

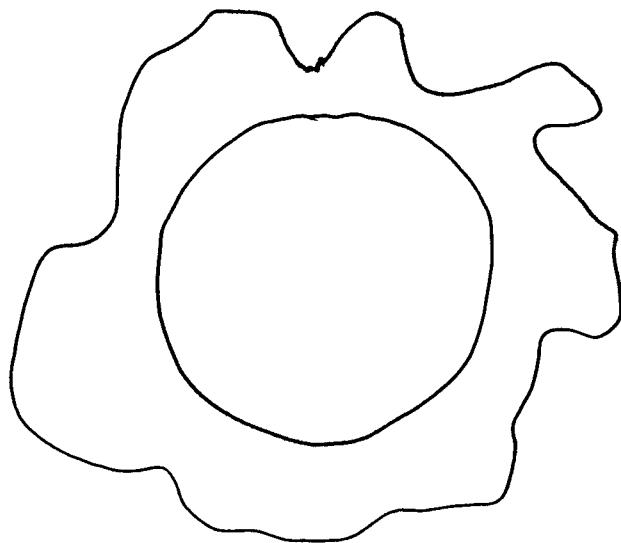
# **Lecture 14      Angular Momentum**

**Math:** Spherical harmonics are a complete set of orthonormal functions on the sphere

**Physics:** Spherical harmonics describe the orbital angular momentum

MATH : SPHERICAL HARMONICS ARE  
A COMPLETE ORTHONORMAL  
SET OF BASIS FUNCTIONS  
ON THE SPHERE.

PHYSICS : SPHERICAL HARMONICS DESCRIBE  
THE ORBITAL ANGULAR MOMENTUM



$$\text{ANY FCN}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta, \varphi)$$

$$a_{lm} = \int \text{ANY FCN}(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega$$
$$\langle \theta, \varphi | A \rangle = \sum_l \sum_m | l, m \rangle \langle l, m | A \rangle$$

## WHERE WE ARE GOING

SOLVE  $\hat{e}_r, \hat{e}_{\theta}, \hat{e}_{\phi}$  PROBLEM FOR ANGULAR MOMENTUM

$$\hat{L}^2 |l, m\rangle = l(l+1) \hbar^2$$

$$\hat{L}_z |l, m\rangle = m \hbar |l, m\rangle$$

$$\hat{L}_{\pm} |l, m\rangle = \sqrt{l(l+1)-m(m\pm 1)} \hbar |l, m\pm 1\rangle$$

## THREE REPRESENTATIONS

DIRAC REP (IN THE HILBERT SPACE)

MATRIX REP (IN ANG MOMENTUM SPACE)

FUNCTION REP (IN POSITION SPACE  $\theta, \phi$ )

$$\ell = 1$$

$$L^2 = \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

every vector is an  $e\vec{v}$

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \frac{i}{\sqrt{2}} \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$L_4 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$e^{\hat{L}_4 t} \Rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

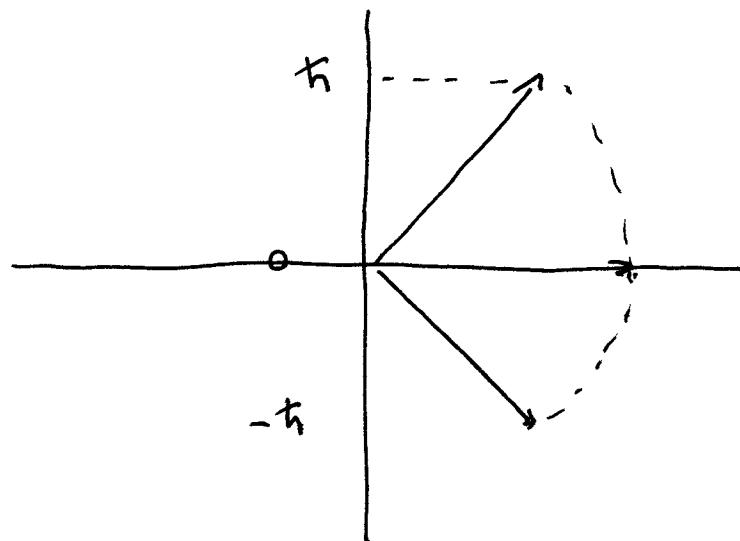
$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_0(x) \\ \psi_-(x) \end{pmatrix}$$

PROB AMP  $S_2 = 1$   
 PROB AMP  $S_2 = 0$   
 PROB AMP  $S_2 = -1$

$|l, l\rangle$   
 $|l, 0\rangle$   
 $|l, -l\rangle$

$$\langle l^2 \rangle = l(l+1)\hbar^2 = 2\hbar^2$$

length =  $\sqrt{2}\hbar$



$$\langle l^2 \rangle = l(l+1)\hbar^2 = 2\hbar^2$$

$$\text{LENGTH} = \sqrt{2}\hbar$$

$$\langle s_z \rangle = m\hbar$$

$$\langle s_x \rangle = 0$$

$$\langle s_y \rangle = 0$$

NEXT STEP ... go into real space

$$|l, m\rangle$$

$$\langle \theta, \varphi | l, m \rangle = Y_{lm}(\theta, \varphi)$$

$$\langle l, m | \theta, \varphi \rangle = Y_{lm}^*(\theta, \varphi)$$

$$\langle l', m' | l, m \rangle = \delta_{ll'} \delta_{mm'}$$

$$\int Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d(\cos \theta) d\varphi = \delta_{ll'} \delta_{mm'}$$

Q: HOW TO FIND ALL THE  $Y_{lm}$ 'S?

