Lecture 13

Equations

Finish simple harmonic oscillator

Start angular momentum

Pictures

Time dependence of superposition states

Spherical harmonics

(*ii*) We see that, for large n, although the absolute value of the momentum is well-defined, its sign is not. This is why ΔP_n is large: for probability distributions with two maxima like that of figure 3, the root-mean-square deviation reflects the distance between the two peaks; it is no longer related to their widths.

2. Evolution of the particle's wave function

Each of the states $| \varphi_n \rangle$, with its wave function $\varphi_n(x)$, describes a stationary state, which leads to time-independent physical predictions. Time evolution appears only when the state vector is a linear combination of several kets $| \varphi_n \rangle$. We shall consider here a very simple case, for which at time $t = 0$ the state vector $| \psi(0) \rangle$ is:

$$| \psi(0) \rangle = \frac{1}{\sqrt{2}} [| \varphi_1 \rangle + | \varphi_2 \rangle] \tag{14}$$

a. WAVE FUNCTION AT THE INSTANT *t*

Apply formula (D-54) of chapter III; we immediately obtain:

$$| \psi(t) \rangle = \frac{1}{\sqrt{2}} [e^{-i \frac{\pi^2 \hbar}{2ma^2} t} | \varphi_1 \rangle + e^{-2i \frac{\pi^2 \hbar}{ma^2} t} | \varphi_2 \rangle] \tag{15}$$

or, omitting a *global* phase factor of $| \psi(t) \rangle$:

$$| \psi(t) \rangle \propto \frac{1}{\sqrt{2}} [| \varphi_1 \rangle + e^{-i\omega_{21} t} | \varphi_2 \rangle] \tag{16}$$

with:

$$\omega_{21} = \frac{E_2 - E_1}{\hbar} = \frac{3\pi^2 \hbar}{2ma^2} \tag{17}$$

b. EVOLUTION OF THE SHAPE OF THE WAVE PACKET

The shape of the wave packet is given by the probability density:

$$|\psi(x, t)|^2 = \frac{1}{2} \varphi_1^2(x) + \frac{1}{2} \varphi_2^2(x) + \varphi_1(x) \varphi_2(x) \cos \omega_{21} t \tag{18}$$

We see that the time variation of the probability density is due to the interference term in $\varphi_1 \varphi_2$. Only one Bohr frequency appears, $\nu_{21} = (E_2 - E_1)/h$, since the initial state (14) is composed only of the two states $| \varphi_1 \rangle$ and $| \varphi_2 \rangle$. The curves corresponding to the variation of the functions φ_1^2, φ_2^2 and $\varphi_1 \varphi_2$ are traced in figures 4-a, b and c.

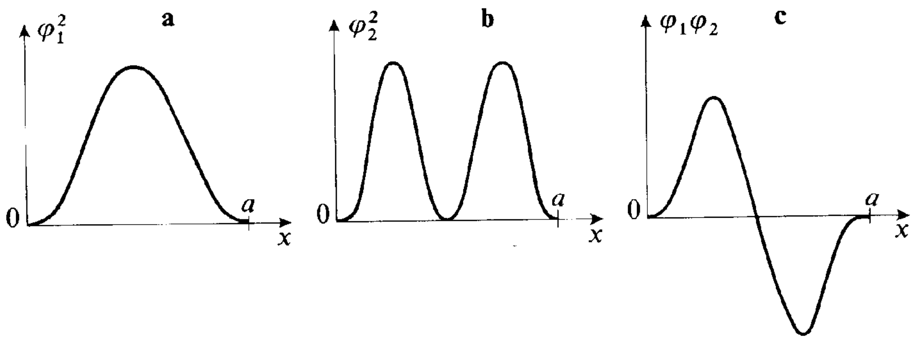


FIGURE 4

Graphical representation of the functions φ_1^2 (the probability density of the particle in the ground state), φ_2^2 (the probability density of the particle in the first excited state) and $\varphi_1\varphi_2$ (the cross term responsible for the evolution of the shape of the wave packet).

