\big),$$

and to the bra $\quad < u \,| \, = \big(-4 - 3i, \; 8i, \; -3 \big)$ there corresponds a ket $\quad | \, u > \, = \begin{pmatrix} -4+3i \\ -8i \\ -3 \end{pmatrix}.$

The **state vectors** of quantum mechanics are kets so kets are the more frequently encountered type of vector and we "form" the corresponding bra when it is needed. This practical procedure is technically incorrect. A technically correct picture is that the corresponding bra already exists among all other possible bras and we simply pick it from all other possible bras when it is needed. We will soon introduce the idea of **Hilbert space**, in which this technically correct concept is founded. There is no preference of kets over bras or bras over kets. You can form bras from kets or kets from bras as a practical matter, though we will have more occasion to form bras from kets because state vectors are kets.

Arithmetic operations previously described for kets apply to bras, where only the format is different since bras are vectors in a different format than kets. Adding a row vector to another row vector results in a row vector, for instance.

(a) $\quad < v \,| \, = \big(2 - i, \; 1 + 3i, \; 3 + 2i \big)$ $\qquad$ (b) $\quad < w \,| \, = \big(-1 + 2i, \; -2 + 2i, \; 1 - i \big)$

(c) $\quad < v \,| + < w \,| \, = \big(2 - 1 - i + 2i, \;\; 1 - 2 + 3i + 2i, \;\; 3 + 1 + 2i - i \big) = \big(1 + i, \; -1 + 5i, \; 4 + i \big)$

(d) $\quad < v \,| - < w \,| \, = \big(2 + 1 - i - 2i, \;\; 1 + 2 + 3i - 2i, \;\; 3 - 1 + 2i + i \big) = \big(3 - 3i, \; 3 + i, \; 2 + 3i \big)$

Postscript: A bra is the vector analog of the complex conjugate of a scalar.

---

8. Using $| \, v >$ and $| \, w >$ from problem 7, find (a) $< v \,|\, v >$, (b) $< w \,|\, w >$, (c) $< v \,|\, w >$, and (d) $< w \,|\, v >$.

Vector-vector multiplication is defined only for a row vector and a column vector with the same number of components. Vector-vector multiplication is not defined for a row vector and a column vector with different numbers of components, for two row vectors, or for two column vectors. The order of the vectors is important. Here we address only the vector-vector product where the row vector is on the left and the column vector is on the right. This order of multiplication is known as an **inner product**. An inner product is a generalization of the dot product of classical mechanics.

The procedure is to sum the product of corresponding components, meaning

$$\big(a_1, \; a_2 \big) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 \qquad \text{or} \qquad \big(a_1, \; a_2, \; a_3 \big) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \, .$$

The procedure generalizes to arbitrary or infinite dimension. An inner product is a scalar.

$$\begin{aligned} \big(5 + 2i, \; -3, \; 2i, \; 1 - 4i \big) \begin{pmatrix} -3 - 2i \\ -4i \\ 3 \\ 3 + 3i \end{pmatrix} &= (5 + 2i)(-3 - 2i) + (-3)(-4i) + 2i \cdot 3 + (1 - 4i)(3 + 3i) \\ &= (-15 - 10i - 6i + 4) + (12i) + (6i) + (3 + 3i - 12i + 12) \\ &= -11 - 16i + 12i + 6i + 15 - 9i = 4 - 7i. \end{aligned}$$

An inner product is a bra times a ket. If we were to denote the vectors of the last example as

$$<v\,| = \big(5+2i,\ -3,\ 2i,\ 1-4i\big) \qquad \text{and} \qquad |\,w> = \begin{pmatrix} -3-2i \\ -4i \\ 3 \\ 3+3i \end{pmatrix}, \quad \text{then} \quad <v\,|\,w> = 4-7i\,.$$

An expression with a bra on the left and a ket on the right is known as a **bracket** or **braket**. The inner product is the first example of a braket. One vertical slash serves to separate the bra and the ket for an inner product. Parts (a) and (b) are easier if you remember that the product of complex conjugates is the sum of the squares of the real and the imaginary parts.

(a) $$<v\,|\,v> = \big(2-i,\ 1+3i,\ 3+2i\big)\begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} = 4+1+1+9+9+4 = 28\,.$$

(b) $$<w\,|\,w> = \big(-1+2i,\ -2+2i,\ 1-i\big)\begin{pmatrix} -1-2i \\ -2-2i \\ 1+i \end{pmatrix} = 1+4+4+4+1+1 = 15\,.$$

(c) $$<v\,|\,w> = \big(2{-}i,\ 1{+}3i,\ 3{+}2i\big)\begin{pmatrix} -1-2i \\ -2-2i \\ 1+i \end{pmatrix} = (2{-}i)(-1{-}2i) + (1{+}3i)(-2{-}2i) + (3{+}2i)(1{+}i)$$

$$= (-2-4i+i-2) + (-2-2i-6i+6) + (3+3i+2i-2) = -4-3i+4-8i+1+5i = 1-6i\,.$$

(d) $$<w\,|\,v> = \big(-1{+}2i,\ -2{+}2i,\ 1{-}i\big)\begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} = (-1{+}2i)(2{+}i){+}(-2{+}2i)(1{-}3i){+}(1{-}i)(3{-}2i)$$

$$= (-2-i+4i-2) + (-2+6i+2i+6) + (3-2i-3i-2) = -4+3i+4+8i+1-5i = 1+6i\,.$$

Postscript: Notice that the inner product of the example and parts (a) through (d) are all scalars. You want to remember that an inner product is a scalar. Notice also that the results of parts (a) and (b) are real scalars. You may also want to notice that the results of parts (c) and (d) are complex conjugates. Also, the inner product is only the first example of a braket. Another object, for instance an operator, can be placed between the bra and the ket in a braket.

---

9. Find the norm of each ket given in problem 7.

The length of a classical vector, a vector in three dimensions with real components, is the square root of its dot product. The **norm** of a vector is a generalization of this concept of length to both an unspecified number of dimensions and complex components. The norm of a vector is the square root of the inner product of the ket and the corresponding bra. A norm is denoted by placing a vertical slash on each side of the vector symbol. For example,

$$|\,u\!> = \begin{pmatrix} -3-2i \\ -4i \\ 3 \\ 3+3i \end{pmatrix} \quad\Rightarrow\quad <\!u\,|\,u\!> = \big(-3+2i,\ 4i,\ 3,\ 3-3i\big)\begin{pmatrix} -3-2i \\ -4i \\ 3 \\ 3+3i \end{pmatrix}$$

$$= 9+4+16+9+9+9 = 56 \quad\Rightarrow\quad \Big|\,|\,u\!>\Big| = \sqrt{56} = 2\sqrt{14}\,.$$

$<\!v\,|\,v\!> = 28$ from problem 8, so $\big|\,|\,v\!>\big| = \sqrt{<\!v\,|\,v\!>} = \sqrt{28} = 2\sqrt{7}$. Similarly, $<\!w\,|\,w\!> = 15$ from problem 8, so $\big|\,|\,w\!>\big| = \sqrt{<\!w\,|\,w\!>} = \sqrt{15}$.

---

10. Normalize the vectors $|\,v\!>$ and $|\,w\!>$ given that

$$|\,v\!> = \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} \quad\text{and}\quad |\,w\!> = \begin{pmatrix} 4-i \\ -3+2i \end{pmatrix}.$$

A vector of norm 1 is said to be **normalized**. The process of normalization is geometrically equivalent to adjusting the norm of the vector to unit "length" while preserving the "direction." A vector is normalized by multiplying it by a scalar. The problem is to find this scale factor. If $|\,v'\!>$ is the vector to be normalized, attach an unknown scale factor A called a **normalization constant** to form $\mathrm{A}\,|\,v'\!>$. Form the corresponding bra remembering to conjugate the normalization constant. Set the inner product of these two vectors equal to one and solve for the normalization constant.

$$<\!v'\,|\,\mathrm{A}^*\,\mathrm{A}\,|\,v'\!> = 1 \quad\Rightarrow\quad |\,\mathrm{A}\,|^2<\!v'\,|\,v'\!> = 1 \quad\Rightarrow\quad |\,\mathrm{A}\,|^2 = \frac{1}{<\!v'\,|\,v'\!>} \quad\Rightarrow\quad \mathrm{A} = \frac{1}{\sqrt{<\!v'\,|\,v'\!>}}\,.$$

The normalized vector is $|\,v\!> = \mathrm{A}\,|\,v'\!> = \dfrac{1}{\sqrt{<\!v'\,|\,v'\!>}}\,|\,v'\!>$. The convention is to use the principal value of the square root. Another convention is to refer to any vector with the same direction by the same symbol so the primes are not usually included. The reason for this is the probabilistic interpretation of the state vector that we will encounter in chapter 2. Any vector with the same "direction" is equivalent for the purposes of quantum mechanics. The process of normalization is similar to the process of forming unit vectors in introductory mechanics.

$$\begin{aligned} <\!v\,|\,\mathrm{A}^*\,\mathrm{A}\,|\,v\!> &= (2-i,\ 1+3i,\ 3+2i)\,\mathrm{A}^*\,\mathrm{A}\begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} \\ &= |\,\mathrm{A}\,|^2\Big((4+1)+(1+9)+(9+4)\Big) \\ &= |\,\mathrm{A}\,|^2\Big(5+10+13\Big) = |\,\mathrm{A}\,|^2\,28 = 1 \end{aligned}$$

$$\Rightarrow \quad A = \frac{1}{\sqrt{28}} \quad \Rightarrow \quad |\,w\!> = \frac{1}{\sqrt{28}} \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix}$$

$$<\!w\,|\,A^*\,A\,|\,w\!> = \big(4+i,\ -3-2i\big)\,A^*\,A \begin{pmatrix} 4-i \\ -3+2i \end{pmatrix} = |\,A\,|^2\big(16+1+9+4\big) = |\,A\,|^2\,30 = 1$$

$$\Rightarrow \quad A = \frac{1}{\sqrt{30}} \quad \Rightarrow \quad |\,w\!> = \frac{1}{\sqrt{30}} \begin{pmatrix} 4-i \\ -3+2i \end{pmatrix}.$$

Postscript: The **normalization condition** is $<v\,|\,v> = 1$. If $<v\,|\,v> \neq 1$, it is wise to normalize it before attempting further calculations. A vector that satisfies the normalization condition is the same as a vector whose components have absorbed the normalization constant, *i.e.*,

$$|\,v\!> = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} A\,a \\ A\,b \\ A\,c \end{pmatrix},$$ for instance. You should notice that

$$|\,v\!> = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad |\,v\!> = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} A\,a \\ A\,b \\ A\,c \end{pmatrix}$$ are equivalent for the purposes of quantum mechanics. The same symbol $|\,v\!>$ is used to denote both normalized and unnormalized vectors.

