

# Lecture 16

## Angular Momentum 3

- (1) The Ladder Operators for angular momentum
- (2) The semiclassical vector model
- (3) Academic geneology



$$[L_z, L_+] = \hbar L_+$$

$$L_z L_+ - L_+ L_z = \hbar L_+$$

$$\begin{aligned} L_z L_+ |\alpha \beta\rangle &= L_+ L_z |\alpha \beta\rangle + \hbar L_+ |\alpha \beta\rangle \\ &= \beta L_+ |\alpha \beta\rangle + \hbar L_+ |\alpha \beta\rangle \\ &= (\beta + \hbar) L_+ |\alpha \beta\rangle \end{aligned}$$

$$L_z (L_+ |\alpha \beta\rangle) = (\beta + \hbar) (L_+ |\alpha \beta\rangle)$$

$L_+ |\alpha \beta\rangle$  IS AN  $e\vec{v}$  OF  $L_z$  WITH  $e\hbar = \beta + \hbar$

$L_-$

$$e\hbar = \beta - \hbar$$

STEP SIZE =  $\beta$



THIS LADDER HAS A TOP AND A BOTTOM

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 - L_z^2 = L_x^2 + L_y^2$$

$$\langle \alpha, \beta | (L^2 - L_z^2) | \alpha, \beta \rangle = \langle \alpha, \beta | (L_x^2 + L_y^2) | \alpha, \beta \rangle$$

$$\langle \alpha, \beta | (\alpha - \beta^2) | \alpha, \beta \rangle$$

$$(\alpha - \beta^2) \langle \alpha, \beta | \alpha, \beta \rangle$$

$$(\alpha - \beta^2) = \langle \alpha, \beta | (L_x^2 + L_y^2) | \alpha, \beta \rangle \geq 0$$

$$\beta^2 \leq \alpha$$

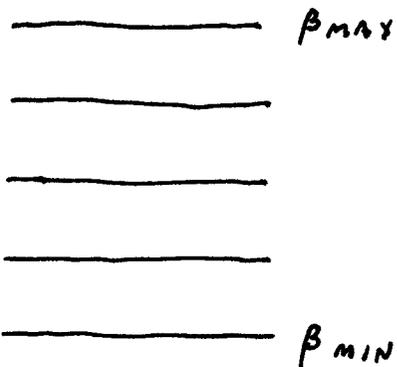
PHYSICALLY: GIVEN TOTAL ANGULAR MOMENTUM  
SQUARED  $\alpha$ , ITS Z PROJECTION  $\beta$   
CANNOT BE GREATER THAN  $\alpha$

TOP AND BOTTOM TO LADDER

$$L_+ | \alpha, \beta_{\max} \rangle = 0$$

$$L_- | \alpha, \beta_{\min} \rangle = 0$$

NEXT STEP: SHOW  $\beta_{\max} = -\beta_{\min}$



STEP SIZE =  $h$

$$\beta_{\max} - \beta_{\min} = 2\beta_{\max} = (\text{integer}) h$$

$$\beta_{\max} = \frac{1}{2} h (\text{integer}) \quad \text{integer} = 0, 1, 2, \dots$$

$$\alpha = (\beta_{\max}) (\beta_{\max} + h) = h^2 \left( \frac{k}{2} \right) \left( \frac{k}{2} + 1 \right)$$

$$\alpha = l(l+1) h^2$$



## ORBITAL ANGULAR MOMENTUM

$$l = 0, 1, 2, 3, \dots$$

$$m = -l, \dots, 0, \dots, l$$

$$L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$L_z |l, m\rangle = m\hbar |l, m\rangle$$

$$L_{\pm} |l, m\rangle = \sqrt{l(l+1) - m(m\pm 1)} \hbar |l, m\pm 1\rangle$$

### SPECIAL CASES:

$$l = 0 \Rightarrow m = 0$$

$$2l+1 = 1$$

SINGLET

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$$\underline{\hspace{10em}} |0, 0\rangle$$