Using these figures and relation (18), it is not difficult to represent graphically the variation in time of the shape of the wave packet (*cf.* fig. 5): we see that the wave packet oscillates between the two walls of the well.

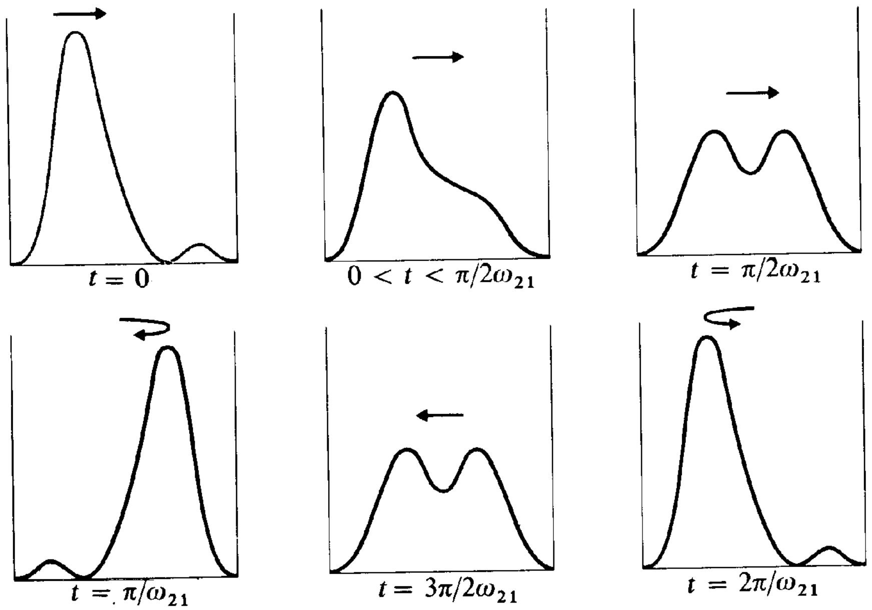


FIGURE 5

Periodic motion of a wave packet obtained by superposing the ground state and the first excited state of a particle in an infinite well. The frequency of the motion is the Bohr frequency $\omega_{21}/2\pi$.

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\,|E_j\rangle + \frac{1}{\sqrt{2}}\,|E_k\rangle$$

$$= \frac{1}{\sqrt{2}}\,|j\rangle + \frac{1}{\sqrt{2}}\,|k\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\,|j\rangle\, e^{-iE_j t/\hbar} + \frac{1}{\sqrt{2}}\,|k\rangle\, e^{-iE_k t/\hbar}$$

$$= \frac{1}{\sqrt{2}}\,|j\rangle\, e^{-i\omega_j t} + \frac{1}{\sqrt{2}}\,|k\rangle\, e^{-i\omega_k t}$$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\,\psi_j(x)\, e^{-i\omega_j t} + \frac{1}{\sqrt{2}}\,\psi_k(x)\, e^{-i\omega_k t}$$

$$P(x,t) = |\psi(x,t)|^2 =$$

$$\left(\frac{1}{\sqrt{2}}\,\psi_j^*(x)\, e^{+i\omega_1 t} + \frac{1}{\sqrt{2}}\,\psi_k^*(x)\, e^{+i\omega_k t}\right)$$

$$\left(\frac{1}{\sqrt{2}}\,\psi_j(x)\, e^{-i\omega_j t} + \frac{1}{\sqrt{2}}\,\psi_k(x)\, e^{-i\omega_k t}\right)$$