A: solve the differential equation ... OR

JUST LIKE SHO, WE HAVE TWO CHOICES

$$L^2 |\ell, m\rangle = \ell(\ell+1) \hbar^2 |\ell, m\rangle$$

$$L_z |\ell, m\rangle = m \hbar |\ell, m\rangle$$

$$-\hbar^2 \left[ \frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{\ell m}(\theta, \varphi) = \ell(\ell+1) \hbar^2 Y_{\ell m}(\theta, \varphi)$$

$$\ell(\ell+1) \hbar^2 Y_{\ell m}(\theta, \varphi)$$

$$-i \hbar \frac{\partial}{\partial \varphi} Y_{\ell m}(\theta, \varphi) = m \hbar Y_{\ell m}(\theta, \varphi)$$

OR

$$L_+ Y_{\ell \ell}(\theta, \varphi) = 0$$

$$\Rightarrow Y_{\ell \ell}$$

$\downarrow L_-$

$$Y_{\ell, \ell-1}$$

$\downarrow L_-$

$$\text{or } L_- Y_{\ell-1, \ell}(\theta, \varphi) = 0 \Rightarrow Y_{\ell-1, \ell}$$

next step find  $F_{em}(\theta)$ 's

$$L + Y_{ee}(\theta, \varphi) = 0$$

~~$$\cancel{k} e^{i\varphi} \left[ \frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right] [F_{ee}(\theta) e^{i\ell\varphi}] = 0$$~~

~~$$\frac{d F_{ee}}{d\theta} \cancel{e^{i\ell\varphi}} + i \cot\theta F_{ee} \cancel{i\ell e^{i\ell\varphi}} = 0$$~~

$$\left[ \frac{d}{d\theta} - \ell \cot\theta \right] F_{ee}(\theta) = 0$$

try  $F_{ee} = A (\sin\theta)^\ell$

$$\frac{d}{d\theta} A (\sin\theta)^\ell = A (\sin\theta)^{\ell-1} \cos\theta$$

$$A (\sin\theta)^{\ell-1} \cos\theta - \ell \frac{\cos\theta}{\sin\theta} A (\sin\theta)^\ell \stackrel{?}{=} 0$$

yep!

$$\Rightarrow Y_{ee}(\theta, \varphi) = A_\ell (\sin\theta)^\ell e^{i\ell\varphi}$$

TO GET THE REST

$$L_- Y_{ee} = a Y_{e e-1}$$

+ then normalize using

$$\int Y_{em}^* Y_{em} d\Omega = 1$$

$$L_- = k e^{-i\varphi} \left( \frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right)$$

SPIRICAL HARMONICS

$$l=0$$

$$Y_{00}(\theta, \varphi) = \langle \theta, \varphi | 0, 0 \rangle = \frac{1}{\sqrt{4\pi}}$$

$$\int \left( \frac{1}{\sqrt{4\pi}} \right)^* \frac{1}{\sqrt{4\pi}} d\Omega = 1 \quad \checkmark$$

$$l=1$$

$$\left. \begin{aligned} Y_{11}(\theta, \varphi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} \\ Y_{10}(\theta, \varphi) &= \sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_{1-1}(\theta, \varphi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi} \end{aligned} \right\} \text{note symmetry}$$

$$l=2$$

$$Y_{2\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} (\sin \theta)^2 e^{\pm 2i\varphi}$$

$$Y_{1\pm 1}(\theta, \varphi) = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\varphi}$$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

# **Spherical Harmonics**

## **The Meaning of the Spherical Harmonics**

[http://infovis.uni-konstanz.de/research/projects/SimSearch3D/images/harmonics\\_img.jpg](http://infovis.uni-konstanz.de/research/projects/SimSearch3D/images/harmonics_img.jpg)

## **The Spherical Harmonics**

<http://oak.ucc.nau.edu/jws8/dpgraph/Yellm.html>

<http://www.bpreid.com/applets/poasDemo.html>

<http://www.du.edu/~jcalvert/math/harmonic/harmonic.htm>

## **Encyclopedia**

[http://en.wikipedia.org/wiki/Spherical\\_harmonics](http://en.wikipedia.org/wiki/Spherical_harmonics)

[http://en.wikipedia.org/wiki/Table\\_of\\_spherical\\_harmonics](http://en.wikipedia.org/wiki/Table_of_spherical_harmonics)

<http://mathworld.wolfram.com/SphericalHarmonic.html>

## **Applications of Spherical Harmonics**

<http://www.falstad.com/qmrotator/>

<http://www.falstad.com/qmatom/>

<http://www.falstad.com/qmatomrad/>

<http://www.falstad.com/qm2dosc/>

<http://www.falstad.com/qm3dosc/>

# **Legendre Polynomials**

## **The Meaning of the Legendre Polynomials**

<http://physics.unl.edu/~tgay/content/multipoles.html>

## **Encyclopedia**

[http://en.wikipedia.org/wiki/Legendre\\_polynomials](http://en.wikipedia.org/wiki/Legendre_polynomials)