$$L^2 |0, 0\rangle = 0 \hbar^2 |0, 0\rangle$$

$$L_z |0, 0\rangle = 0 \hbar |0, 0\rangle$$

$$l=1 \quad m = -1, 0, +1$$

$$2l+1 = 3$$

TRIPLET

$$\text{-----} \quad |1, 1\rangle$$

$$\text{-----} \quad |1, 0\rangle$$

$$\text{-----} \quad |1, -1\rangle$$

$$L^2 |1, m\rangle = 1(1+1) \hbar^2 |1, m\rangle = 2 \hbar^2 |1, m\rangle$$

$$L_z |1, m\rangle = m \hbar |1, m\rangle$$

$$l=2 \quad m = -2, -1, 0, +1, +2$$

$$2l+1 = 5$$

QUINTET

$$\text{-----} \quad |2, 2\rangle$$

$$\text{-----} \quad |2, 1\rangle$$

$$\text{-----} \quad |2, 0\rangle$$

$$\text{-----} \quad |2, -1\rangle$$

$$\text{-----} \quad |2, -2\rangle$$

$$L^2 |2, m\rangle = 6 \hbar^2 |2, m\rangle$$

$$L_z |2, m\rangle = m \hbar |2, m\rangle$$

$$S = \frac{1}{2}$$

$$m_S = -\frac{1}{2}, \frac{1}{2}$$

$$2S + 1 = 2$$

DOUBLET

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$$|\frac{1}{2}, \frac{1}{2}\rangle$$

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$$|\frac{1}{2}, -\frac{1}{2}\rangle$$

$$S = \frac{3}{2}$$

$$m_S = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$$

$$2S + 1 = 4$$

QUARTET

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$$|\frac{3}{2}, \frac{3}{2}\rangle$$

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$$|\frac{3}{2}, \frac{1}{2}\rangle$$

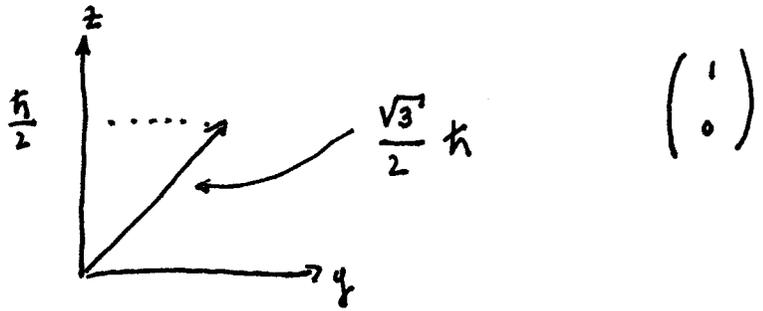
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$$|\frac{3}{2}, -\frac{1}{2}\rangle$$

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$$|\frac{3}{2}, -\frac{3}{2}\rangle$$





$$\langle S_z \rangle = (1 \ 0)^* \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle S_z \rangle = \frac{\hbar}{2} = m\hbar$$

$$\langle S_x \rangle = (1 \ 0)^* \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle S_x \rangle = 0$$

$$\langle S_y \rangle = (1 \ 0)^* \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle S_y \rangle = 0$$

MEASURE  $S_x$        $\frac{1}{2}$  TIME       $+\frac{\hbar}{2}$

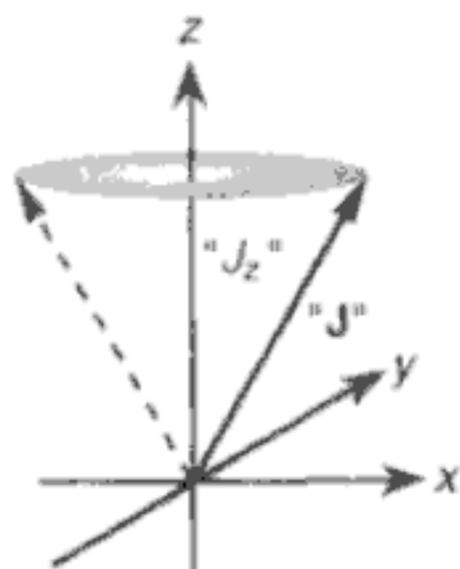
$\frac{1}{2}$  TIME       $-\frac{\hbar}{2}$

