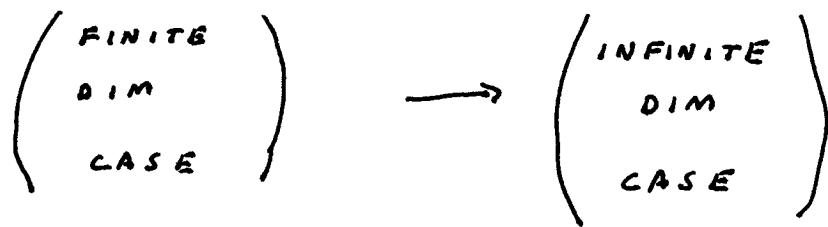


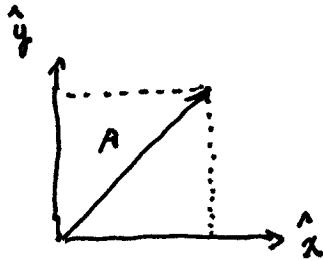
Lecture 4

LEFT OVERS : PROJECTION OPERATORS

RESOLUTIONS OF THE IDENTITY



PROJECTION OPERATORS



$$x \text{ COMPONENT } (\hat{x} \cdot \vec{A}) \hat{x}$$

$$y \text{ COMPONENT } (\hat{y} \cdot \vec{A}) \hat{y}$$

$$\hat{x} \hat{x} \cdot \vec{A}$$

$$\hat{x} (\hat{x} \cdot \vec{A}) \quad |i\rangle \langle i| A \rangle$$

↑ ↑
BASIS INNER
VECTOR PRODUCT

$$(\hat{x} \hat{x} \cdot) \vec{A} \quad (|i\rangle \langle i|) |\vec{A}\rangle$$

↑ ↓
OUTER VECTOR
PRODUCT

$$|e_1\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|e_2\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P_1 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\neq 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_1 = |1\rangle \langle 1|$$

$$P_2 = |2\rangle \langle 2|$$

$$P_2 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|1\rangle \langle 1| A \rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) \begin{pmatrix} a \\ b \end{pmatrix}$$

inner
product
first

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} a$$

outer
product
first

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

COMPLETE SET

$$\sum_i |i\rangle \langle i| = I$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\sum_i |i\rangle \langle i|$ IS CALLED A
RESOLUTION OF
THE IDENTITY

EXPANSION IN A DIFFERENT BASES

$$|A\rangle = |A\rangle = I |A\rangle$$

$$|A\rangle = \sum_i |i\rangle \underbrace{\langle i|}_{}_{a_i} |A\rangle$$

$$|A\rangle = \sum_i a_i |i\rangle$$

$$|A\rangle = \sum_i b_i |i\rangle$$

THM 10: FOR EVERY HERMITIAN OPERATOR, THERE EXISTS
A BASIS OF ITS ORTHONORMAL EIGENVECTORS

x, p, H are Hermitian

\Rightarrow unique expansion in $\frac{p}{E}$

ANY COMMON EIGENVECTORS?

THM 13: IF ~~NOT BOTH~~ R and A are commuting operators
then there exists a common basis of eigenvectors
see diagram in the notes.

COMMUTATOR

$$[R, A] = 0 = RA - AR$$

THREE CASES:

(1) COMPATIBLE: SHARE ALL \vec{e}_n

(2) INCOMPATIBLE: SHARE NO \vec{e}_n

(3) MIXED: SHARE SOME \vec{e}_n BUT NOT ALL

LEFT OVERS: PROJECTION OPERATORS

RESOLUTIONS OF THE IDENTITY

FINITE DIMENSIONAL
CASE



INFINITE
DIMENSIONAL
CASE

VECTORS

FUNCTIONS

MATRICES

OPERATORS

(
FINITE
DISCRETE SET
OF ψ_n 'S AND
 $e^{\lambda n}$ 'S
)