<http://mathworld.wolfram.com/LegendrePolynomial.html>

# **Wolfram Demonstrations**

<http://demonstrations.wolfram.com/SphericalHarmonics/>

<http://demonstrations.wolfram.com/VisualizingAtomicOrbitals/>

<http://demonstrations.wolfram.com/HydrogenOrbitals/>

<http://demonstrations.wolfram.com/PlotsOfLegendrePolynomials/>

<http://demonstrations.wolfram.com/PolarPlotsOfLegendrePolynomials/>

<http://demonstrations.wolfram.com/DipoleAntennaRadiationPattern/>

## **Other Examples**

### **The Earth's Magnetic Field**

[http://cgc.rncan.gc.ca/geomag/nmp/early\\_nmp\\_e.php?p=1](http://cgc.rncan.gc.ca/geomag/nmp/early_nmp_e.php?p=1)  
[http://en.wikipedia.org/wiki/Earth%27s\\_magnetic\\_field](http://en.wikipedia.org/wiki/Earth%27s_magnetic_field)  
<http://www.ngdc.noaa.gov/geomag/WMM/DoDWMM.shtml>  
[http://www.geomag.us/info/Declination/magnetic\\_lines\\_2010.gif](http://www.geomag.us/info/Declination/magnetic_lines_2010.gif)

### **The Earth's Gravitational Field**

<http://en.wikipedia.org/wiki/Geoid>  
[http://www.esri.com/news/arcuser/0703/graphics/geoid1\\_lg.gif](http://www.esri.com/news/arcuser/0703/graphics/geoid1_lg.gif)  
<http://www.geomag.us/models/pomme5.html>  
<http://earth-info.nga.mil/GandG/images/ww15mgh2.gif>  
<http://op.gfz-potsdam.de/champ/>  
[http://www.gfy.ku.dk/~pditlev/annual\\_report/matematiker.jpg](http://www.gfy.ku.dk/~pditlev/annual_report/matematiker.jpg)

### **The Universe**

[http://abyss.uoregon.edu/~js/21st\\_century\\_science/lectures/lec27.html](http://abyss.uoregon.edu/~js/21st_century_science/lectures/lec27.html)  
[http://wmap.gsfc.nasa.gov/media/080997/080997\\_5yrFullSky\\_WMAP\\_4096B.tif](http://wmap.gsfc.nasa.gov/media/080997/080997_5yrFullSky_WMAP_4096B.tif)

### **Computer Lighting and Games**

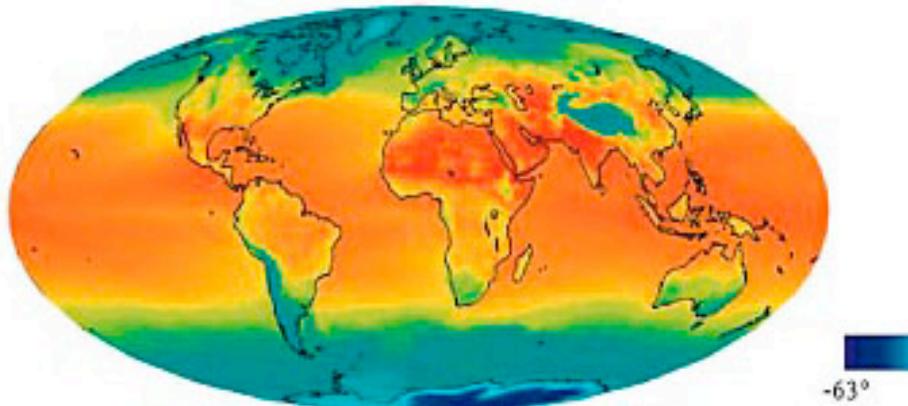
<https://buffy.eecs.berkeley.edu/PHP/resabs/images/2006//101194-5.jpg>  
[http://www.cg.tuwien.ac.at/research/publications/2008/Habel\\_08\\_SSH/](http://www.cg.tuwien.ac.at/research/publications/2008/Habel_08_SSH/)  
<http://www.planetlara.com/underworld/render/lara/full.jpg>  
<http://casuallyhardcore.com/blog/index.php?s=shader>

### **Art**

<http://www.math.hawaii.edu/~dale/bleecker/bleecker.html>  
<http://cricketdiane.files.wordpress.com/2009/04/cricketdiane-castle-in-the-sky-2006-1.jpg>