---

11. Argue that the set of integers constitutes a linear vector space and that the set of real numbers between -1 and 1 does not constitute a linear vector space.

This problem is an introduction to a **complex linear vector space**. It should also introduce you to some notation appropriate to vector spaces. The fundamental concept is that of a **linear vector space** which is defined by the two properties of closure and the eight algebraic properties described below. A complex linear vector space is a linear vector space where the components of the vectors are complex numbers. This problem uses real numbers because they are the simplest. Scalars can be considered one dimensional vectors and integers are simply real scalars.

A **linear vector space** *is a set of vectors which is closed with respect to vector addition and scalar multiplication and obeys the axioms of addition and scalar multiplication in equations (3) through (10).*

Symbolically, if the linear vector space is denoted $\mathbf{C}$ and $|\,v\!>$ and $|\,w\!>$ are any vectors in the space, the properties of closure with respect to vector addition can be expressed

$$|\,v\!>,\ |\,w\!> \in \mathbf{C} \quad \Rightarrow \quad |\,v\!> + \,|\,w\!> \in \mathbf{C}, \tag{1}$$

and if α is any scalar, the property of closure with respect to scalar multiplication is

$$|\,v\!> \in \mathbf{C} \quad \Rightarrow \quad \alpha\,|\,v\!> \in \mathbf{C}. \tag{2}$$

A linear vector space needs to obey the axioms of equations (3) through (10). The definition of addition must satisfy commutivity in addition, equation (3); associativity in addition, equation (4); the existence of an additive identity, equation (5); and the existence of an additive inverse, equation (6). Symbolically,

$$|\,v\!>+\;|\,w\!> \,=\, |\,w\!>+\;|\,v\!>, \tag{3}$$

$$\Big(\,|\,v\!>+\;|\,w\!>\Big)+|\,z\!> \,=\, |\,w\!>+\Big(\,|\,v\!>+\;|\,z\!>\Big), \tag{4}$$

$$|\,v\!>+\;|\,0\!> \,=\, |\,v\!>, \tag{5}$$

$$|\,v\!>+\;|\,-v\!> \,=\, |\,0\!>. \tag{6}$$

The definition of scalar multiplication must satisfy the existence of a multiplicative identity, equation (7); distribution in vectors, equation (8); distribution in scalars, equation (9); and associativity in scalar multiplication, equation (10). Symbolically,

$$1\cdot|\,v\!> \,=\, |\,v\!>, \tag{7}$$

$$\alpha\Big(\,|\,v\!>+\;|\,w\!>\Big)=\alpha\,|\,v\!>+\,\alpha\,|\,w\!>, \tag{8}$$

$$\big(\alpha+\beta\big)\,|\,v\!>=\alpha\,|\,v\!>+\,\beta\,|\,v\!>, \tag{9}$$

$$\alpha\Big(\beta\,|\,v\!>\Big)=\alpha\,\beta\,|\,v\!>, \tag{10}$$

The set of numbers that are the components of the vectors that constitute the space is called a **field**. The numbers of interest to quantum mechanics are complex numbers. Quantum mechanics requires a linear vector space in the field of complex numbers, or a complex linear vector space. Complex linear vector spaces are subsets of linear vector spaces, and real numbers are a subset of the complex numbers, so sets of one dimensional vectors whose components are solely real numbers can be considered as long as the set satisfies conditions (1) through (10).

Since scalars can be regarded as one dimensional vectors, a simple example of a linear vector space is the set of integers. If we add or multiply any two integers, we attain another integer so the set is closed under addition and multiplication. Integers are commutative, associative, the set contains 0 so possesses an additive identity, and contains the negative numbers so also possesses additive inverses, thus satisfy all the required properties of addition. The set also satisfies the multiplicative properties. Since 1 is a member of the set, it possesses a multiplicative identity, integers are distributive (so satisfy the properties of equations (8) and (9)), and associative with respect to multiplication. All ten conditions are satisfied, so the set of integers is a linear vector space.

The inclusive set of numbers between -1 and 1 also does not satisfy the conditions required for a linear vector space. It is not closed under addition. For instance, $0.75 + 0.75 = 1.5$ is not a member of the set.

Postscript: Complex numbers are the field required for quantum mechanics calculations. Since we have defined addition in kets and scalar multiplication of kets, and we know the properties of complex numbers, the two properties of closure and the eight algebraic properties follow for vector addition and scalar multiplication for vectors with complex components. Of course, all the kets in any complex linear vector space must have the same number of components. For instance, all two dimensional kets are a linear vector space over the field of complex numbers. The set of all three dimensional kets is similarly a linear vector space over the field of complex numbers. The set that is the union of all two and three dimensional kets does not form a linear vector space because addition of a two and three dimensional ket is not defined, therefore, none of the additive properties of a linear vector space can be satisfied.

---

12. Consider the set of all entities of the form $<v\,| = (a,\ b,\ c)$, where a, b, and c are real numbers, and where vector addition and scalar multiplication are defined as follows:

$$(a,\ b,\ c) + (d,\ e,\ f\,) = (a + d,\ b + e,\ c + f\,)$$

$$\alpha\,(a,\ b,\ c) = (\alpha\, a,\ \alpha\, b,\ \alpha\, c)$$

where α is also a real number. Verify that this constitutes a linear vector space.

This problem is designed to develop your understanding of the definition of a linear vector space encountered in the previous problem.

You need to show that the two properties of closure, and the eight algebraic axioms are true for the rules of vector addition and scalar multiplication that are defined in the statement of the problem. You should first verify the two properties of closure using the fact that the sum or product of two real numbers will be another real number. Then, for example, to verify the axiom that scalar multiplication is distributive over vector addition, equation (8) from problem 11, you might write:

$$\begin{aligned}
\alpha\Big(<v\,| + <w\,|\Big) &= \alpha\Big((a,\ b,\ c) + (d,\ e,\ f\,)\Big)\\
&= \alpha\,(a + d,\ b + e,\ c + f\,)\\
&= \big(\alpha\,(a + d),\ \alpha\,(b + e),\ \alpha\,(c + f\,)\big)\\
&= (\alpha\, a + \alpha\, d,\ \alpha\, b + \alpha\, e,\ \alpha\, c + \alpha\, f\,)\\
&= (\alpha\, a,\ \alpha\, b,\ \alpha\, c) + (\alpha\, d,\ \alpha\, e,\ \alpha\, f\,)\\
&= \alpha\,(a,\ b,\ c) + \alpha\,(d,\ e,\ f\,)\\
&= \alpha <v\,| + \alpha <w\,|\,.
\end{aligned}$$

You should use the definitions in the statement of the problem forwards and backwards to verify the two properties of closure and other seven algebraic axioms analogously. We use the symbol $\mathbf{R}$ to denote the set of real numbers. You should also know that $|\,0>$ means zero appropriate to the space. For instance, $<0\,| = (0,\ 0,\ 0)$ for this problem.

To show that this is a linear vector space, we must show that the ten properties described in problem 11 are satisfied individually.

- Closure under vector addition, equation (1): Here,

$$<v\,| + <w\,| \ = (a,\ b,\ c) \ + \ (d,\ e,\ f\,) = (a + d,\ b + e,\ c + f\,)\,.$$

If $a, b, c, d, e, f \in \mathbf{R}$, then $a + d,\ b + e,\ c + f \in \mathbf{R}$, and the addition operation results in other elements in the field of real numbers. The resulting vector has three component that are real numbers so is a member of the three dimensional space. Therefore, this space is closed under vector addition.

- Closure under scalar multiplication, equation (2): $\alpha < v\,| = \alpha\,(a,\ b,\ c) = (\alpha\, a,\ \alpha\, b,\ \alpha\, c)$. If $\alpha \in \mathbf{R}$, and $a,\ b,\ c \in \mathbf{R}$, then the products $\alpha\, a,\ \alpha\, b,$ and $\alpha\, c \in \mathbf{R},$ and the operation of scalar multiplication results in other elements in the field of real numbers. The resulting vector has three elements that are real numbers, so is a member of the three dimensional space. Therefore, this space is closed under scalar multiplication.

- Commutivity in addition, equation (3):

$$\begin{aligned} <v\,| + <w\,| &= (a,\ b,\ c) + (d,\ e,\ f) \\ &= (a+d,\ b+e,\ c+f) \\ &= (d+a,\ e+b,\ f+c) \\ &= (d,\ e,\ f) + (a,\ b,\ c) \\ &= <w\,| + <v\,|, \end{aligned}$$

for all real numbers. Therefore, vector addition satisfies the property of commutivity in addition.

- Associativity in addition, equation (4):

$$\begin{aligned} \Big(<v\,| + <w\,| \Big) + <z\,| &= \Big((a,\ b,\ c) + (d,\ e,\ f) \Big) + (g,\ h,\ i) \\ &= (a+d,\ b+e,\ c+f) + (g,\ h,\ i) \\ &= (a+d+g,\ b+e+h,\ c+f+i) \\ &= (a,\ b,\ c) + (d+g,\ e+h,\ f+i) \\ &= (a,\ b,\ c)\ + \Big((d,\ e,\ f) + (g,\ h,\ i) \Big) \\ &= <v\,| + \Big(<w\,| + <z\,| \Big). \end{aligned}$$

Therefore, vector addition is associative.

- Existence of an additive identity, equation (5):

$$<v\,| + <0\,| = (a,\ b,\ c) + (0,\ 0,\ 0) = (a+0,\ b+0,\ c+0) = (a,\ b,\ c) = <v\,|,$$

therefore, an additive identity does exist.

- Existence of an additive inverse, equation (6):

$$<v\,| + <-v\,| = (a,\ b,\ c) + (a',\ b',\ c') = (a+a',\ b+b',\ c+c') = (0,\ 0,\ 0) = <0\,|$$

if and only if $a' = -a,\ \ b' = -b,\ \ c' = -c \ \ \Rightarrow\ \ <-v\,| = (-a,\ -b,\ -c)$. Therefore, there is a unique additive inverse.

- Existence of a multiplicative identity, equation (7):

$$1\cdot <v\,| = 1\cdot(a,\ b,\ c) = (1\cdot a,\ 1\cdot b,\ 1\cdot c) = (a,\ b,\ c) = <v\,|.$$

Therefore, a multiplicative identity does exist for scalar multiplication.