FOUR TERMS (2 DIRECT TERMS)

$$\frac{1}{\sqrt{2}}\,\psi_j^*(x)\, e^{+i\omega_j t}\,\frac{1}{\sqrt{2}}\,\psi_j(x)\, e^{-i\omega_j t} = \frac{1}{2}\,|\psi_j(x)|^2$$

$$\frac{1}{\sqrt{2}}\,\psi_k^*(x)\, e^{+i\omega_k t}\,\frac{1}{\sqrt{2}}\,\psi_k(x)\, e^{-i\omega_k t} = \frac{1}{2}\,|\psi_k(x)|^2$$

2 CROSS TERMS

$$\left(\frac{1}{\sqrt{2}}\ \psi_j^*(x)\ e^{+i\omega_j t}\right)\left(\frac{1}{\sqrt{2}}\ \psi_k(x)\ e^{-i\omega_k t}\right)$$

$$\left(\frac{1}{\sqrt{2}}\ \psi_k^*(x)\ e^{+i\omega_k t}\right)\left(\frac{1}{\sqrt{2}}\ \psi_j(x)\ e^{-i\omega_j t}\right)$$

$$\frac{1}{2}\ \psi_j(x)\ \psi_k(x)\ \ e^{-i(\omega_k-\omega_j)t}$$

$$\frac{1}{2}\ \psi_j(x)\ \psi_k(x)\ \ e^{+i(\omega_k-\omega_j)t}$$

$$e^{i\theta}+e^{-i\theta} = 2\cos\theta$$

$$\psi_j(x)\,\psi_k(x)\ \ \cos\,(\omega_k-\omega_j)\,t$$

$$\psi_j(x)\,\psi_k(x)\ \ \cos\,(\ \omega_{kj}\ t\,)$$

$$\rho(x,t) = \frac{1}{2}\left|\psi_j(x)\right|^2 + \frac{1}{2}\left|\psi_k(x)\right|^2 + \psi_j(x)\,\psi_k(x)\cos(\Omega t)$$

1. Quantitative Aspects of the Harmonic Oscillator

Consider a particle moving in a simple harmonic oscillator well with the zero-time state vector

$$\mid \psi(t=0) > = \; N \, \big[\mid n=3 > \; + \; \mid n=4 > \big].$$

(a) Calculate the normalization constant N. Write down the equation for the normalized zero-time state vector $\mid \psi(0) >$ in terms of the energy eigenkets $\mid n >$. Using your equation for $\mid \psi(0) >$ in terms of the energy eigenkets $\mid n >$ write down the equation for the corresponding normalized time-dependent state vector $\mid \psi(t) >$ in terms of the energy eigenkets $\mid n >$. Convert your equation for the time-dependent state vector $\mid \psi(t) >$ in terms of the energy eigenkets $\mid n >$ into the corresponding equation for the time-dependent position-space wavefunction $\psi(x,t)$ in terms of the position-space stationary states $\psi_n(x)$.

(b) If you measure E at $t = 0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< E(t) >$. Calculate the time-dependent uncertainty $\Delta E(t)$. Explain the time-dependence, or lack thereof, of $< E(t) >$ and $\Delta E(t)$.

(c) If you measure x at $t = 0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< x(t) >$. Calculate the time-dependent uncertainty $\Delta x(t)$. Simulate the time evolution of this system using http://falstad.com. Do your expressions for $< x(t) >$ and $\Delta x(t)$ agree with your simulation?

(d) If you measure p at $t = 0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< p(t) >$. Calculate the time-dependent uncertainty $\Delta p(t)$. Simulate the time evolution of this system using http://falstad.com. Do your expressions for $< p(t) >$ and $\Delta p(t)$ agree with your simulation?

(e) Sketch the $t = 0$ probability density distributions $P(E,0)$, $P(x,0)$, and $P(p,0)$. Add your calculated expectation values and uncertainties to your sketches. Do they agree?