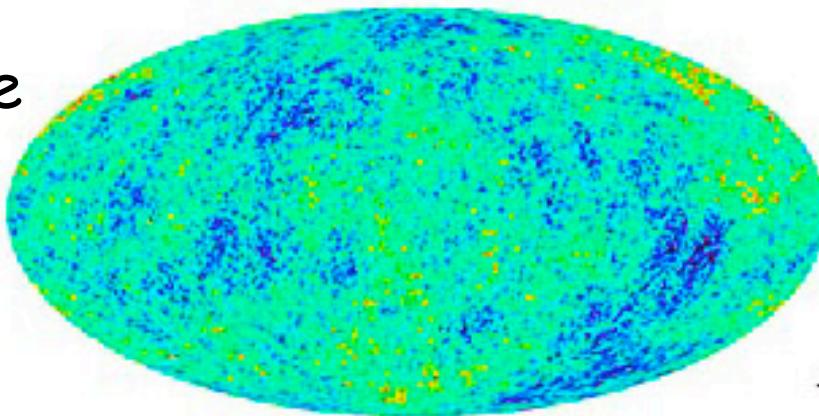
# Temperature Maps

Earth



-63°      -13°      37°  
Centigrade  
JUNE 1992

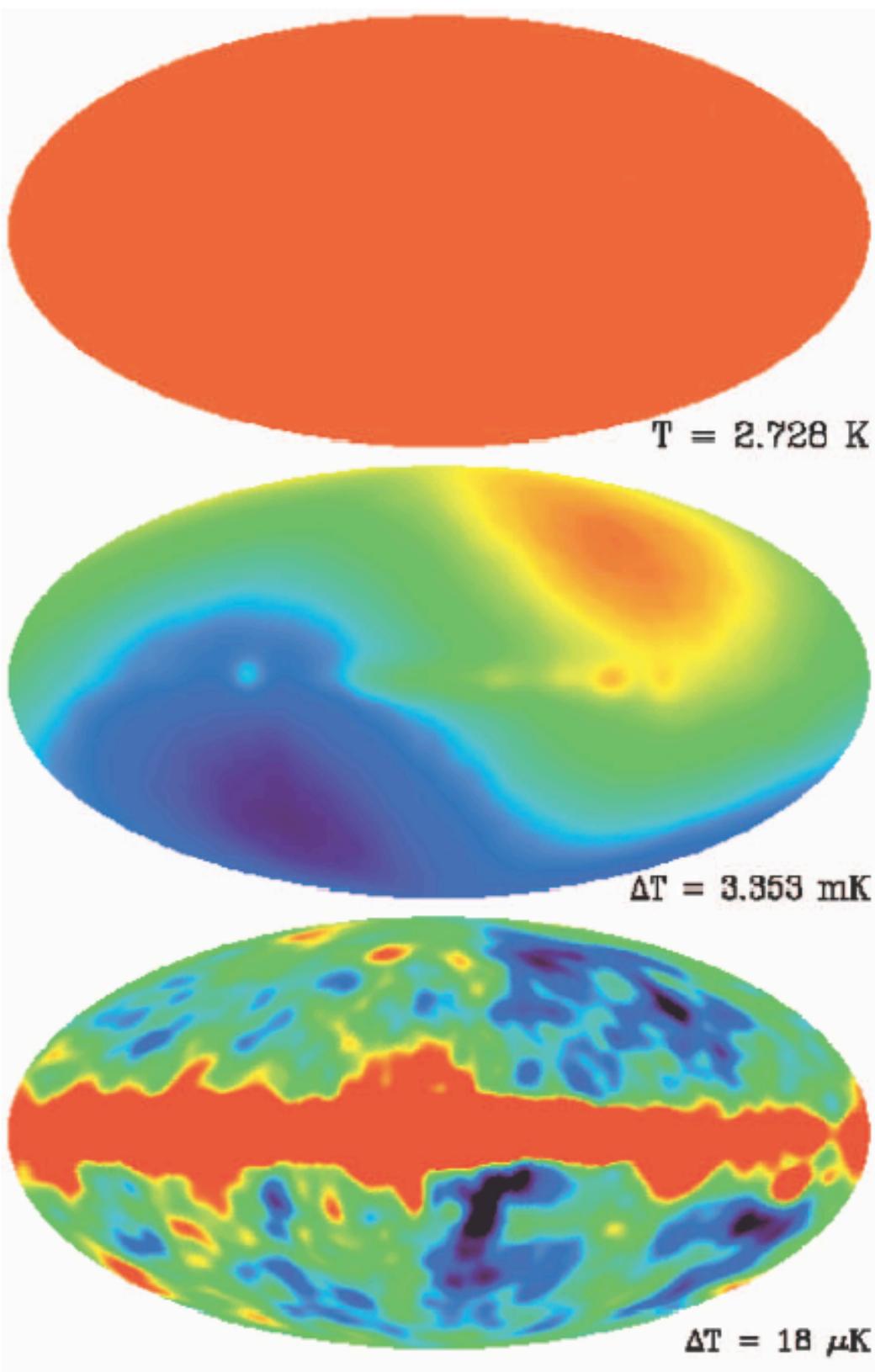
Universe

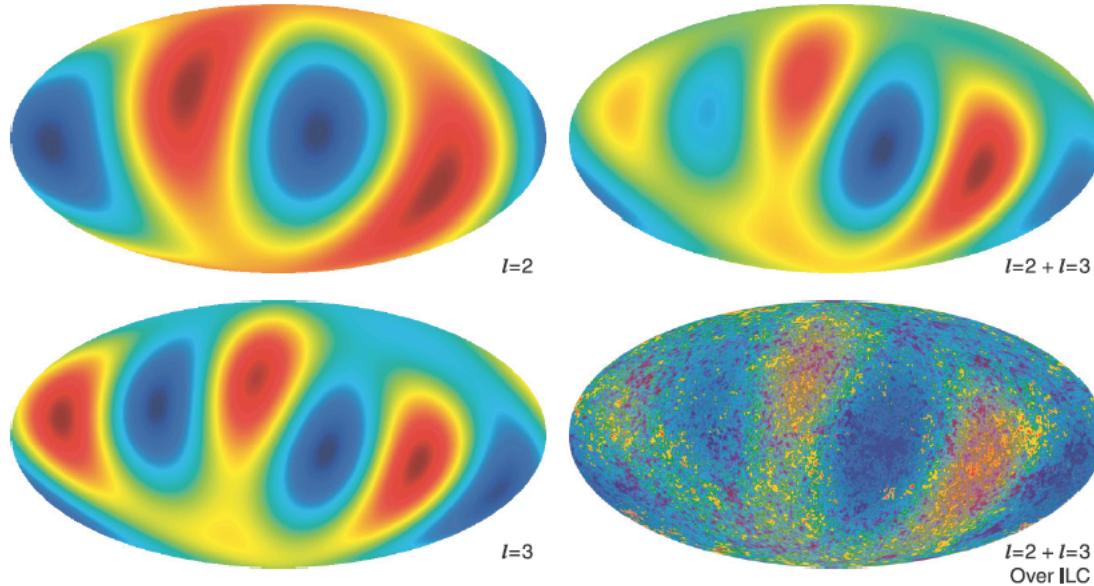


-300      +300  
microKelvin  
300,000 Years ABB

About 1% of TV White Noise is due to CMB







**Figure 14.** The  $l = 2$  quadrupole map and  $l = 3$  octupole map are