- Distribution in vectors, equation (8):

$$\begin{aligned}
\alpha\Big(<v\,|+<w\,|\Big) &= \alpha\Big((a,\ b,\ c)+(d,\ e,\ f)\Big)\\
&= \alpha\,(a+d,\ b+e,\ c+f)\\
&= \Big(\alpha\,(a+d),\ \alpha\,(b+e),\ \alpha\,(c+f)\Big)\\
&= (\alpha\,a+\alpha\,d,\ \alpha\,b+\alpha\,e,\ \alpha\,c+\alpha\,f)\\
&= (\alpha\,a,\ \alpha\,b,\ \alpha\,c)+(\alpha\,d,\ \alpha\,e,\ \alpha\,f)\\
&= \alpha\,(a,\ b,\ c)+\alpha\,(d,\ e,\ f)\\
&= \alpha<v\,|\ +\alpha<w\,|\,.
\end{aligned}$$

Therefore, scalar multiplication is distributive over vector addition.

- Distribution in scalars, equation (9):

$$\begin{aligned}
(\alpha+\beta)<v\,| &= (\alpha+\beta)\,(a,\ b,\ c)\\
&= \Big((\alpha+\beta)\,a,\ (\alpha+\beta)\,b,\ (\alpha+\beta)\,c\Big)\\
&= (\alpha\,a+\beta\,a,\ \alpha\,b+\beta\,b,\ \alpha\,c+\beta\,c)\\
&= (\alpha\,a,\ \alpha\,b,\ \alpha\,c)+(\beta\,a,\ \beta\,b,\ \beta\,c)\\
&= \alpha\,(a,\ b,\ c)+\beta\,(a,\ b,\ c)=\alpha<v\,|\ +\beta<v\,|\,.
\end{aligned}$$

Therefore, vector multiplication is distributive over scalar addition.

- Associativity in scalar multiplication, equation (10):

$$\alpha\,(\beta<v\,|\,)=\alpha\Big(\beta\,(a,\ b,\ c)\Big)=\alpha\,(\beta a,\ \beta b,\ \beta c)=(\alpha\,\beta\,a,\ \alpha\,\beta\,b,\ \alpha\,\beta\,c)=\alpha\,\beta\,(a,\ b,\ c)=\alpha\,\beta<v\,|\,.$$

Therefore, scalar multiplication is associative.

Since the given definitions satisfy all ten conditions over the field of real numbers, we conclude the definitions of addition and scalar multiplication constitute a linear vector space in $\mathbf{R}^3$.

Postscript: The symbol $\mathbf{R}^3$ means the space of vectors with three real components. The symbol $\mathbf{C}^3$ means the space of vectors with three complex components. The symbol $\mathbf{C}^\infty$ means the space of vectors that have an infinite number of complex components.

---

13. Use the definitions of vector addition and scalar multiplication given in problem 12 to show that the set of all vectors of the form $<u\,|\ =(a,\ b,\ 1)$ does not constitute a linear vector space.

This problem is a counterexample of a system that is not a linear vector space. It is intended to provide contrast and context to those systems that do form linear vector spaces. The set and definitions of vector addition and scalar multiplication must satisfy all ten properties to constitute a linear vector space. Check to see if this system is closed under vector addition; add two vectors of the form given. Does their sum have the same form?

For vectors of the form $<u\,| = (a,\ b,\ 1)$,

$$<v\,| + <w\,| = (a,\ b,\ 1) + (c,\ d,\ 1) = (a+c,\ b+d,\ 1+1) = (e,\ f,\ 2)$$

which is not of the form $(a,\ b,\ 1)$ because of the third component, *i.e.*, $2 \neq 1$, so the sum $(e,\ f,\ 2)$ is not contained within the space. Therefore, the space is not closed under vector addition, and this is sufficient to show that all vectors of the form $<u\,| = (a,\ b,\ 1)$ do not form a linear vector space using common definitions of addition and scalar multiplication.

---

14. (a) Show that $|\,u_1> = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $|\,u_2> = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and $|\,u_3> = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are linearly dependent.

(b) Show that $|\,v_1> = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $|\,v_2> = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $|\,v_3> = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ are linearly dependent.

(c) Show that $|\,w_1> = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $|\,w_2> = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $|\,w_3> = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are linearly independent.

A set of vectors is **linearly independent** if and only if $\sum_{i=1}^{n} \alpha_i\,|\,v_i> \neq |\,0>$ unless all $\alpha_i = 0$. Linear independence is very important to quantum mechanical calculation though it is rarely a focal issue because linear independence is usually assumed. In fact, we will see that linear independence is inherent in the first postulate of quantum mechanics.

Likely the easiest way to determine linear independence is proof by contradiction. For example, assume the set of vectors given in part (a) is linearly independent. Then

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. But the components form the simultaneous equations

$$\begin{aligned} \alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ \alpha_1 + 3\alpha_2 &= 0 \\ 2\alpha_2 + \alpha_3 &= 0 \end{aligned}$$

for which $\alpha_1 = 3$, $\alpha_2 = -1$ and $\alpha_3 = 2$ is a solution. This contradicts our assumption. Therefore, the given vectors are not linearly independent, or are **linearly dependent**.

(b) Assume the vectors are linearly independent. Then

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $a_1 = a_2 = a_3 = 0$ would be the only coefficients that could make this equation true. Performing the scalar multiplications, the above condition means

$$\begin{pmatrix} a_1 \\ a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ 0 \\ a_2 \end{pmatrix} + \begin{pmatrix} 3a_3 \\ 2a_3 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} a_1 + a_2 + 3a_3 = 0 \\ a_1 + 2a_3 = 0 \\ a_2 + a_3 = 0 \end{aligned}$$

that is, the individual components must sum to zero. This particular system has an infinite number of solutions such that $a_i \neq 0$, for example, $a_1 = 1$, $a_2 = 1/2$, and $a_3 = -1/2$. Therefore, in contradiction to the assumption, $|\, v_1 >$, $|\, v_2 >$, and $|\, v_3 >$ are linearly dependent.

(c) However, consider

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ 0 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ a_3 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} a_1 + a_2 &= 0 \\ a_1 + a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned} \qquad \Rightarrow \quad \begin{aligned} a_1 &= -a_2 \\ a_1 &= -a_3 \\ a_2 &= -a_3 \end{aligned} \qquad \Rightarrow \quad \begin{aligned} a_1 &= -a_2 \\ -a_2 &= -a_3 \\ a_3 &= -a_3 \end{aligned}$$

The last equation can be true if and only if $a_3 = 0 \;\Rightarrow\; a_2 = 0 \;\Rightarrow\; a_1 = 0$. Since all $a_i = 0$, $|\, w_1 >$, $|\, w_2 >$, and $|\, w_3 >$ are linearly independent.

Postscript: The definition of linear independence has an upper limit on the summation of n. The upper limit can be any number including ∞. In other words, though the vectors in this problem are in three dimensions, the concept generalizes to any including infinite dimensions.

The condition of linear independence means that you cannot form any of the other vectors in the set using a linear combination of two or more vectors in the set.

If the number of vectors in a linearly independent set is the same as the dimension of the linear vector space, all of the other vectors in the space can be formed using a linear combination of this basic set, or set of **basis** vectors. There are an infinite number of sets of vectors that will constitute a basis for any linear vector space. The choice of a basis set is often suggested by the context of the problem just as you might chose the unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$; or $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$, for a mechanics application based on the symmetry of the problem. When all of the other vectors in the space can be formed using a linear combination of a set of vectors, that set of vectors is said to **span** the space. All basis sets span the space for which they are a basis.

The number of components in a column or row vector effectively determines the dimension of the space for spaces composed of vectors as we have defined vectors. We will find that sets of objects other than vectors, operators for instance, can constitute a linear vector space. The dimension of a linear vector space is more generally defined by the maximum number of linearly independent objects in the space.

---

15. Which of the following sets of vectors are orthogonal?

$$\text{(a)} \quad |\, 1 > = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad |\, 2 > = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \qquad |\, 3 > = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(b) $|1> = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}, \quad |2> = \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix}, \quad |3> = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$

(c) $|1> = \begin{pmatrix} 1+i \\ i \\ -2-i \end{pmatrix}, \quad |2> = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad |3> = \begin{pmatrix} 1+i \\ 1 \\ -1 \end{pmatrix}$

(d) $|1> = \begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix}, \quad |2> = \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix}, \quad |3> = \begin{pmatrix} 1+i \\ 3 \\ -i \end{pmatrix}$

This exercise is designed to introduce the property of **orthogonality**. Orthogonality is a property that leads to **orthonormality**. Orthonormality is essential to any set of basis vectors that is used for quantum mechanical calculation.

The vectors $3\,\hat{\imath},\ 2\,\hat{\jmath},\ 4\,\hat{k}$ are mutually perpendicular. The fact that the dot product of any of these vectors with any other is zero demonstrates this property of mutual perpendicularity. The term "perpendicular" does not extend to more than three dimensions. The concept is generalized to higher dimensions using the higher dimensional analog of the dot product, the inner product. If the inner product of every vector in the set with every other vector in the set is zero, then that set of vectors is **orthogonal**.

To show that any of these sets of vectors are orthogonal, you must show that the inner product between any two members of the set is zero, *i.e.*, you must show that $<i\,|\,j> = 0$ for all $i \neq j$. If any of the $<i\,|\,j> \neq 0$, then the system is not orthogonal. For the three vectors given for part (c) for instance,

$$<1\,|\,2> = \begin{pmatrix} 1-i, & -i, & -2+i \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = (2-2i) \ + \ (i) \ + \ (-2+i) = 2-2-2i+i+i = 0\,,$$

so these two vectors are orthogonal. However,

$$<1\,|\,3> = \begin{pmatrix} 1-i, & -i, & -2+i \end{pmatrix} \begin{pmatrix} 1+i \\ 1 \\ -1 \end{pmatrix} = (1+1)+(-i)+(2-i) = 4-2i \neq 0\,,$$

so the set of three vectors given in part (c) is not orthogonal. Remember to conjugate the components when forming bras. By the way, if $<i\,|\,j> = 0$, then $<j\,|\,i> = 0$.

(a)
$$<1\,|\,2> = \begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 2-1-1 = 0$$

$$<1\,|\,3> = \begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0+1-1 = 0$$

$$<2\,|\,3> = \begin{pmatrix} 2, & -1, & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0 - 1 + 1 = 0$$

therefore, this set is orthogonal because $<i\,|\,j> = 0$.

(b) $$<1\,|\,2> = \begin{pmatrix} 1, & -i, & -i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix} = 1 - 2i + 1 = 2 - 2i \neq 0$$

therefore, this set is not orthogonal because $<1\,|\,2> \neq 0$.