Three Ways to Solve:

(1) Use Matrix Method
(2) Do the integrals in x-space
(3) Use Dirac Notation

Matrix Method

1) Simple harmonic Oscillator:

$$|\psi(t=0)\rangle = N[\,|N=3\rangle + |N=4\rangle\,]$$

✓ a) $\langle \psi(0)|\psi(0)\rangle = (0\ 0\ 0\ 1\ 1\ 0\ 0\ \ldots)\, N \cdot N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = 1$

$- |N|^2 (1+1) = 2|N|^2$

$\Rightarrow \boxed{N = \frac{1}{\sqrt{2}}}$

- $\boxed{|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}[\,|N=3\rangle + |N=4\rangle\,]}$

$E_N = (N + 1/2)\hbar\omega$

$\Rightarrow E_N = \frac{7}{2}\hbar\omega$ or

$\frac{9}{2}\hbar\omega$

- $\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-7i\omega t/2}|3\rangle + e^{-9i\omega t/2}|4\rangle\right]}$

- $\langle x|\psi(t)\rangle = \psi(x,t)$

$= \frac{1}{\sqrt{2}}\left(\langle x|3\rangle e^{-7i\omega t/2} + \langle x|4\rangle e^{-9i\omega t/2}\right)$

$= \boxed{\frac{1}{\sqrt{2}}\left(\psi_3(x)\, e^{-7i\omega t/2} + \psi_4(x)\, e^{-9i\omega t/2}\right)}$

1b) ✓ E at $t=0$, possibilities & probabilities?

possibilities: $E_N = \frac{7}{2}\hbar\omega, \frac{9}{2}\hbar\omega$ → only possible eigenvalues @ $t=0$.

Probabilities:

$$P(7/2\,\hbar\omega) = \left|\langle N=3|\psi(t=0)\rangle\right|^2 = \left|(0\,0\,0\,1\,0\ldots)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\\1\\0\\\vdots\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(1\cdot1 + 0\cdot1\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \boxed{\frac{1}{2}}$$

$$P(9/2\,\hbar\omega) = \left|\langle N=4|\psi(t=0)\rangle\right|^2 = \left|(0\,0\,0\,0\,1\,0\ldots)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\\1\\0\\\vdots\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(0\cdot1 + 1\cdot1)\right|^2 = \left|\frac{1}{\sqrt{2}}\right| = \boxed{\frac{1}{2}}$$

- $\langle E(t)\rangle = \langle\psi(t)|\mathcal{H}|\psi(t)\rangle$

$$= \langle\psi(t)|\begin{array}{c|ccccc c}
0 & 1/2 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 3/2 & 0 & 0 & 0 & \cdots \\
2 & 0 & 0 & 5/2 & 0 & 0 & \cdots \\
3 & 0 & 0 & 0 & 7/2 & 0 & \\
4 & 0 & 0 & 0 & 0 & 9/2 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\;\hbar\omega\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\e^{-i\cdot7\omega t/2}\\e^{-i\cdot9\omega t/2}\\\vdots\end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\left(0\;\;0\;\;0\;\;\cancel{e^{7i\omega t/2}}\;\;\cancel{e^{9i\omega t/2}}\;\ldots\right)\begin{pmatrix}0\\0\\0\\7/2\,\cancel{e^{-i7\omega t/2}}\\9/2\,\cancel{e^{-i9\omega t/2}}\\\vdots\end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\left[7/2 + 9/2\right] = \boxed{4\hbar\omega}$$

1b) cont

$$\Delta E(t) = \left[\langle \psi(t) | \mathcal{H}^2 | \psi(t) \rangle - \langle \psi(t) | \mathcal{H} | \psi(t) \rangle^2 \right]^{1/2}$$