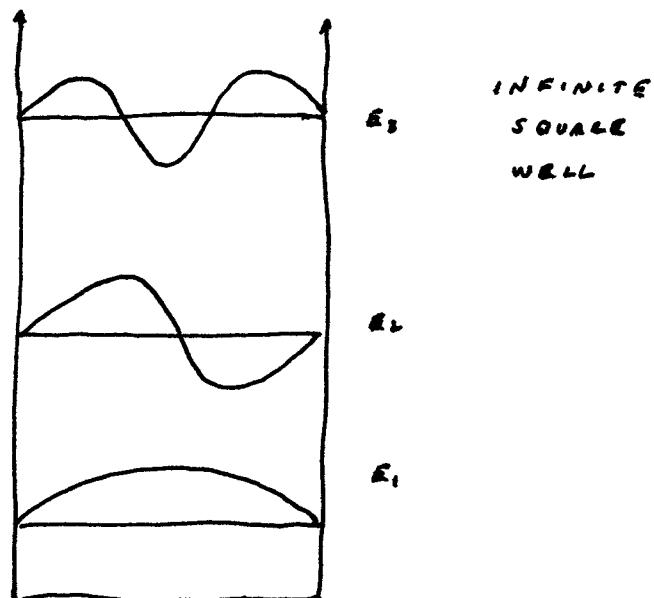
(
INFINITE DISCRETE
SET OF ψ_n 'S AND
 ~~$e^{\lambda n}$~~ $e^{\lambda f}$ 'S
)

OR

(
INFINITE CONTINUOUS
SET OF ψ_n 'S AND
 $e^{\lambda f}$ 'S
)