(c) $<1\,|\,3> \neq 0$ and $<2\,|\,3> \neq 0$, therefore, this set is not orthogonal. Of course, either non-zero inner product is sufficient to show that the set is not orthogonal.

(d) $$<1\,|\,2> = \begin{pmatrix} 1, & 0, & 1-i \end{pmatrix} \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} = -1 + i - 0 + 1 - i = 0$$

$$<1\,|\,3> = \begin{pmatrix} 1, & 0, & 1-i \end{pmatrix} \begin{pmatrix} 1+i \\ 3 \\ -i \end{pmatrix} = 1 + i + 0 - i - 1 = 0$$

$$<2\,|\,3> = \begin{pmatrix} -1-i, & i, & 1 \end{pmatrix} \begin{pmatrix} 1+i \\ 3 \\ -i \end{pmatrix} = -1 - i - i + 1 + 3i - i = 0$$

therefore, this set is orthogonal because $<i\,|\,j> = 0$.

Postscript: A set of basis vectors that is useful for quantum mechanical calculation does not include a zero vector. All basis vectors must be non-zero.

As indicated, if $<i\,|\,j> = 0$, then $<j\,|\,i> = 0$. But if $<i\,|\,j> \neq 0$, then generally $<i\,|\,j> \neq <j\,|\,i>$.

None of the vectors given in this problem are normalized. A set of vectors that is both orthogonal and normalized has the property of **orthonormality**.

---

16. Show that the two dimensional systems

(a) $$|\,1> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\,2> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$ and

(b) $$|\,v> = \frac{1}{2} \begin{pmatrix} 1-i \\ \sqrt{2} \end{pmatrix} \qquad |\,w> = \frac{1}{2} \begin{pmatrix} -1+i \\ \sqrt{2} \end{pmatrix}$$ are orthonormal.

This problem is an introduction to **orthonormality**. If the inner product of dissimilar vectors is zero and the inner product of a ket with its bra is 1, then that set is **orthonormal**. In other words, an orthonormal set of vectors is both orthogonal and normalized. All orthogonal sets can be made orthonormal by the process of normalization. In fact, any linearly independent set of vectors can be made orthogonal and thus orthonormal. A basis must be orthonormal to be useful for quantum mechanics. The unit vectors $\hat{\imath}\,,\ \hat{\jmath}\,,\ \hat{k}\,,$ are orthonormal in three dimensions, for instance. Show that the inner product of each ket with its corresponding bra is 1, and that the inner product of the two vectors is zero. Symbolically, show that $<i\,|\,i> = 1$ and $<i\,|\,j> = 0\,.$

(a) The inner products of each vector with itself are

$$\big(1,\ 1\big)\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2\big(1+1\big) = \frac{1}{2}\,(2) = 1\,,$$

$$\big(1,-1\big)\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2\big(1+1\big) = \frac{1}{2}\,(2) = 1\,,$$

from which we conclude they are normalized, and the inner product of the two vectors is

$$\big(1,\ 1\big)\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2\big(1-1\big) = \frac{1}{2}\,(0) = 0\,,\quad \text{so the system is orthonormal.}$$

(b) The inner products of each vector with itself are

$$<v\,|\,v> = \big(1+i,\ \sqrt{2}\,\big)\,\frac{1}{2}\,\frac{1}{2}\begin{pmatrix}1-i\\ \sqrt{2}\end{pmatrix} = \frac{1}{4}\big(1+1+2\big) = 1\,,$$

$$<w\,|\,w> = \big(-1-i,\ \sqrt{2}\,\big)\,\frac{1}{2}\,\frac{1}{2}\begin{pmatrix}-1+i\\ \sqrt{2}\end{pmatrix} = \frac{1}{4}\big(1+1+2\big) = 1\,.$$

The inner product of the two vectors is

$$<v\,|\,w> = \big(1+i,\ \sqrt{2}\,\big)\,\frac{1}{2}\,\frac{1}{2}\begin{pmatrix}-1+i\\ \sqrt{2}\end{pmatrix} = \frac{1}{4}\big(-1+i-i-1+2\big) = 0\,,\quad \text{so the system is orthonormal.}$$

Postscript: The **orthonormality condition** is often expressed using the **Kronecker delta**. The Kronecker delta is defined

$$\delta_{ij} = \begin{cases} 1\,, & \text{for } i = j\,,\\ 0\,, & \text{for } i \neq j\,. \end{cases}$$

Using Dirac notation and the Kronecker delta, orthonormality is denoted

$$<i\,|\,j> = \delta_{ij}\,.$$

This problem demonstrates how a bra is formed from a ket that has a normalization constant. The normalization constant is placed to the right of a bra. If the normalization constant is complex, it must be conjugated in forming the bra.

---

17. Orthonormalize the vectors

$$|\,I> = \begin{pmatrix}1\\-3\end{pmatrix} \quad \text{and} \quad |\,II> = \begin{pmatrix}3\\4\end{pmatrix},$$

and check to ensure that your resulting vectors are orthonormal.

You can make an orthonormal basis from any linearly independent set of vectors using the **Gram-Schmidt orthonormalization procedure**. Given a linearly independent basis, the Gram-Schmidt orthonormalization procedure has you

(1) choose a vector $\mid I>$ at random and normalize it to attain the unit vector $\mid 1>$.

(2) Choose a second vector $\mid II>$ and subtract its projection on the first vector. An inner product times either unit vector is the projection in the direction of the unit vector, per figure 1–1. This means

$$\mid 2> \,=\, \mid II> - \mid 1><1 \mid II> \,. \qquad \text{Normalize this vector.}$$

Figure 1 − 1. The dot or inner product can be interpreted geometrically as the projection of one vector along the other. Multiply the projection by either normalized vector and you have a vector that is the projection of the other vector along it.

(3) Choose another vector $\mid III>$ and subtract its projection along the first two vectors, or

$$\mid 3> \,=\, \mid III> - \mid 1><1 \mid III> - \mid 2><2 \mid II> \,. \qquad \text{Normalize this vector.}$$

(4) Repeat the procedure until you exhaust all the vectors in the linearly independent basis.

The vectors

$$\mid I> \,=\, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{and} \quad \mid II> \,=\, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

are linearly independent. They can be made orthonormal using the Gram-Schmidt orthonormalization procedure. First, normalize one vector. Attach a normalization constant to $\mid I>$ and

$$<I \mid \mathrm{A}^*\mathrm{A} \mid I> \,=\, \big(1,\ -3\big)\,\mathrm{A}^*\,\mathrm{A}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \big|\,\mathrm{A}\,\big|^2\big(1+9\big) = \big|\,\mathrm{A}\,\big|^2\,10 = 1\,,$$

$$\Rightarrow \quad \mathrm{A} = \frac{1}{\sqrt{10}} \quad \Rightarrow \quad \mid 1> \,=\, \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

is normalized. Next subtract the projection of the second along the first,

$$\begin{aligned}
\mid 2> \,&=\, \mid II> - \mid 1><1 \mid II> \\
&= \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}\left[\big(1,\ -3\,\big)\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right]\frac{1}{\sqrt{10}} \\
&= \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} 1 \\ -3 \end{pmatrix}\big(3-12\big) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \frac{9}{10}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \\
&= \begin{pmatrix} 3+9/10 \\ 4-27/10 \end{pmatrix} = \begin{pmatrix} 39/10 \\ 13/10 \end{pmatrix} = \frac{13}{10}\begin{pmatrix} 3 \\ 1 \end{pmatrix},
\end{aligned}$$

where the square brackets in the second line are used only to indicate that we are going to do that vector multiplication first. Normalizing this vector,

$$\frac{13}{10}\left(3,\ 1\right) \mathrm{A}^* \frac{13}{10}\begin{pmatrix}3\\1\end{pmatrix} = \left|\,\mathrm{A}\,\right|^2 \frac{169}{100}\left(9+1\right) = \left|\,\mathrm{A}\,\right|^2 \frac{169}{10} = 1$$

$$\Rightarrow \quad \mathrm{A} = \frac{\sqrt{10}}{13} \quad \Rightarrow \quad |\,2> = \frac{\sqrt{10}}{13}\,\frac{13}{10}\begin{pmatrix}3\\1\end{pmatrix} = \frac{\sqrt{10}}{10}\begin{pmatrix}3\\1\end{pmatrix}$$

is the normalized vector. Here, we have exhausted all the vectors in the linearly independent, two dimensional basis, so have completed the Gram-Schmidt procedure.

Are these vectors now orthonormal? Checking

$$<1\,|\,2> = \left(1,\ -3\right) \frac{1}{\sqrt{10}}\,\frac{\sqrt{10}}{10}\begin{pmatrix}3\\1\end{pmatrix} = \frac{1}{10}\left(3-3\right) = \frac{1}{10}\left(0\right) = 0\,,$$

so they are orthogonal, and since they are normalized, they are also orthonormal.

Postscript: You should understand that any linearly independent set of vectors can be converted into an orthonormal basis, but we will never have occasion to employ the mathematical mechanics explicit in the Gram-Schmidt orthonormalization procedure. We will encounter linearly independent systems for which you now know that an orthonormal basis exists. The existence of the orthonormal basis provides the foundation for further calculation. The concept that any linearly independent set of vectors can be made into an orthonormal basis is the point of this problem, and the mathematical mechanics are a detail that is not particularly important to the physics.

---

18. (a) Show that the vectors

$$|\,I> = \begin{pmatrix}3\\0\\0\end{pmatrix}, \quad |\,II> = \begin{pmatrix}0\\1\\2\end{pmatrix}, \quad |\,III> = \begin{pmatrix}0\\2\\5\end{pmatrix} \quad \text{are linearly independent.}$$

(b) Show that the system of three vectors in part (a) is not orthogonal.

(c) Use the Gram-Schmidt process to form an orthonormal basis from the vectors in part (a).

(d) Show that the vectors resulting from part (c) are orthonormal.

This problem is an example of the fact that linear independence is a sufficient condition for orthonormality. The reason a student of quantum mechanics wants to understand linear independence is explicitly because it is a sufficient condition for orthonormality. An orthonormal basis is a practical necessity for quantum mechanical calculation. This problem also shows how a linearly independent system can be made orthonormal using the Gram-Schmidt procedure. Notice that orthogonality is not necessary (though an orthogonal system can be made orthonormal directly through the process of normalization).