- 1st calculate $\mathcal{H}^2$: $\langle \mathcal{H} \rangle = \left(a^+a + \frac{1}{2}\right) \hbar w \left(a^+a + \frac{1}{2}\right) \hbar w$

$$= \left(a^+a\,a^+a + \frac{1}{2} a^+a + a^+a \frac{1}{2} + \frac{1}{4} \right) \hbar^2 w^2 \qquad a^+a = N$$

$$= \boxed{(n^2 + n + 1/4)\hbar^2 w^2}$$

- $\langle \psi(t) | \mathcal{H}^2 | \psi(t) \rangle = \frac{\hbar^2 w^2}{2} \left[\langle 3 | e^{7iwt/2} + \langle 4 | e^{9iwt/2} \right] \cdot$

$$(n^2 + N + 1/4) \cdot \left[|3\rangle e^{-7iwt/2} + |4\rangle e^{-9iwt/2} \right]$$

$$= \frac{\hbar^2 w^2}{2} \left[\langle 3 | e^{7iwt/2} + \langle 4 | e^{9iwt/2} \right] \cdot \left[(3^2 + 3 + 1/4) |3\rangle e^{-7iwt/2} + \right.$$

$$\left. (4^2 + 4 + 1/4) |4\rangle e^{-9iwt/2} \right] = \frac{\hbar^2 w^2}{2} \left[\langle 3 | \cancel{e^{7iwt/2}} + \langle 4 | \cancel{e^{9iwt/2}} \right]$$

$$\cdot \left[\frac{49}{4} |3\rangle \cancel{e^{-7iwt/2}} + \frac{81}{4} |4\rangle \cancel{e^{-9iwt/2}} \right]$$

$$= \frac{\hbar^2 w^2}{2} \left[\cancelto{1}{\langle 3 | 3 \rangle} \cancelto{1}{e^0} \cdot \frac{49}{4} + \frac{81}{4} \cancelto{1}{\langle 4 | 4 \rangle} \cancelto{1}{e^0} \right] = \boxed{\frac{65}{4} \hbar^2 w^2}$$

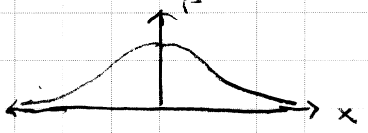
$$\Delta E(t) = \left[\frac{65}{4} \hbar^2 w^2 - (4 \hbar w)^2 \right]^{1/2} = \left[\frac{65}{4} \hbar^2 w^2 - \frac{64}{4} \hbar^2 w^2 \right]^{1/2}$$

$$= \boxed{\frac{\hbar w}{2}}$$

- Both $\langle E(t) \rangle$ + $\Delta E(t)$ should be free of time dependence, because of the symmetrical Bra/Ket relation for the Hamiltonian + the orthonormality of all energy eigenvectors.

1c) ✓ The possible values for x @ t=0 are any and all values of x between $-\infty + \infty$. This is due to the continuous nature of position in the universe. The probabilities of the position values will be governed by the gaussian distrubution, given by the square of the wave fxn, However, to obtain a single precise value of x will cause the probability to go to zero, as shown by the equation below, because are $dx \to 0$.

$$N.\int_a^a |\Psi(x,t)|^2 dx = 0 \Rightarrow$$ No possible probability except zero for a single position measurement.



- $\langle x(t)\rangle = \langle \Psi(t)|X|\Psi(t)\rangle$

$$= \langle\Psi(t)|\left(\frac{\hbar}{2m\omega}\right)^{1/2}\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & \sqrt{1} & 0 & 0 & 0\cdots \\ 1 & \sqrt{1} & 0 & \sqrt{2} & 0 & 0\cdots \\ 2 & 0 & \sqrt{2} & 0 & \sqrt{3} & 0\cdots \\ 3 & 0 & 0 & \sqrt{3} & 0 & \sqrt{4}\cdots \\ 4 & 0 & 0 & 0 & \sqrt{4} & 0\cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\ \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\e^{-i7\omega t/2}\\e^{-i9\omega t/2}\\\vdots\end{pmatrix}$$

$$= \langle\Psi(t)|\frac{1}{\sqrt{2}}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\cdot\begin{pmatrix}0\\0\\0\\\sqrt{4}e^{-i7\omega t/2}\\\sqrt{4}e^{-i9\omega t/2}\\\vdots\end{pmatrix}$$