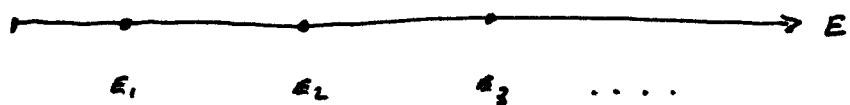
OR \rightarrow AND

DISCRETE ~~CONT~~ EN's



INFINITE
SQUARE
WELL

DISCRETE
SPECTRUM



INSIDE THE WELL
CAN BUILD ANY STATE OUT OF e_i 's

FOURIER SERIES EXPANSION

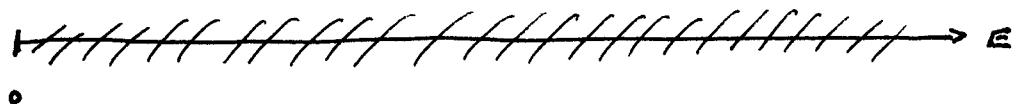
CONTINUOUS & ev's



FREE
PARTICLE



CONTINUOUS SPECTRUM



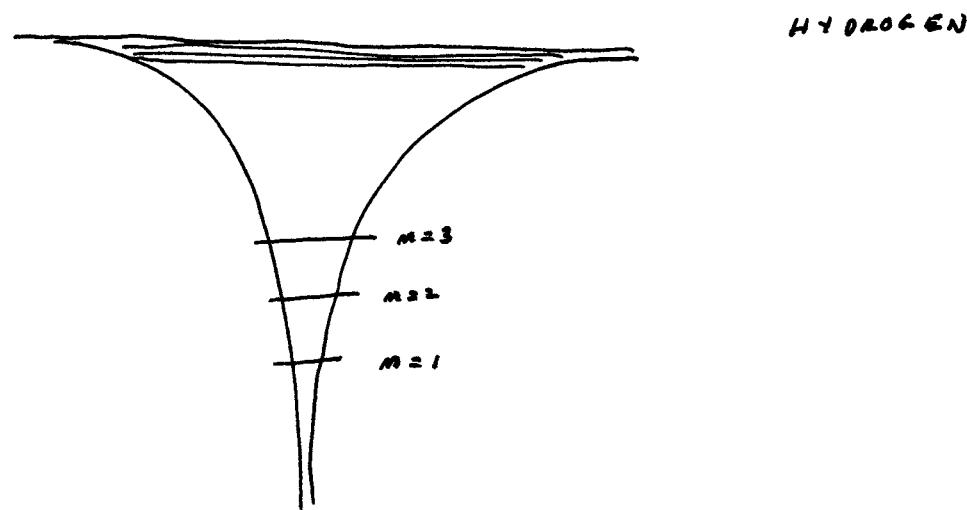
CAN BUILD ANY STATE OUT OF e^f 's

e^{ikx}

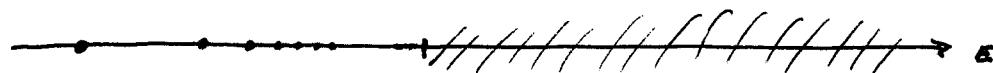
e

FOURIER TRANSFORM EXPANSION

BOTH



SPECTRUM



CAN BUILD ANY STATE OUT OF e^f 'S

HYDROGEN EIGENFCN EXPANSION

GENERALIZE THE INNER PRODUCT

DISCRETE:

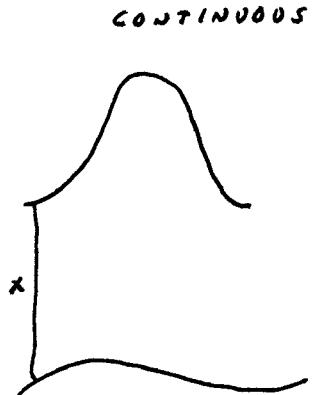
$$\langle f | g \rangle = \sum_{i=1}^m f_i^* g_i \quad \begin{array}{l} \text{DEPEND ON BASIS} \\ \text{BUT SUM DOES NOT} \end{array}$$

CONTINUOUS:

$$\langle f | g \rangle = \int_a^b f^*(x) g(x) dx \quad \begin{array}{l} \text{DEPEND ON BASIS} \\ \text{BUT INTEGRAL DOES NOT} \end{array}$$

DISCRETE

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot + & \cdot + & \cdot + & \cdot + \end{matrix}$$



INTEGRATE

MULTIPLY ~~POINT BY POINT~~

COMPONENT BY COMPONENT

SUM

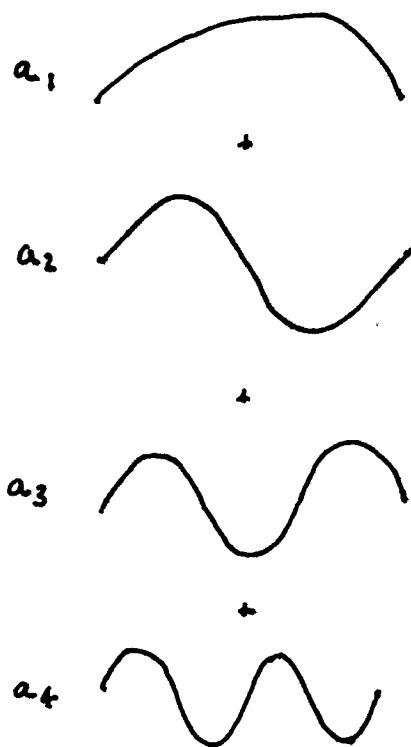
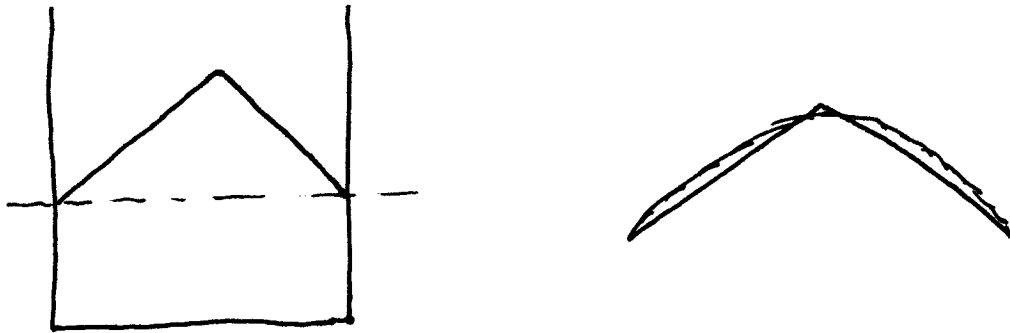
MULTIPLY