Use the strategy of demonstrating linear independence seen in problem 14 for part (a). Part (b) requires you to form inner products. This system is not orthogonal, but only one of the nine possible inner products disqualifies it. Part (c) is the Gram-Schmidt process. You should find

$$|\,1\!> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |\,2\!> = \begin{pmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \qquad |\,3\!> = \begin{pmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

Part (d) answers the question does this Gram-Schmidt process really work? Yes it does, and you really need only check that the inner product analogous to that which was non-zero in part (b) is zero after the Gram-Schmidt process has been applied. The other properties of orthonormality are satisfied because of the nature of normalized vectors and two of the results of part (b).

(a) The condition of linear independence is

$$\sum_i^n a_i\,|\,v_i\!> \;\neq\; |\,0\!> \qquad \text{unless all} \quad a_i = 0\,. \quad \text{Here this means}$$

$$a_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 3a_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ 2a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2a_3 \\ 5a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 3a_1 \quad &= 0 \\ a_2 + 2a_3 \quad &= 0 \\ 2a_2 + 5a_3 \quad &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} a_1 \quad &= \;\; 0 \\ a_2 \quad &= -2a_3\,. \\ 2a_2 \quad &= -5a_3 \end{aligned}$$

The first line says a_1 is zero. Substituting the equation from the second line into the equation of the third line,

$$2a_2 = -5a_3 \quad \Rightarrow \quad 2\,(-2a_3) = -4a_3 = -5a_3\,,$$

which can be true only if $a_3 = 0$. Then since $a_2 = -2a_3$, a_2 is also zero. The unique solution for all coefficients is that all $a_i = 0$, therefore, the three vectors are linearly independent.

(b) To show the three vectors are not orthogonal, we need to show only that any inner product of different vectors does not vanish. In fact

$$<\!II\,|\,III\!> = \big(0,\; 1,\; 2\big) \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = 0 + 2 + 10 = 12 \neq 0\,,$$

therefore, this system of three vectors is not orthogonal. By the way, $<\!I\,|\,II\!> = <\!I\,|\,III\!> = 0$.

(c) First normalize $|\,1\!>$,

$$<\!1\,|\,\mathrm{A}^*\mathrm{A}\,|\,1\!> = \Big(3,\; 0,\; 0\Big)\,\mathrm{A}^*\mathrm{A} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = |\,\mathrm{A}\,|^2 \Big(9 + 0 + 0\Big) = |\,\mathrm{A}\,|^2\, 9 = 1$$

$$\Rightarrow \quad \mathrm{A} = \frac{1}{3} \quad \Rightarrow \quad |\,1\!> = \frac{1}{3}\begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Then construct $|\,2'\!>$,

$$|\,2'\!> = \begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\Big(1,0,0\Big)\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}(0+0+0) = \begin{pmatrix}0\\1\\2\end{pmatrix}$$

and then normalize $|\,2'\!>$ to obtain $|\,2\!>$,

$$<\!2\,|\,\mathrm{A}^*\mathrm{A}\,|\,2\!> = \Big(0,\ 1,\ 2\Big)\,\mathrm{A}^*\mathrm{A}\begin{pmatrix}0\\1\\2\end{pmatrix} = \big|\,\mathrm{A}\,\big|^2\Big(0+1+4\Big) = \big|\,\mathrm{A}\,\big|^2\,5 = 1$$

$$\Rightarrow \quad \mathrm{A} = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad |\,2\!> = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix}.$$

Finally, construct $|\,3'\!>$,

$$\begin{aligned}
|\,3'\!> \ &= \begin{pmatrix}0\\2\\5\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\Big(1,0,0\Big)\begin{pmatrix}0\\2\\5\end{pmatrix} - \begin{pmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}\left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)\begin{pmatrix}0\\2\\5\end{pmatrix}\\
&= \begin{pmatrix}0\\2\\5\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}(0+0+0) - \begin{pmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}\left(0+\frac{2}{\sqrt{5}}+\frac{10}{\sqrt{5}}\right)\\
&= \begin{pmatrix}0\\2\\5\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}(0) - \begin{pmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}\frac{12}{\sqrt{5}}\\
&= \begin{pmatrix}0\\2\\5\end{pmatrix} - \begin{pmatrix}0\\12/5\\24/5\end{pmatrix} = \begin{pmatrix}0\\-2/5\\1/5\end{pmatrix}
\end{aligned}$$

and then normalize $|\,3'\!>$ to obtain $|\,3\!>$,

$$<\!3\,|\,\mathrm{A}^*\mathrm{A}\,|\,3\!> = \left(0,\ -\frac{2}{5},\ \frac{1}{5}\right)\,\mathrm{A}^*\mathrm{A}\begin{pmatrix}0\\-2/5\\1/5\end{pmatrix} = \big|\,\mathrm{A}\,\big|^2\left(0+\frac{4}{25}+\frac{1}{25}\right) = \big|\,\mathrm{A}\,\big|^2\,\frac{1}{5} = 1$$

$$\Rightarrow \quad \mathrm{A} = \sqrt{5} \quad \Rightarrow \quad |\,3\!> = \sqrt{5}\begin{pmatrix}0\\-2/5\\1/5\end{pmatrix} = \begin{pmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{pmatrix}.$$

Summarizing, the application of the Gram-Schmidt procedure yields the three vectors

$$|\,1\!> = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |\,2\!> = \begin{pmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}, \quad |\,3\!> = \begin{pmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{pmatrix}.$$

(d) The vectors are normalized, so each inner product with itself is 1. $<I\,|\,II> \, = \, <I\,|\,III> \, = 0$, as before, because all inner products of the form

$$\begin{pmatrix} a, & 0, & 0 \end{pmatrix} \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} = 0 + 0 + 0 = 0.$$

However, $<II\,|\,III>$ was non-zero, but

$$<2\,|\,3> \, = \begin{pmatrix} 0, & 1/\sqrt{5}, & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = -\frac{2}{5} + \frac{2}{5} = 0,$$

so the three vectors in the summary of part (c) are orthonormal.

Postscript: Again, this problem is an example that linear independence is a sufficient condition for orthonormality. The practical reason for this is that if linearly independence can be demonstrated, an orthonormal basis is assumed. Our interest is in the orthonormal basis. Linear independence establishes the existence of the orthonormal basis. The validity of this assumption can be based on the Gram-Schmidt procedure, though we will never have reason to explicitly employ the Gram-Schmidt process.

---

19. Find the adjoint of the operator

$$\mathcal{A} = \begin{pmatrix} -4+2i & 2+i & -1-3i \\ 6 & 2+3i & -3i \\ -3-i & 2-3i & -5 \end{pmatrix}.$$

An adjoint is a **transpose conjugate**. Symbolically,

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \Rightarrow \quad \mathcal{A}^\dagger = \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{pmatrix},$$

where the dagger symbol, †, located at the upper right of the operator indicates adjoint. Simply make the rows into columns and the columns into rows and complex conjugate all of the elements. The mechanics of forming an adjoint operator is easy. The reason to form an adjoint is deeper, and we will attempt to explain the deeper significance in the postscript.

$$\mathcal{A} = \begin{pmatrix} -4+2i & 2+i & -1-3i \\ 6 & 2+3i & -3i \\ -3-i & 2-3i & -5 \end{pmatrix} \quad \Rightarrow \quad \mathcal{A}^\dagger = \begin{pmatrix} -4-2i & 6 & -3+i \\ 2-i & 2-3i & 2+3i \\ -1+3i & 3i & -5 \end{pmatrix}.$$

Postscript: A **dual space** contains complex conjugates or complex conjugate analogies to the objects that comprise the original space. The concept is that the complex conjugate or analogy to a complex conjugate of any scalar, vector, or matrix operator exists in the dual space, and when necessary, the appropriate object is simply picked from all of the other objects in the dual space. A complex conjugate is in the dual space of the corresponding scalar, a bra is in the dual space of the corresponding ket, and an adjoint operator is in the dual space of the corresponding operator.

$$\text{scalars} \quad \overbrace{\alpha \longleftrightarrow \alpha^*}^{\text{complex conjugate}} \qquad \text{vectors} \quad \overbrace{|\,v\!> \longleftrightarrow \,<\!v\,|}^{\text{adjoint}} \qquad \text{operators} \quad \overbrace{\mathcal{A} \longleftrightarrow \mathcal{A}^\dagger}^{\text{adjoint}}$$

Notice that a bra formed from a ket satisfies the definition of an adjoint. "Transpose" means to make the rows into columns and the columns into rows. A bra is a transpose conjugate of a ket.

Among the reasons to consider a dual space is that a complex conjugate or complex conjugate analogy exists for any object that can be considered for quantum mechanical calculation. The complex linear vector space of actual quantum mechanical calculation is often of infinite dimension. It is impossible to form an infinity of complex conjugates or adjoints, yet, we need to know that these objects also exist. It is straightforward to show that they exist in a dual space comprised of complex conjugates or objects analogous to complex conjugates.

Secondly, the physical measurements of any branch of physics including quantum mechanics must be real quantities. The product of any scalar and its complex conjugate is a real number. The product of any ket and its corresponding bra is a real number. We will find that the product of any matrix operator and its adjoint is a matrix operator that has elements that are real. The dual space comprised of complex conjugates/complex conjugate analogies provides a mathematically tractable mechanism for obtaining real quantities from intrinsically complex objects. A significant byproduct of the dual space is that there is a geometric analog to the length of a vector, the norm, regardless of the number of dimensions.

The concept of a space and a corresponding dual space is a precursor to the **Hilbert space**. A Hilbert space is defined by three properties, namely: 1) linearity, 2) orthonormality, and 3) completeness. Physical phenomena that can be described by quantum mechanics appears to be linear, you should understand orthonormality as it applies to kets and bras at this point, and completeness means that the set spans the space. The physicist's view of the Hilbert space is not in concert with the mathematician's view of a Hilbert space. There are an infinite number of spaces that satisfy these three properties. Physicists collect every possible set that meets the definition of a Hilbert space and call that the Hilbert space. Mathematics discusses a specific Hilbert space because there are an infinite number, while physics discusses the Hilbert space because the physicist sees only one. Infinite and finite dimensional operators and their adjoints, infinite and finite dimensional kets and their bras, all scalars and their complex conjugates, as well as orthogonal functions likes sines and cosines, and orthogonal polynomials like Hermite, Laguerre, and Legendre polynomials, and many other objects are included in the physicist's view of the Hilbert space. Most problems can be addressed using a subset or subspace of the Hilbert space. We will say more about the Hilbert space. Hilbert space is discussed in most books on functional analysis such as Shilov[1] and Riesz and Nagy[2]. Chapter 5 of Byron and Fuller[3] is an

[1] Shilov *Elementary Functional Analysis,* Dover Publications, 1974, pp. 54–68.
[2] Riesz and Nagy *Functional Analysis,* Dover Publications, 1990, pp. 197–199.
[3] Byron and Fuller *The Mathematics of Classical and Quantum Physics*, Dover Publications, 1969.

excellent discussion of Hilbert space. These references provide more extensive information than needed for our presentation. For our purposes, think of the Hilbert space as the collection of all orthonormal sets.