$$= \frac{\cancel{2}}{\cancel{2}}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\cdot\left(0\ \ 0\ \ 0\ \ e^{+i7\omega t/2}\ \ e^{+i9\omega t/2}\cdots\right)\begin{pmatrix}0\\0\\0\\e^{-i9\omega t/2}\\e^{-i9\omega t/2}\\\vdots\end{pmatrix}$$

$$=\left[\frac{\hbar}{2m\omega}\right]^{1/2}\left(e^{i7\omega t/2}\cdot e^{-i9\omega t/2} + e^{i9\omega t/2}\cdot e^{-i7\omega t/2}\right) = \left[\frac{\hbar}{2m\omega}\right]^{1/2}\frac{\left(e^{i\omega t}+e^{-i\omega t}\right)\cdot 2}{2}$$

$$= \boxed{\left[\frac{2\hbar}{m\omega}\right]^{1/2}\cos(\omega t)}$$

1c)$_{\text{cont}}$ $$\Delta x(t) = \left[\langle \psi(t) | X^2 | \psi(t) \rangle - \left(\langle \psi(t) | X | \psi(t) \rangle \right)^2 \right]^{1/2}$$

$$X^2 = X \cdot X = \left(\frac{\hbar}{2m\omega} \right) \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \\ \hline 0 & 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 1 & 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ 2 & \sqrt{2} & 0 & 5 & 0 & \sqrt{12} & \cdots \\ 3 & 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 4 & 0 & 0 & \sqrt{12} & 0 & 9 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

– from HW 5, ch.8 soln's

$$\langle \psi(t) | X^2 | \psi(t) \rangle = \frac{1}{2} \cdot \frac{\hbar}{2m\omega} \langle \psi(t) | \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} \\ 0 & \sqrt{6} & 0 & 7 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i7\omega t/2} \\ e^{-i9\omega t/2} \end{pmatrix}$$

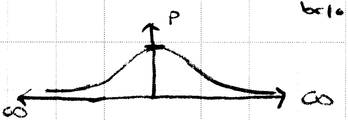
$$= \frac{\hbar}{4m\omega} \left(0 \quad 0 \quad 0 \quad \cancel{e^{7i\omega t/2}} \quad \cancel{e^{9i\omega t/2}} \ \ldots \right) \begin{pmatrix} 0 \\ \sqrt{6}\, e^{-7i\omega t/2} \\ \sqrt{12}\, e^{-9i\omega t/2} \\ 7 \cancel{e^{-i\omega 7t/2}} \\ 9 \cancel{e^{-9i\omega t/2}} \end{pmatrix} = \boxed{\frac{4\hbar}{m\omega}}$$

$$\Delta x(t) = \left[\frac{4\hbar}{m\omega} - \left(\left[\frac{2\hbar}{m\omega} \right]^{1/2} \cos(\omega t) \right)^2 \right] = \boxed{\left(\frac{2\hbar}{m\omega} \left[2 - \cos^2(\omega t) \right] \right)^{1/2}}$$

– according to http://falstad.com, my values of $\langle x(t) \rangle$ + $\Delta x(t)$ are consistent w/ the simulation.

1d)
8

similiar to the position possibilitres + probabilitics, measurement of p @ time t=0 could have an infinite number of possibilitics, in the interval -∞ to ∞. (for p) IF we were able to Pinpoint one valuefor p then the Corresponding probability will goto zero for the reasons listed in part 1c. This all boils down to p being directly proportional to $\dot{x}$ (ie velocity), and x's continuous nature in the real spectrum, P will still have a gaussian probability centered at zero + falling off exponential as shown below