POINT BY POINT

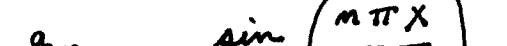
INTEGRATE

FOURIER SERIES

National Brand



$$\sum_m a_m \sin\left(\frac{m\pi x}{L}\right)$$


 EXPANSION COEFF'S FCNS

$$|\psi\rangle = |\psi\rangle = I|\psi\rangle$$

$$= \sum_{m=1}^{\infty} |E_m\rangle \langle E_m| \psi \rangle$$

\sim
 a_m

$$|\psi\rangle = \sum_{m=1}^{\infty} a_m |E_m\rangle$$

$$\langle x | \psi \rangle = \sum_{m=1}^{\infty} a_m \langle x | E_m \rangle$$

$$\psi(x) = \sum_{m=1}^{\infty} a_m \psi_m(x)$$

\uparrow
EXPANSION
COEFF'S

\nwarrow
EXPANSION FCNS
"
ENERGY EIGEN FCNS

EXPANSION

FINDING THE COEFF'S

$$|\psi\rangle = \sum_{m=1}^{\infty} a_m |E_m\rangle$$

$$\langle E_m |$$

$$\langle E_m | \psi \rangle = \sum_{m=1}^{\infty} a_m \langle E_m | E_m \rangle$$

$$= a_m$$

$$a_m = \langle E_m | \psi \rangle = \langle E_m | I | \psi_m \rangle$$

$$= \int \langle E_m | I x \rangle \langle x | \psi \rangle dx$$

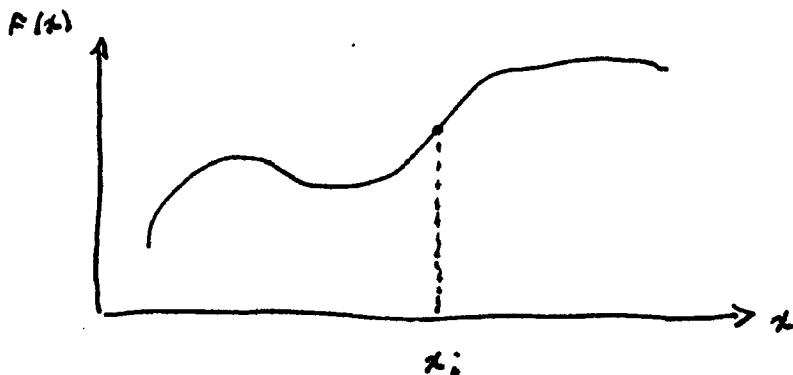
$$= \int \psi_m^*(x) \psi(x) dx$$

$$\sin\left(\frac{m\pi x}{L}\right)$$

DIRAC DELTA FCNS

FOURIER TRANSFORMS

DIRAC DELTA FUNCTION

(THING) THAT PICKS OUT $f(x_i)$

DIRAC:

$$\int (\text{THING}) f(x) dx = f(x_i)$$

\downarrow

$$\delta(x - x_i)$$

\nearrow NUMBER

KRONENCKER:

$$\sum (\text{THING}) a_i \delta_{ij} = a_j$$

\downarrow

$$\delta_{ij}$$

\nearrow NUMBER

DIRAC THING MAP: FUNCTION \rightarrow NUMBERKRONENCKER THING MAP: VECTOR \rightarrow NUMBEROLD WISDOM: DIRAC δ -FCNS ONLY MAKE SENSE INSIDE AN INTEGRAL

NEW WISDOM: FUNCTIONAL NOT A FUNCTION

TWO KINDS OF FUNCTIONS

GOOD OLD FUNCTIONS

D. PHYSICAL FCNS

P: PROPER FCNS ↗

M: SQUARE INTEGRABLE L^2

$$\langle f | f \rangle \text{ FINITE}$$

GENERALIZED FCNS

$$\langle x | x' \rangle \text{ INFINITE} \rightarrow \text{NOT SQUARE INTEGRABLE}$$

$$\langle x | f \rangle \text{ FINITE}$$

P: IMPROPER FCNS

M: DISTRIBUTIONS

DIRAC DELTA FCN

$$\langle x | g \rangle = \int (\text{THING}) g(x') dx' = g(x)$$

$$\delta(x-x')$$

well done!