---

20. Given that

$$\mathcal{A} = \begin{pmatrix} -4+2i & 2+i & -1-3i \\ 6 & 2+3i & -3i \\ -3-i & 2-3i & -5 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 3-i & 2-2i & -4+2i \\ 1-3i & 2+4i & 5-2i \\ -1-6i & 7 & -4+i \end{pmatrix}, \quad \text{find}$$

(a) $\mathcal{A}+\mathcal{B}$,

(b) $\mathcal{A}-\mathcal{B}$,

(c) $\mathcal{A}^\dagger + \mathcal{B}^\dagger$, and

(d) $\left(\mathcal{A}+\mathcal{B}\right)^\dagger$.

Operators are added/subtracted by adding/subtracting corresponding elements. Symbolically,

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

for addition. Subtract corresponding elements if the indicated operation is subtraction. You know how to find adjoints from problem 19. You should find that the answers to parts (c) and (d) are identical—the sum of the adjoints is the same as the adjoint of the sum.

(a)
$$\begin{aligned} \mathcal{A}+\mathcal{B} &= \begin{pmatrix} -4+2i+(3-i) & 2+i+(2-2i) & -1-3i+(-4+2i) \\ 6+(1-3i) & 2+3i+(2+4i) & -3i+(5-2i) \\ -3-i+(-1-6i) & 2-3i+(7) & -5+(-4+i) \end{pmatrix} \\ &= \begin{pmatrix} -1+i & 4-i & -5-i \\ 7-3i & 4+7i & 5-5i \\ -4-7i & 9-3i & -9+i \end{pmatrix}. \end{aligned}$$

(b)
$$\begin{aligned} \mathcal{A}-\mathcal{B} &= \begin{pmatrix} -4+2i-(3-i) & 2+i-(2-2i) & -1-3i-(-4+2i) \\ 6-(1-3i) & 2+3i-(2+4i) & -3i-(5-2i) \\ -3-i-(-1-6i) & 2-3i-(7) & -5-(-4+i) \end{pmatrix} \\ &= \begin{pmatrix} -7+3i & 3i & 3-5i \\ 5+3i & -i & -5-i \\ -2+5i & -5-3i & -1-i \end{pmatrix}. \end{aligned}$$

(c)
$$\begin{pmatrix} -4-2i & 6 & -3+i \\ 2-i & 2-3i & 2+3i \\ -1+3i & 3i & -5 \end{pmatrix} + \begin{pmatrix} 3+i & 1+3i & -1+6i \\ 2+2i & 2-4i & 7 \\ -4-2i & 5+2i & -4-i \end{pmatrix} = \begin{pmatrix} -1-i & 7+3i & -4+7i \\ 4+i & 4-7i & 9+3i \\ -5+i & 5+5i & -9-i \end{pmatrix}.$$

(d) Using the result of part (a),

$$\left(\mathcal{A}+\mathcal{B}\right)^\dagger = \begin{pmatrix} -1+i & 4-i & -5-i \\ 7-3i & 4+7i & 5-5i \\ -4-7i & 9-3i & -9+i \end{pmatrix}^\dagger = \begin{pmatrix} -1-i & 7+3i & -4+7i \\ 4+i & 4-7i & 9+3i \\ -5+i & 5+5i & -9-i \end{pmatrix}.$$

---

21. Given that

$$\mathcal{A} = \begin{pmatrix} -4+2i & 2+i & -1-3i \\ 6 & 2+3i & -3i \\ -3-i & 2-3i & -5 \end{pmatrix} \quad \text{and} \quad |\,v> = \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix}, \quad \text{find } \mathcal{A}\,|\,v> .$$

An operator times a vector is a vector. A symbolic example of $\mathcal{A}\,|\,v>$ in 3 dimensions is

$$\mathcal{A}\,|\,v> = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{pmatrix} = |\,w> .$$

In layman's terms, multiply each row of the operator by the column vector as if it was a row vector. The sum of the products is the component of the new vector $|\,w>$ at the same level as the horizontal row of the operator that was used.

$$\begin{aligned}
\mathcal{A}\,|\,v> &= \begin{pmatrix} -4+2i & 2+i & -1-3i \\ 6 & 2+3i & -3i \\ -3-i & 2-3i & -5 \end{pmatrix} \begin{pmatrix} 2+i \\ 1-3i \\ 3-2i \end{pmatrix} \\
&= \begin{pmatrix} (-4+2i)(2+i) + (2+i)(1-3i) + (-1-3i)(3-2i) \\ 6(2+i) + (2+3i)(1-3i) + (-3i)(3-2i) \\ (-3-i)(2+i) + (2-3i)(1-3i) + (-5)(3-2i) \end{pmatrix} \\
&= \begin{pmatrix} (-8-4i+4i-2) + (2-6i+i+3) + (-3+2i-9i-6) \\ (12+6i) + (2-6i+3i+9) + (-9i-6) \\ (-6-3i-2i+1) + (2-6i-3i-9) + (-15+10i) \end{pmatrix} \\
&= \begin{pmatrix} -10+5-5i-9-7i \\ 12+6i+11-3i-9i-6 \\ -5-5i-7-9i-15+10i \end{pmatrix} = \begin{pmatrix} -14-12i \\ 17-6i \\ -27-4i \end{pmatrix}.
\end{aligned}$$

Postscript: This is called matrix-vector multiplication in the language of linear algebra, but physics uses the terminology $\mathcal{A}$ **operates** on $|\,v>$. Since an operator operating on a ket is another ket, we can write

$$|\,w> = \mathcal{A}\,|\,v> = |\mathcal{A}\,v>,$$

where the last combinations of symbols is new but is clear because we know the product of an operator and a ket is a ket.

---

22. Find the vector-operator product $<v\,|\,\mathcal{A}^\dagger$ using $\mathcal{A}$ and $|\,v>$ from problem 21.

The inverse order of the two objects from problem 21, $|\,v> \mathcal{A}$, by itself is undefined. However, if the object on the left is a row vector, a bra, then each column of the operator can be treated as a ket for multiple multiplications like vector-vector multiplication. The result is a new bra vector. Symbolically,

$$\begin{aligned} <v\,|\,\mathcal{A} &= (\,b_1,\; b_2,\; b_3\,) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= (\,b_1 a_{11} + b_2 a_{21} + b_3 a_{31},\quad b_1 a_{12} + b_2 a_{22} + b_3 a_{32},\quad b_1 a_{13} + b_2 a_{23} + b_3 a_{33}\,) = \,<w\,|\,. \end{aligned}$$

Notice that this problem asks for the product of the bra of $|\,v>$ and the adjoint of $\mathcal{A}$.

$$\begin{aligned} <v\,|\,\mathcal{A}^\dagger &= (\,2-i,\; 1+3i,\; 3+2i\,) \begin{pmatrix} -4-2i & 6 & -3+i \\ 2-i & 2-3i & 2+3i \\ -1+3i & 3i & -5 \end{pmatrix} \\ &= \Big((2-i)(-4-2i) + (1+3i)(2-i) + (3+2i)(-1+3i), \\ &\qquad (2-i)(6) + (1+3i)(2-3i) + (3+2i)(3i), \\ &\qquad\qquad (2-i)(-3+i) + (1+3i)(2+3i) + (3+2i)(-5) \Big) \\ &= \Big((-8-4i+4i-2) + (2-i+6i+3) + (-3+9i-2i-6), \\ &\qquad (12-6i) + (2-3i+6i+9) + (9i-6), \\ &\qquad\qquad (-6+2i+3i+1) + (2+3i+6i-9) + (-15-10i) \Big) \\ &= \Big(-10+5+5i-9+7i,\quad 12-6i+11+3i+9i-6,\quad -5+5i-7+9i-15-10i \Big) \\ &= \Big(-14+12i,\quad 17+6i,\quad -27+4i \Big)\,. \end{aligned}$$

Postscript: An operator can operate to the right on a ket or to the left on a bra. Just as $|\,v> \mathcal{A}$ is undefined, the combination $\mathcal{A} <v\,|$ is also undefined.

We have chosen objects that are adjoints to those in problem 21. Notice that the result of problem 22 is the adjoint of the result of problem 21. Analogous operations with adjoints result in adjoint objects in the dual space.

Since an operator operating on a bra is another bra, we can write

$$< w \,| \; = \; < v \,|\, \mathcal{A}^{\dagger} \; = \; < v \, \mathcal{A} \,| \, ,$$

where the last combinations of symbols is new. Notice that "$\dagger$" is not included when the operator is inside the bra. An adjoint is understood for any object inside the "$<$" and the "$|$" that denote the bra.

---

23. Find the product $\mathcal{A}\,\mathcal{B}$ given that

$$\mathcal{A} = \begin{pmatrix} 2+5i & 3-2i \\ -1-4i & 4-i \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} -3-2i & -1-6i \\ 4+5i & -2+3i \end{pmatrix}.$$

The product of two operators is another operator. For general $\mathcal{A}$ and $\mathcal{B}$ in three dimensions,

$$\mathcal{A}\,\mathcal{B} \; = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}.$$

This is essentially a generalization of operator-vector multiplication. It is like the second operator is composed of three column vectors. The operator-vector like product of each row of the first operator and the first column of the second operator becomes the first column of the new operator, the operator-vector like product of each row of the first operator and the second column of the second operator becomes the second column of the new operator, and so on. Operator-operator multiplication can be generalized to any including infinite dimension.

$$\mathcal{A}\,\mathcal{B} = \begin{pmatrix} 2+5i & 3-2i \\ -1-4i & 4-i \end{pmatrix} \begin{pmatrix} -3-2i & -1-6i \\ 4+5i & -2+3i \end{pmatrix}$$

$$= \begin{pmatrix} (2+5i)(-3-2i)+(3-2i)(4+5i) & (2+5i)(-1-6i)+(3-2i)(-2+3i) \\ (-1-4i)(-3-2i)+(4-i)(4+5i) & (-1-4i)(-1-6i)+(4-i)(-2+3i) \end{pmatrix}$$

$$= \begin{pmatrix} (-6-4i-15i+10)+(12+15i-8i+10) & (-2-12i-5i+30)+(-6+9i+4i+6) \\ (3+2i+12i-8)+(16+20i-4i+5) & (1+6i+4i-24)+(-8+12i+2i+3) \end{pmatrix}$$

$$= \begin{pmatrix} 4-19i+22+7i & 28-17i+13i \\ -5+14i+21+16i & -23+10i-5+14i \end{pmatrix} = \begin{pmatrix} 26-12i & 28-4i \\ 16+30i & -28+24i \end{pmatrix}.$$

---

24. For
$$|\,v\!> = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad |\,w\!> = \begin{pmatrix} 1-i \\ 2+3i \end{pmatrix}, \qquad \text{and}$$
$$\mathcal{A} = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 1+2i & -2-i \\ -2+i & 1-3i \end{pmatrix},$$

(a) find $<\!v\,|\,w\!>$ and $<\!w\,|\,v\!>$, then compare these two inner products.