MEASURE  $S_y$        $\frac{1}{2}$  TIME       $+\frac{\hbar}{2}$

$\frac{1}{2}$  TIME       $-\frac{\hbar}{2}$

### Rotations

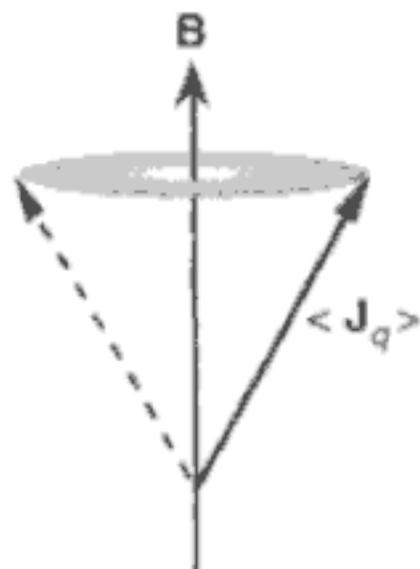
Operator eigenvalues



$$\begin{aligned}\langle jm | J_x | jm \rangle &= 0 \\ \langle jm | J_y | jm \rangle &= 0 \\ \langle jm | J_z | jm \rangle &= m \\ \langle jm | \mathbf{J}^2 | jm \rangle &= j(j+1)\end{aligned}$$

### Quantum mechanics

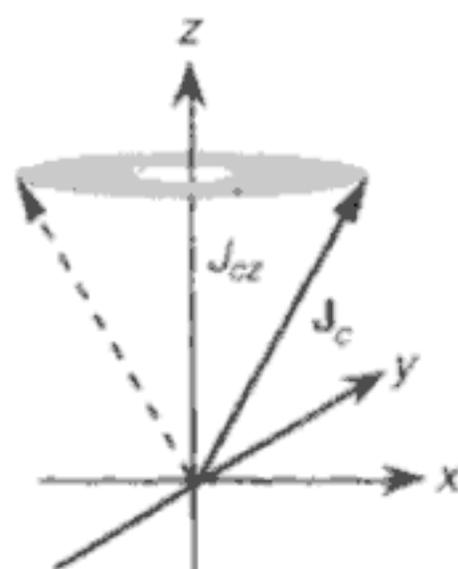
Larmor precession



$$\frac{d\langle \mathbf{J}_q \rangle}{dt} = \omega_L \langle \mathbf{J}_q \rangle \times \mathbf{B}/B$$

### Classical mechanics

Uniform precession



Time averages  
 $J_{cx} = 0 \quad J_{cy} = 0$

# The vector model

This is a useful semi-classical model of the quantum results.

**Imagine  $\mathbf{L}$  precesses around the z-axis.** Hence the magnitude of  $\mathbf{L}$  and the z-component  $L_z$  are constant while the x and y components can take a range of values and average to zero, just like the quantum eigenfunctions.