$$\delta(x-x') = \langle x | x' \rangle = \langle x' | x \rangle$$

$$\delta(p-p') = \langle p | p' \rangle = \langle p' | p \rangle$$

PROPERTIES OF DIRAC DELTA FCN

$$(1) \quad \delta(x-x') = 0 \text{ WHEN } x \neq x'$$

$$(2) \quad \int_a^b \delta(x-x') dx' = 1 \quad \text{WHEN } a \leq x \leq b$$

$$(3) \quad \delta(x-x') = \delta(x'-x) \quad \text{EVEN FCN}$$

GAUSSIAN REPRESENTATION

$$g_\Delta(x-x') = \frac{1}{\sqrt{\pi} \Delta} e^{-{(x-x')}^2/\Delta^2}$$

$$\delta(x-x') = \lim_{\Delta \rightarrow 0} g_\Delta(x-x')$$

ACTION

$$\int \delta(x-x') f(x') dx' = f(x)$$

OLD WISDOM: DELTA FCNS ONLY MAKE SENSE

INSIDE INTEGRALS



NEW WISDOM: DIRAC'S δ

IS NOT A FCN, it is a FCNAL
FCN: number \rightarrow number
FCNAL: number \rightarrow FCN

CAN ALSO DEFINE DERIVATIVES OF DELTA

$$\delta'(x-x') \text{ def} = \frac{d}{dx} [\delta(x-x')]$$

$$= -\frac{d}{dx'} [\delta(x-x')]$$

↑
↓



ODD Fcn



ACTION OF δ'

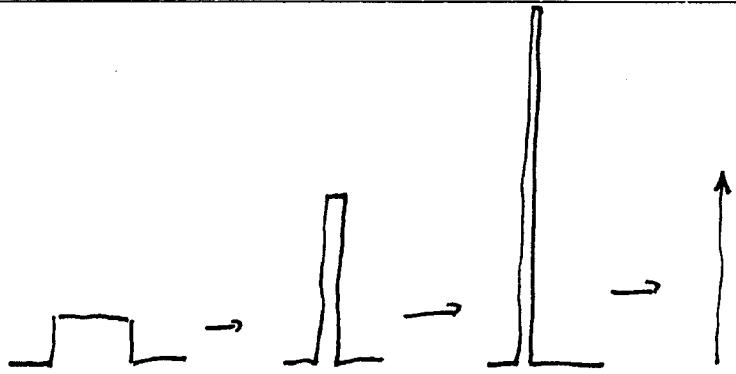
$$\int \delta'(x-x') f(x') dx' = \frac{df(x)}{dx}$$

GENERALIZE

$$\frac{d^n [\delta(x-x')]}{dx^m} = \delta(x-x') \frac{d^n}{dx'^n}$$

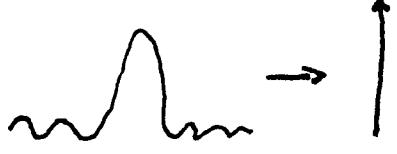
OTHER REPS

RECTANGLE FCN



SINC FCN

$$\text{SINC}(x) = \frac{\sin x}{x}$$



EXP FCN

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(x-x')\xi} d\xi$$

Representations of the delta function

The delta function can be viewed as the limit of a sequence of functions

$$\delta(x) = \lim_{a \rightarrow 0} \delta_a(x),$$

where $\delta_a(x)$ is sometimes called a *nascent delta function*. This limit is in the sense that

$$\lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \delta_a(x) f(x) dx = f(0)$$

for all continuous f .

The term *approximate identity* has a particular meaning in harmonic analysis, in relation to a limiting sequence to an identity element for the convolution operation (also on groups more general than the real numbers, e.g. the unit circle). There the condition is made that the limiting sequence should be of positive functions.