(b) Find $\mathcal{A}\,|\,v\!>$ and $<\!v\,|\,\mathcal{A}^\dagger$. Compare these products.

(c) Find $\mathcal{A}^\dagger\,|\,v\!>$ and $<\!v\,|\,\mathcal{A}$. Compare these products. Compare the results of part (c) with those of part (b).

(d) Find $\mathcal{B}\,|\,w\!>$, and $<\!w\,|\,\mathcal{B}^\dagger$, and compare these products.

(e) Find $\mathcal{A}\,\mathcal{B}$ and $\mathcal{B}\,\mathcal{A}$, and compare these products.

(f) Find $\mathcal{B}^\dagger\,\mathcal{A}^\dagger$ and $\mathcal{A}^\dagger\,\mathcal{B}^\dagger$. Compare these products. Compare the results of part (f) with those of part (e).

This problem summarizes the different types of multiplication appropriate to a complex linear vector space. More importantly, this problem is designed to highlight some features of adjoint objects and dual spaces, and introduce the concept of **non-commutivity**. Commutivity is the algebraic property of multiplication that indicates the order of multiplication does not matter. If $a \cdot b = b \cdot a$, then a and b are said to commute. Scalars commute. Matrix operators do not generally commute. Order matters in matrix multiplication. Symbolically, $\mathcal{A}\,\mathcal{B} \neq \mathcal{B}\,\mathcal{A}$ in general.

You should find that $<\!v\,|\,w\!>$ and $<\!w\,|\,v\!>$ are complex conjugates in part (a). The operator-vector and vector-operator products of parts (b), (c), and (d) will be adjoints. You should see that there are no simple connections between the results of parts (b) and (c). There are no equalities nor are there any adjoints in spite of the similarity of objects being multiplied. Though there are two sets of adjoints in the results of parts (e) and (f), you should notice that $\mathcal{A}\,\mathcal{B} \neq \mathcal{B}\,\mathcal{A}$ and $\mathcal{A}^\dagger\,\mathcal{B}^\dagger \neq \mathcal{B}^\dagger\,\mathcal{A}^\dagger$. These operators do not commute. Finally, notice that for any product formed, if we take the adjoints of each factor and reverse the order of the multiplication, we attain the adjoint of the initial product. This is true for vector-vector, vector-operator, operator-vector, and operator-operator multiplication.

(a) An inner product is a scalar, and
$$<\!v\,|\,w\!> = \big(2,\ -1\big)\begin{pmatrix} 1-i \\ 2+3i \end{pmatrix} = 2\big(1-i\big) - 1\big(2+3i\big) = 2-2i-2-3i = -5i\,.$$
$$<\!w\,|\,v\!> = \big(1+i,\ 2-3i\big)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \big(1+i\big)2 + \big(2-3i\big)(-1) = 2+2i-2+3i = 5i\,.$$
These two inner products are scalars that are complex conjugates.

(b) The operator-vector product indicated is
$$\mathcal{A}\,|\,v\!> = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 + (-1)(-1) \\ (-3)2 + 1(-1) \end{pmatrix} = \begin{pmatrix} 4+1 \\ -6-1 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}.$$

The vector-operator product indicated is

$$<v\,|\,\mathcal{A}^\dagger = \begin{pmatrix} 2, & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 2+(-1)(-1), & 2(-3)+(-1)1 \end{pmatrix} = \begin{pmatrix} 4+1, & -6-1 \end{pmatrix} = \begin{pmatrix} 5, & -7 \end{pmatrix}.$$

The results are adjoints, a ket and its corresponding bra.

(c) The operator-vector product is

$$\mathcal{A}^\dagger\,|\,v> = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2\cdot 2+(-3)(-1) \\ (-1)2+1(-1) \end{pmatrix} = \begin{pmatrix} 4+3 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}.$$

The vector-operator product indicated is

$$<v\,|\,\mathcal{A} = \begin{pmatrix} 2, & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 2+(-1)(-3), & 2(-1)+(-1)1 \end{pmatrix} = \begin{pmatrix} 4+3, & -2-1 \end{pmatrix} = \begin{pmatrix} 7, & -3 \end{pmatrix}.$$

The results are adjoints, a ket and its corresponding bra. Besides the fact that these are kets and bras in a two dimensional space, we see no simple connection between the vectors attained in parts (b) and (c). There are no results of parts (b) and (c) that are equal, nor are there any adjoints.

(d) The next operator-vector product is

$$\begin{aligned}
\mathcal{B}\,|\,w> &= \begin{pmatrix} 1+2i & -2-i \\ -2+i & 1-3i \end{pmatrix} \begin{pmatrix} 1-i \\ 2+3i \end{pmatrix} = \begin{pmatrix} (1+2i)(1-i)+(-2-i)(2+3i) \\ (-2+i)(1-i)+(1-3i)(2+3i) \end{pmatrix} \\
&= \begin{pmatrix} 1-i+2i+2-4-6i-2i+3 \\ -2+2i+i+1+2+3i-6i+9 \end{pmatrix} \\
&= \begin{pmatrix} 1+2-4+3+(-1+2-6-2)i \\ -2+1+2+9+(2+1+3-6)i \end{pmatrix} = \begin{pmatrix} 2-7i \\ 10 \end{pmatrix}.
\end{aligned}$$

The corresponding vector-operator product is

$$\begin{aligned}
<w\,|\,\mathcal{B}^\dagger &= \begin{pmatrix} 1+i, & 2-3i \end{pmatrix} \begin{pmatrix} 1-2i & -2-i \\ -2+i & 1+3i \end{pmatrix} \\
&= \begin{pmatrix} (1+i)(1-2i)+(2-3i)(-2+i), & (1+i)(-2-i)+(2-3i)(1+3i) \end{pmatrix} \\
&= \begin{pmatrix} 1-2i+i+2-4+2i+6i+3, & -2-i-2i+1+2+6i-3i+9 \end{pmatrix} \\
&= \begin{pmatrix} 1+2-4+3+(-2+1+2+6)i, & -2+1+2+9+(-1-2+6-3)i \end{pmatrix} \\
&= \begin{pmatrix} 2+7i, & 10 \end{pmatrix},
\end{aligned}$$

and the two products are adjoint vectors, a ket and its corresponding bra.

(e) The indicated operator-operator product is

$$\begin{aligned}
\mathcal{A}\mathcal{B} &= \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1+2i & -2-i \\ -2+i & 1-3i \end{pmatrix} \\
&= \begin{pmatrix} 2(1+2i)-1(-2+i) & 2(-2-i)-1(1-3i) \\ -3(1+2i)+1(-2+i) & -3(-2-i)+1(1-3i) \end{pmatrix} \\
&= \begin{pmatrix} 2+4i+2-i & -4-2i-1+3i \\ -3-6i-2+i & 6+3i+1-3i \end{pmatrix} = \begin{pmatrix} 4+3i & -5+i \\ -5-5i & 7 \end{pmatrix}.
\end{aligned}$$

$$\mathcal{B}\,\mathcal{A} = \begin{pmatrix} 1+2i & -2-i \\ -2+i & 1-3i \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (1+2i)2+(-2-i)(-3) & (1+2i)(-1)+(-2-i)1 \\ (-2+i)2+(1-3i)(-3) & (-2+i)(-1)+(1-3i)1 \end{pmatrix}$$
$$= \begin{pmatrix} 2+4i+6+3i & -1-2i-2-i \\ -4+2i-3+9i & 2-i+1-3i \end{pmatrix} = \begin{pmatrix} 8+7i & -3-3i \\ -7+11i & 3-4i \end{pmatrix}.$$

As indicated earlier, you should notice that these operators do not commute, *i.e.*, $\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$.

(f) The operator-operator product of the adjoints in inverse order is

$$\mathcal{B}^\dagger\,\mathcal{A}^\dagger = \begin{pmatrix} 1-2i & -2-i \\ -2+i & 1+3i \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (1-2i)2+(-2-i)(-1) & (1-2i)(-3)+(-2-i)1 \\ (-2+i)2+(1+3i)(-1) & (-2+i)(-3)+(1+3i)1 \end{pmatrix}$$
$$= \begin{pmatrix} 2-4i+2+i & -3+6i-2-i \\ -4+2i-1-3i & 6-3i+1+3i \end{pmatrix} = \begin{pmatrix} 4-3i & -5+5i \\ -5-i & 7 \end{pmatrix}.$$

$$\mathcal{A}^\dagger\,\mathcal{B}^\dagger = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1-2i & -2-i \\ -2+i & 1+3i \end{pmatrix}$$
$$= \begin{pmatrix} 2(1-2i)-3(-2+i) & 2(-2-i)-3(1+3i) \\ -1(1-2i)+1(-2+i) & -1(-2-i)+1(1+3i) \end{pmatrix}$$
$$= \begin{pmatrix} 2-4i+6-3i & -4-2i-3-9i \\ -1+2i-2+i & 2+i+1+3i \end{pmatrix} = \begin{pmatrix} 8-7i & -7-11i \\ -3+3i & 3+4i \end{pmatrix}.$$

Notice that $\mathcal{A}^\dagger\mathcal{B}^\dagger \neq \mathcal{B}^\dagger\mathcal{A}^\dagger$. These operators do not commute. $\mathcal{B}^\dagger\mathcal{A}^\dagger$ from part (f) is the adjoint of $\mathcal{A}\mathcal{B}$ from part (e), and $\mathcal{A}^\dagger\mathcal{B}^\dagger$ from part (f) is the adjoint of $\mathcal{B}\mathcal{A}$ from part (e).

Postscript: Some operators and even some classes of operators do commute. Operators that do commute are particularly important to quantum mechanical calculation.