added. The combined map is then shown superposed on the ILC map from Figure 2. Note that the quadrupole and octupole components arrange themselves to match the cool fingers and the warm regions in between. The fingers and the alignment of the  $l = 2$  and  $l = 3$  multipoles are intimately connected.

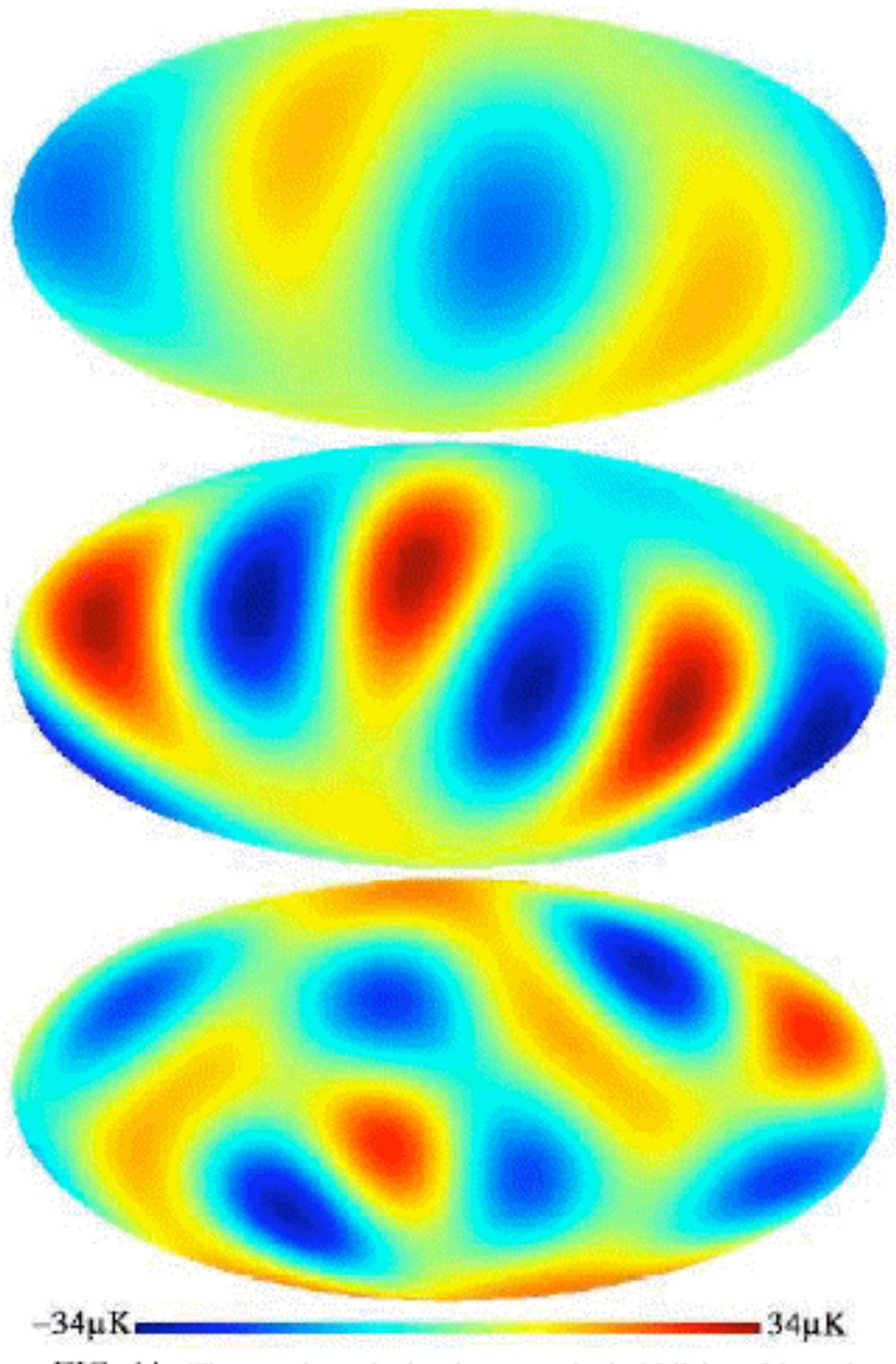
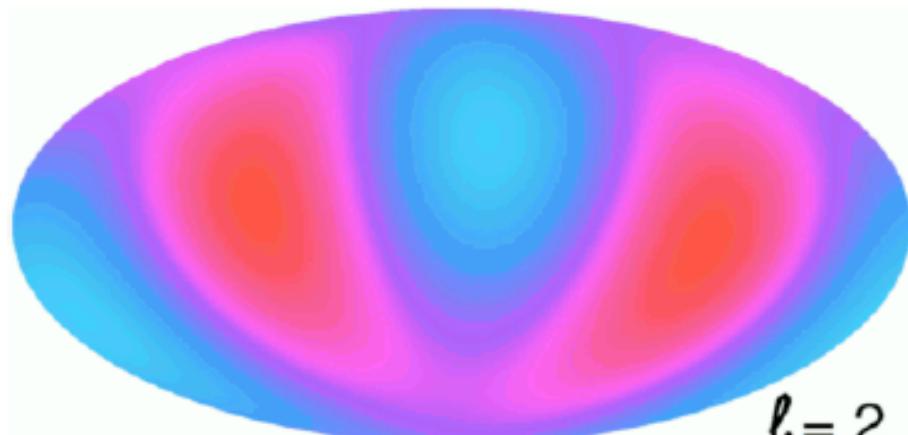