Some nascent delta functions are:

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \quad \text{Limit of a Normal distribution}$$

$$\delta_a(x) = \frac{1}{\pi} \frac{a}{a^2 + x^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - |ak|} dk \quad \text{Limit of a Cauchy distribution}$$

$$\delta_a(x) = \frac{e^{-|x/a|}}{2a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + a^2 k^2} dk \quad \text{Cauchy } \varphi(\text{see note below})$$

$$\delta_a(x) = \frac{\text{rect}(x/a)}{a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{sinc}\left(\frac{ak}{2\pi}\right) e^{ikx} dk \quad \text{Limit of a rectangular function}$$

$$\delta_a(x) = \frac{1}{\pi x} \sin\left(\frac{x}{a}\right) = \frac{1}{2\pi} \int_{-1/a}^{1/a} \cos(kx) dk \quad \text{rectangular function } \varphi(\text{see note below})$$

$$\delta_a(x) = \partial_x \frac{1}{1 + e^{-x/a}} = -\partial_x \frac{1}{1 + e^{x/a}} \quad \text{Derivative of the sigmoid (or Fermi-Dirac) function}$$

$$\delta_a(x) = \frac{a}{\pi x^2} \sin^2\left(\frac{x}{a}\right)$$

$$\delta_a(x) = \frac{1}{a} A_i\left(\frac{x}{a}\right) \quad \text{Limit of the Airy function}$$

$$\delta_a(x) = \frac{1}{a} J_{1/a} \left(\frac{x+1}{a} \right)$$

Limit of a Bessel function

Note: If $\delta(a, x)$ is a nascent delta function which is a probability distribution over the whole real line (i.e. is always non-negative between $-\infty$ and $+\infty$) then another nascent delta function $\delta_\varphi(a, x)$ can be built from its characteristic function as follows:

$$\delta_\varphi(a, x) = \frac{1}{2\pi} \frac{\varphi(1/a, x)}{\delta(1/a, 0)}$$

where

$$\varphi(a, k) = \int_{-\infty}^{\infty} \delta(a, x) e^{-ikx} dx$$

is the characteristic function of the nascent delta function $\delta(a, x)$. This result is related to the localization property of the continuous Fourier transform.

The Dirac comb

Main article: Dirac comb

A so-called uniform "pulse train" of Dirac delta measures, which is known as a Dirac comb, or as the shah distribution, creates a sampling function, often used in digital signal processing (DSP) and discrete time signal analysis.