A given quantum number  $l$  determines the magnitude of the vector  $\mathbf{L}$  via

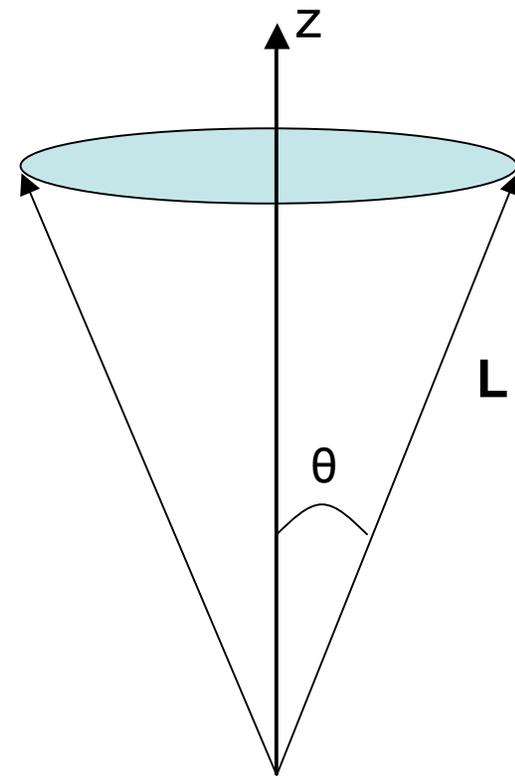
$$L^2 = l(l+1)\hbar^2$$

$$|\mathbf{L}| = \sqrt{l(l+1)}\hbar$$

The z-component can have the  $2l+1$  values corresponding to

$$L_z = m\hbar, \quad -l \leq m \leq l$$

In the vector model this means that only particular special angles between the angular momentum vector and the z-axis are allowed



# The vector model (2)

Example:  $l=2$

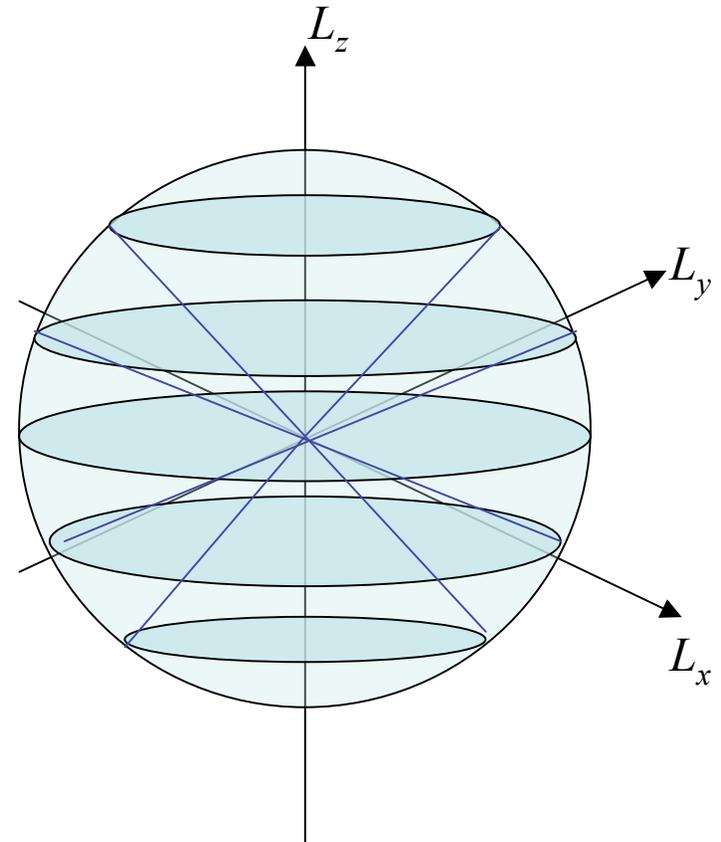
Magnitude of the angular momentum is

$$L^2 = l(l+1)\hbar^2 = 6\hbar^2$$

$$|\mathbf{L}| = \sqrt{l(l+1)}\hbar = \sqrt{6}\hbar$$

Component of angular momentum  
in z- direction can be

$$-l \leq m \leq l \Rightarrow L_z = -2\hbar, -\hbar, 0, \hbar, 2\hbar$$



Quantum eigenfunctions correspond to a cone of solutions for  $\mathbf{L}$  in the vector model