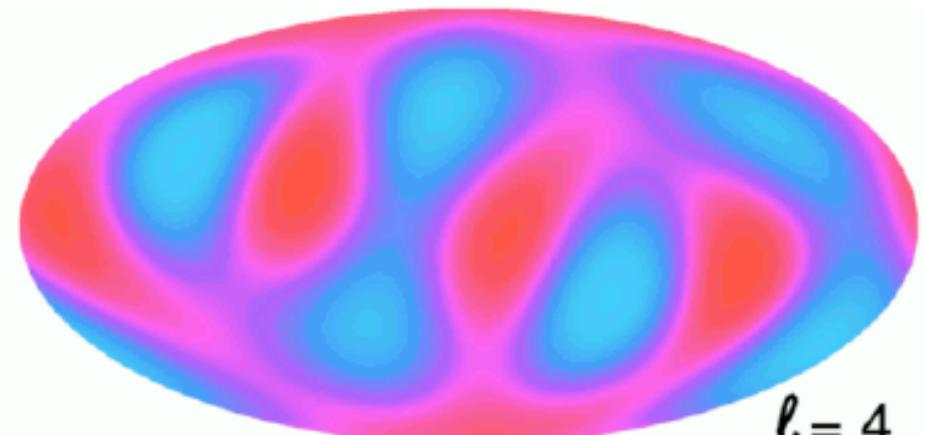
FIG. 14. The quadrupole (top), octopole (middle) and hexadecapole (bottom) components of our cleaned all-sky CMB map from Figure 1 are shown on a common temperature scale. Note that not only is the quadrupole power low, but both it and the octopole have almost all their power perpendicular to a common axis in space, as if some process has suppressed large-scale power in the direction of this axis. We computed the corresponding images for the WMAP team ILC map as well, and found them to be very similar.

# Spherical Harmonic Decomposition

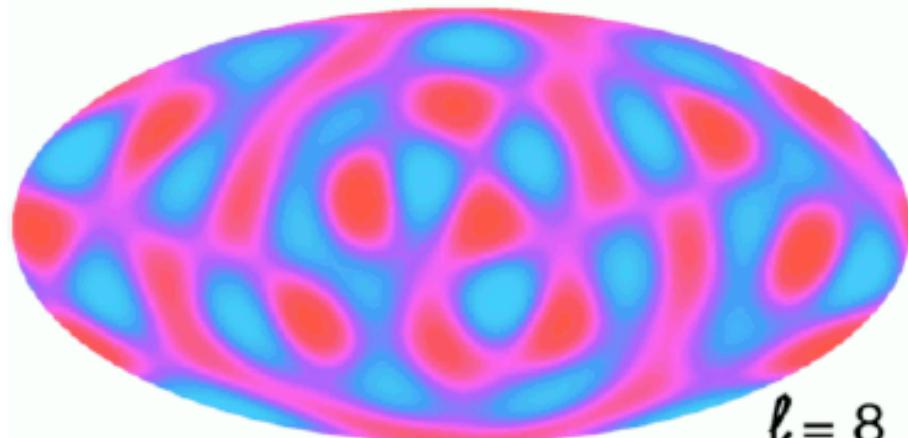
$$\frac{\delta T}{T}(\theta, \phi) = \sum_{l,m} a_{l,m} Y_{l,m}(\theta, \phi)$$