See also

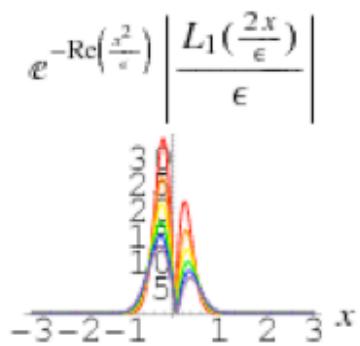
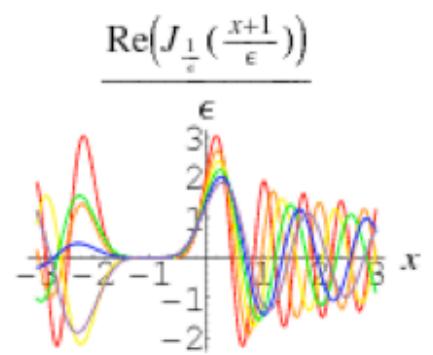
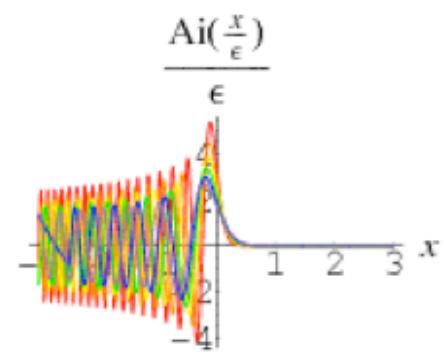
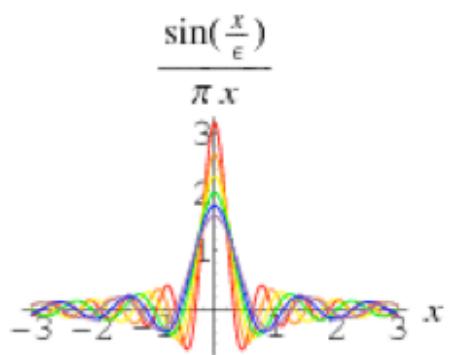
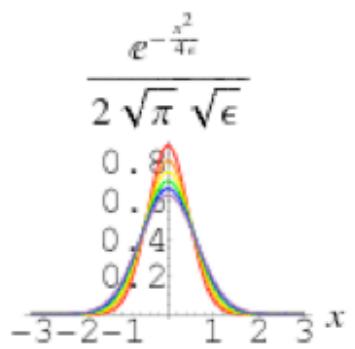
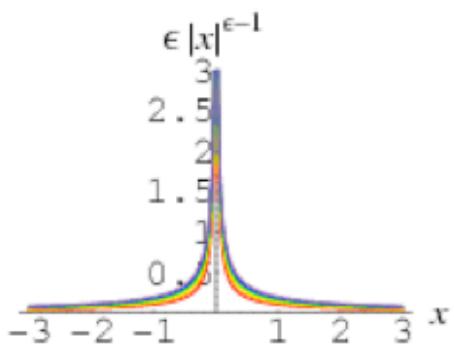
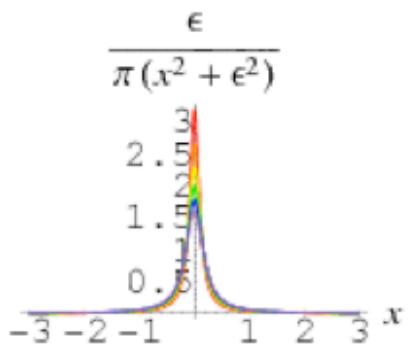
- Kronecker delta
- Dirac comb
- Logarithmically-spaced Dirac comb
- Green's function
- Dirac measure

External links

- Delta Function (<http://mathworld.wolfram.com/DeltaFunction.html>) on MathWorld
- Dirac Delta Function (<http://planetmath.org/encyclopedia/DiracDeltaFunction.html>) on PlanetMath
- The Dirac delta measure is a hyperfunction (<http://www.osaka-kyoiku.ac.jp/~ashino/pdf/chinaproceedings.pdf>)
- We show the existence of a unique solution and analyze a finite element approximation when the source term is a Dirac delta measure (<http://pubs.siam.org/sam-bin/dbq/article/43178>)
- Non-Lebesgue measures on R. Lebesgue-Stieltjes measure, Dirac delta measure. (<http://www.mathematik.uni-muenchen.de/~lerdos/WS04/FA/content.html>)

The delta function can be defined as the following limits as $\epsilon \rightarrow 0$,

$$\begin{aligned}
\delta(x) &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{x^2 + \epsilon^2}, \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{2} \epsilon |x|^{\epsilon-1} \\
&= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2 \sqrt{\pi \epsilon}} e^{-x^2/(4\epsilon)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \text{Ai}\left(\frac{x}{\epsilon}\right) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} J_{1/\epsilon}\left(\frac{x+1}{\epsilon}\right) \\
&= \lim_{\epsilon \rightarrow 0} \left| \frac{1}{\epsilon} e^{-x^2/\epsilon} L_n\left(\frac{2x}{\epsilon}\right) \right|,
\end{aligned}$$



~~DISCRETE~~ FOURIER TRANSFORMS

TIME

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

SYMMETRIC CONVENTION

ω TEMPORAL FREQUENCY

SPACE

κ

κ SPATIAL FREQUENCY

$$F(\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\kappa) e^{i\kappa x} d\kappa$$

$$\epsilon = \kappa \omega$$

$$p = \hbar \kappa$$

$$dp = \hbar dk$$