Supplementary Problems

25. Show that $<v\,|\,w> = <w\,|\,v>^*$ in $\mathbf{C}^3$.

This problem is intended to convince you that the complex conjugate of an inner product is the same as the inner product in inverse order. If you can prove this in $\mathbf{C}^3$, you can see that the process is easily extended to larger dimension. Form two general vectors in $\mathbf{C}^3$, such as

$$|\,v> = \begin{pmatrix} a_1+b_1i \\ a_2+b_2i \\ a_3+b_3i \end{pmatrix} \qquad \text{and} \qquad |\,w> = \begin{pmatrix} c_1+d_1i \\ c_2+d_2i \\ c_3+d_3i \end{pmatrix}.$$

Form the inner products $<v\,|\,w>$ and $<w\,|\,v>^*$. Compare them. They will be identical.

---

26. If $\alpha = 3$, $\beta = 5 - 2i$, $<v\,| = \big(2-4i,\ -6,\ 2+3i\big)$, and $|\,w> = \begin{pmatrix} 3+i \\ 4-3i \\ -2 \end{pmatrix}$, find

(a) $\alpha\,|\,v>$,

(b) $<w\,\beta\,|$,

(c) $\beta\,|\,v> + \alpha\,|\,w>$, and

(d) $<w\,|\,v>$.

Remember to conjugate the components when forming a bra from a ket or a ket from a bra. Remember also that if a symbol is between the $<$ and $|$ of a bra, it is understood to be conjugated or adjoint, so $<w\,\beta\,| = <w\,|\,\beta^*$ in part (b).

---

27. Normalize

(a) $|\,1> = \begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix}$,

(b) $|\,2> = \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix}$, and

(c) $|\,3> = \begin{pmatrix} 1+i \\ 3 \\ -i \end{pmatrix}$.

(d) Do the three normalized vectors constitute an orthonormal basis in $\mathbf{C}^3$?

Normalize each ket using the procedures of problem 10. Problem 15 may be helpful for part (d).

---

28. Given that the components of two-dimensional vectors are complex numbers, the definitions of vector addition and scalar multiplication

$$(a,\ b) + (c,\ d) = (2a+2c,\ 2b+2d) \qquad \text{and} \qquad \alpha\,(a,\ b) = (2\alpha a,\ 2\alpha b)$$

do not satisfy the requirements for a complex linear vector space. Which properties of a linear vector space are satisfied by these definitions, and which properties are not satisfied?

The definitions are known as double addition and double multiplication. Each component is a complex number, so you can employ the algebraic properties of complex numbers for component addition and multiplication. As an example of what is intended,

$$\Big[(a,\ b) + (c,\ d)\Big] + (e,\ f) = (2a+2c,\ 2b+2d) + (e,\ f) = (4a+4c+2e,\ 4b+4d+2f), \quad \text{but}$$

$$(a,\ b)+\Big[(c,\ d)+(e,\ f)\Big]=(a,\ b)+(2c+2e,\ 2d+2f)=(2a+4c+4e,\ 2b+4d+4f)\,,$$

and the results are not the same so double addition is not associative. You should find the definitions meet four of the remaining nine properties of a linear vector space. See problems 11, 12, and 13.

---

29. Which of the following sets of vectors are linearly independent?

(a) $\begin{pmatrix}1\\0\\1\end{pmatrix},\ \begin{pmatrix}0\\1\\-1\end{pmatrix},\ \begin{pmatrix}1\\1\\0\end{pmatrix}$

(b) $\begin{pmatrix}i\\-1\\1\end{pmatrix},\ \begin{pmatrix}1\\-1\\-i\end{pmatrix},\ \begin{pmatrix}1\\-i\\1\end{pmatrix}$

(c) $\begin{pmatrix}i\\i\\0\end{pmatrix},\ \begin{pmatrix}i\\i\\i\end{pmatrix},\ \begin{pmatrix}i\\0\\i\end{pmatrix}$

(d) $\begin{pmatrix}-i\\1\\-1\end{pmatrix},\ \begin{pmatrix}i\\-1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix}$

Use the procedures of problem 14. The vectors of parts (b) and (c) are linearly independent. The vectors of parts (a) and (d) are linearly dependent.

---

30. Show that the Pauli spin matrices are linearly independent.

The Pauli spin matrices exist in the subspace $\mathbf{C}^2$ of the general complex linear vector space $\mathbf{C}^\infty$. They are

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \text{and}\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

If they are linearly independent, they must satisfy the condition of linear independence,

$$a\begin{pmatrix}0&1\\1&0\end{pmatrix}+b\begin{pmatrix}0&-i\\i&0\end{pmatrix}+c\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

if and only if $a=b=c=0$. Examine the last matrix equation and you will find that these conditions are satisfied. The Pauli spin matrices are focal in the study of spin angular momentum.

---

31. Are the following sets of three 2×2 matrices linearly independent?

(a) $\mathcal{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad \mathcal{B}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad \mathcal{C}=\begin{pmatrix}-2&-1\\0&-2\end{pmatrix}$

(b) $\mathcal{D}=\begin{pmatrix}1&2\\1&1\end{pmatrix}\quad \mathcal{E}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad \mathcal{F}=\begin{pmatrix}1&2\\0&0\end{pmatrix}$

Problems 30 and 31 suggest that the concept of a vector space extends to objects not commonly identified as vectors. The ten properties of a linear vector space are all satisfied by 2×2 matrices with real elements using the procedures of matrix addition and scalar-matrix multiplication that have been defined. As such, 2×2 matrices with real elements form a linear vector space.

Assume linear independence. This means

$$a\,\mathcal{A} + \; b\,\mathcal{B} + \; c\,\mathcal{C} = |\,0> = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for part (a). Solve the resulting simultaneous equations. The system is linearly independent if and only if the coefficients are all zero, *i.e.*, $a = b = c = 0$. If any of the coefficients is non-zero, you have contradicted the assumption and can conclude that the set is linearly dependent. Notice that the 2×2 zero matrix plays the role of zero in the space composed of 2×2 matrices.

---

32. Orthonormalize the following sets of vectors:

$$|\,1> = \begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix}, \quad |\,2> = \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix}, \quad |\,3> = \begin{pmatrix} 1+i \\ 3 \\ -i \end{pmatrix}.$$

You found in problem 15 that these vectors are orthogonal. Therefore, you need only to normalize each vector to attain an orthonormal basis.

---

33. (a) Show that the vectors

$$|\,I> = \begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}, \quad |\,II> = \begin{pmatrix} i \\ 1+i \\ 0 \end{pmatrix}, \quad |\,III> = \begin{pmatrix} 1-i \\ 0 \\ 2+i \end{pmatrix} \quad \text{are linearly independent.}$$

(b) Show that the system of three vectors in part (a) is not orthogonal.

(c) Use the Gram-Schmidt process to form an orthonormal basis from the vectors in part (a).

(d) Show the resulting vectors are orthonormal.

This problem is the complex number analogy to problem 18. See problem 18 for procedures. The actual value of this problem and problem 18 is to reinforce that linear independence is a sufficient condition for orthonormality. The Gram-Schmidt procedure is a mathematical mechanism through which an orthonormal set can be realized given a linearly independent set. Orthonormality is a practical necessity for quantum mechanical calculations. Linear independence is the precondition.

---

34. Given that

$$\mathcal{A} = \begin{pmatrix} -3+4i & 5+2i & -2-i \\ 2 & 1+4i & -6i \\ -1-2i & 4-2i & -8 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 2-2i & 3-i & -4+5i \\ 3-2i & 5+i & 3-i \\ -4-3i & 5 & -1+5i \end{pmatrix}, \quad \text{find}$$

(a) $\mathcal{A} + \mathcal{B}$,

(b) $\mathcal{A} - \mathcal{B}$,

(c) $\mathcal{A}^\dagger + \mathcal{B}^\dagger$, and

(d) $\big(\mathcal{A} + \mathcal{B}\big)^\dagger$. Compare the results from parts (c) and (d).

See problem 20 for both intent and procedures.

---

35. Find the product $\mathcal{A}\,\mathcal{B}$ given that

$$\mathcal{A} = \begin{pmatrix} 3+2i & 4-i \\ -5-2i & 3-2i \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} -2-7i & -3-2i \\ 2+4i & -3+6i \end{pmatrix}.$$

See problem 23 for both intent and procedures.

---

36. For $\quad | \, v \!> \, = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad | \, w \!> \, = \begin{pmatrix} 3-i \\ 2+4i \end{pmatrix}, \qquad$ and

$$\mathcal{A} = \begin{pmatrix} 3 & -4 \\ -1 & 6 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 3+i & -5-3i \\ -3+2i & 4-i \end{pmatrix},$$

(a) find $<\! v \,|\, w \!>$ and $<\! w \,|\, v \!>$, then compare these two inner products.

(b) Find $\mathcal{A} \,|\, v \!>$ and $<\! v \,|\, \mathcal{A}^\dagger$. Compare these products.

(c) Find $\mathcal{A}^\dagger \,|\, v \!>$ and $<\! v \,|\, \mathcal{A}$. Compare these products. Compare the results of part (c) with those of part (b).

(d) Find $\mathcal{B} \,|\, w \!>$, and $<\! w \,|\, \mathcal{B}^\dagger$, and compare these products.

(e) Find $\mathcal{A}\,\mathcal{B}$ and $\mathcal{B}\,\mathcal{A}$, and compare these products.

(f) Find $\mathcal{B}^\dagger \mathcal{A}^\dagger$ and $\mathcal{A}^\dagger \mathcal{B}^\dagger$. Compare these products. Compare the results of part (f) with those of part (e).

See problem 24 for intent and procedures.

---

37. Show that $<\! v \,|\, \mathcal{A}^\dagger$ is the adjoint of $\mathcal{A} \,|\, v \!>$ in $\mathbf{C}^2$.

Problem 24 indicates that the adjoints multiplied in the inverse order yields the adjoint of the original product. Problems 24 and 36 provide numerical examples. Problem 25 illustrates this fact in general for inner products. This problem is intended to illustrate this fact for operator-vector and vector-operator products. If you can do this in two dimensions, you can easily extend the process to higher or infinite dimension.

Use an arbitrary vector and operator like

$$| v> = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix} \qquad \text{and} \qquad \mathcal{A} = \begin{pmatrix} c_1 + d_1 i & e_1 + f_1 i \\ c_2 + d_2 i & e_2 + f_2 i \end{pmatrix}.$$

Form $\mathcal{A} \,| v>$ and $< v \,| \mathcal{A}^\dagger$ and you will find that they are the same.

Operators are a primary subject of part 2. The operator-operator analogy of adjoints multiplied in the inverse order yielding the adjoint of the original product, $\left(\mathcal{A}\,\mathcal{B}\right)^\dagger = \mathcal{B}^\dagger\,\mathcal{A}^\dagger$, is addressed in part 2.