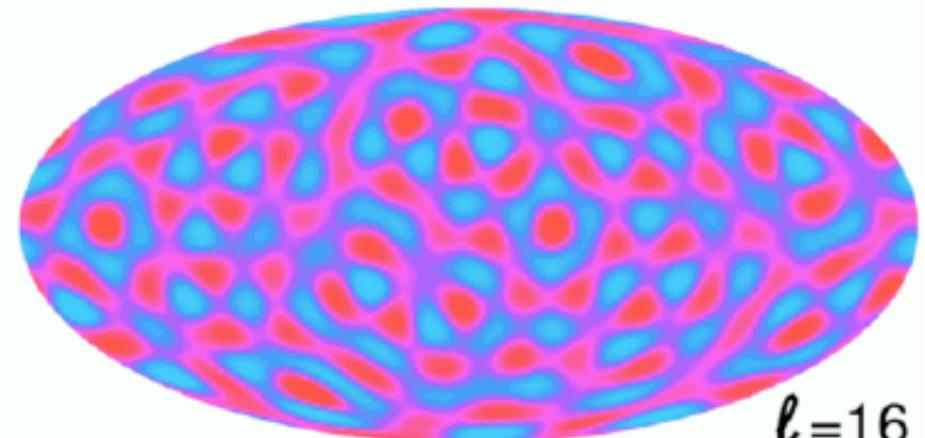
$\ell = 2$



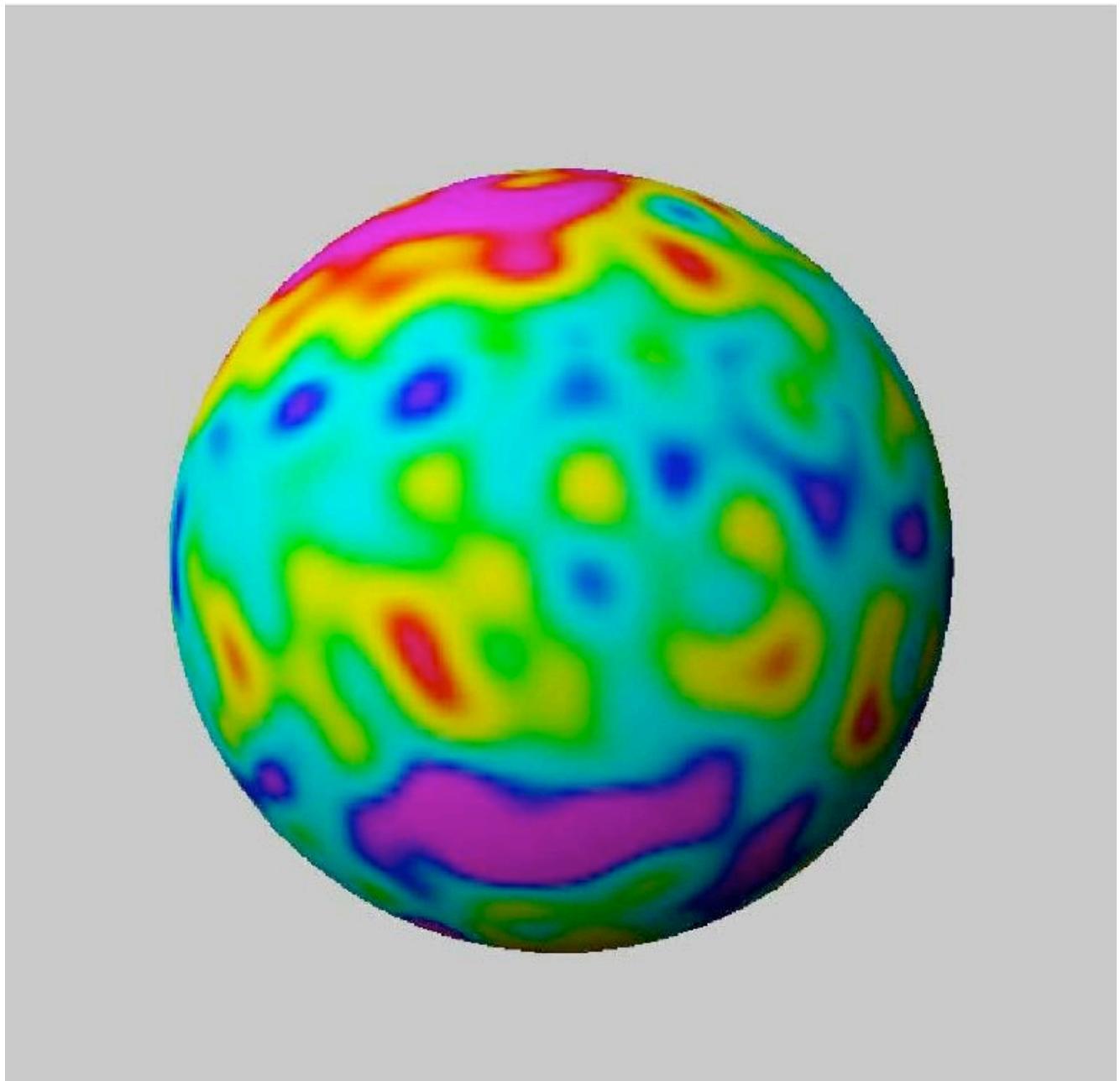
$\ell = 4$



$\ell = 8$

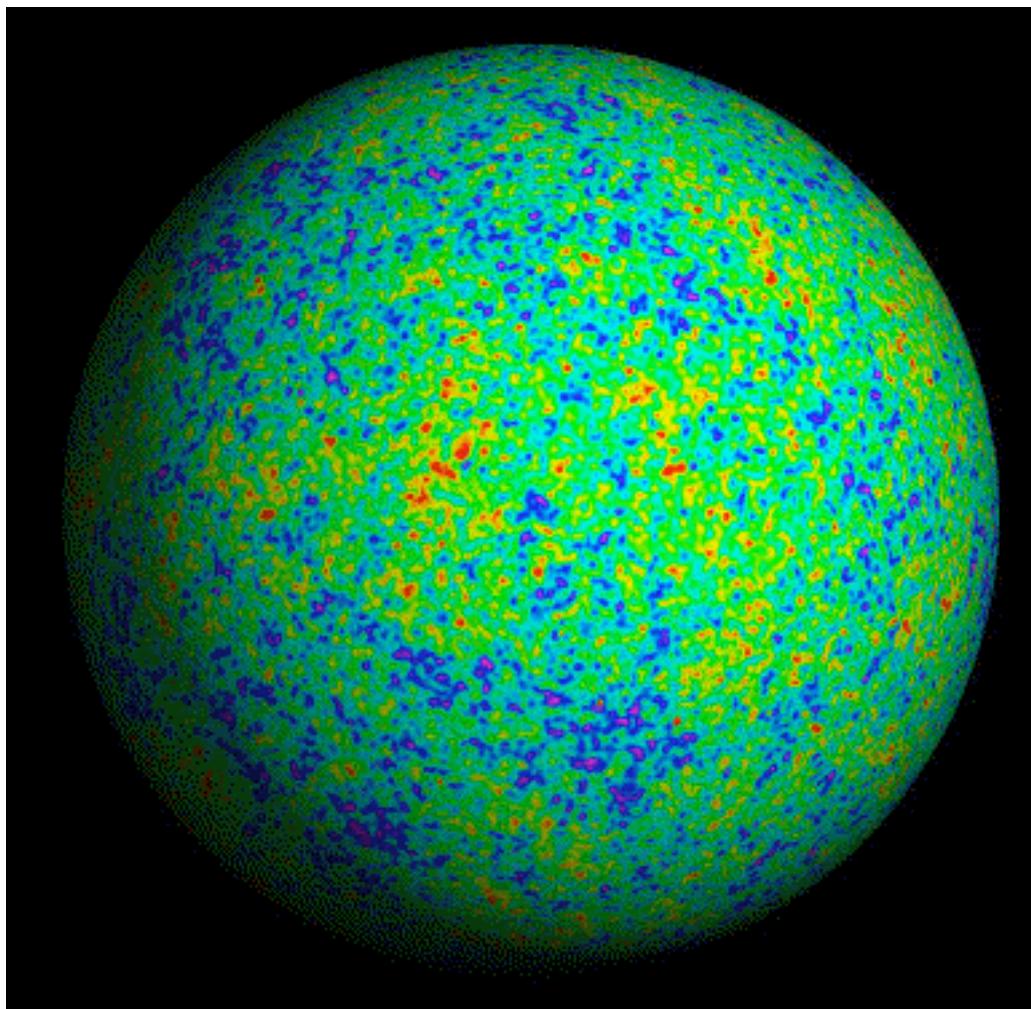


$\ell = 16$



northCMBhemisphere.gif...

<http://www.valdostamus...>



2. Consider a system initially in the state  $|\psi(0)\rangle$  with the Hamiltonian  $H$ , where

$$|\psi(0)\rangle = N \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ in the } L_z \text{ basis, and where } H = (2\omega/\hbar)L^2 + (3\omega)L_z.$$

- (a) What angular momentum is described by the 3-component vector  $|\psi(0)\rangle$ ? What is the length of this vector? What are the allowed  $z$ -projections?
- (b) Calculate the normalization constant  $N$  and the Hamiltonian matrix.

Hint:  $L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$  and  $L_z |l, m\rangle = m\hbar |l, m\rangle$ .

- (c) Calculate the eigenvalues and the eigenvectors of the Hamiltonian.

Hint:  $H$ ,  $L^2$  and  $L_z$  all commute.

- (d) Calculate the time evolution of the state vector  $|\psi(t)\rangle$  by expanding  $|\psi(0)\rangle$  as a sum of energy eigenvectors and using the time evolution of the energy eigenvectors.
- (e) If the energy is measured at time  $t$ , what results can be found and with what probabilities will these results be found?
- (f) Calculate  $\langle E \rangle$  and  $\Delta E$ . Plot  $P(E)$  vs.  $E$  and indicate  $\langle E \rangle$  and  $\Delta E$  on your plot.
- (g) If  $L^2$  and  $L_z$  are measured at time  $t$ , what results can be found and with what probabilities will these results be found?

FOR  $\ell = 1$

$$H = A L^2 + B L_z$$

$$L^2 \rightarrow \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_z \rightarrow \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$H \rightarrow \begin{pmatrix} 2A\hbar^2 + B\hbar & 0 & 0 \\ 0 & 2A\hbar^2 & 0 \\ 0 & 0 & 2A\hbar^2 - B\hbar \end{pmatrix}$$