$$- \langle p(t) \rangle = \langle \Psi(t) | P | \Psi(t) \rangle$$

$$= \frac{1}{2} \langle \Psi(t) | \, i \left(\frac{m\omega\hbar}{2}\right)^{1/2} \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & i & 0 & 0 & 0 \cdots \\ 1 & \sqrt{1} & 0 & i\sqrt{2} & 0 & 0 \cdots \\ 2 & 0 & \sqrt{2} & 0 & i\sqrt{3} & 0 \cdots \\ 3 & 0 & 0 & \sqrt{3} & 0 & i\sqrt{4} \cdots \\ 4 & 0 & 0 & 0 & \sqrt{4} & 0 \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-7i\omega t/2} \\ e^{-9i\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \frac{i}{2}\left(\frac{m\omega\hbar}{2}\right)^{1/2} \left(\cancel{0} \;\; 0 \;\; \cancel{0} \;\; \cancel{e^{7i\omega t/2}} \;\; e^{9i\omega t/2}\right) \begin{pmatrix} \cancel{0} \\ \cancel{0} \\ \cancel{i\sqrt{3}}e^{-9i\omega t/2} \\ 2i\,e^{-9i\omega t/2} \\ 2e^{-7i\omega t/2} \end{pmatrix}$$

$$= \frac{i}{\cancel{2}}\left(\frac{m\omega\hbar}{2}\right)^{1/2} \left(e^{-i\omega t} + e^{i\omega t}\right) \cdot \frac{2}{2}$$

$$= \boxed{\cancel{i}\left(\frac{+m\omega\hbar}{2}\right)^{1/2} \underset{2\sin(\omega t)}{\cos(\omega t)}}$$

$$- \Delta p(t) = \left[\langle \Psi(t) | P^2 | \Psi(t) \rangle - \left(\langle \Psi(t) | P | \Psi(t) \rangle \right)^2 \right]^{1/2}$$

1d) cont

$$p^2 = p \cdot p = i^2\left(\frac{m\omega\hbar}{2}\right) \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \\ \hline 0 & 0 & i & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \sqrt{2}i & 0 & 0 & \cdots \\ 2 & 0 & -\sqrt{2} & 0 & i\sqrt{3} & 0 & \cdots \\ 3 & 0 & 0 & \sqrt{3} & 0 & 2i & \cdots \\ 4 & 0 & 0 & 0 & 2 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2}i & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3}i & 0 \\ 0 & 0 & \sqrt{3} & 0 & 2i \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 3i & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 5i & 0 & -2\sqrt{3} \\ 0 & -\sqrt{3}\cdot\sqrt{2} & 0 & 7i & 0 \\ 0 & 0 & 2\sqrt{3} & 0 & 4i \end{pmatrix}$$

- $$\langle\psi(t)|\,p^2\,|\psi(t)\rangle = \frac{1}{2}\cdot\frac{-m\omega\hbar}{2}\langle\psi(t)| \begin{pmatrix} i & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 3i & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 5i & 0 & -2\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 7i & 0 \\ 0 & 0 & 2\sqrt{3} & 0 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-7i\omega t/2} \\ e^{-9i\omega t/2} \end{pmatrix}$$

$$= \frac{-m\omega\hbar}{4}\langle \cancel{0}\ \cancel{0}\ \cancel{0}\ \cancel{e^{7i\omega t/2}}\ \cancel{e^{9i\omega t/2}}| \begin{pmatrix} \cancel{0} \\ \cancel{-\sqrt{3}\sqrt{2}\,e^{-7i\omega t/2}} \\ \cancel{-2\sqrt{3}\,e^{-9i\omega t/2}} \\ 7i\,\cancel{e^{-7i\omega t/2}} \\ 4i\,\cancel{e^{-9i\omega t/2}} \end{pmatrix}$$

$$= \frac{-m\omega\hbar}{4}\left(7i + 4i\right)$$

$$\Rightarrow \Delta p(t) = \left[-\frac{11}{4}im\omega\hbar - \left(i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\cos(\omega t)\right)^2\right]^{1/2}$$

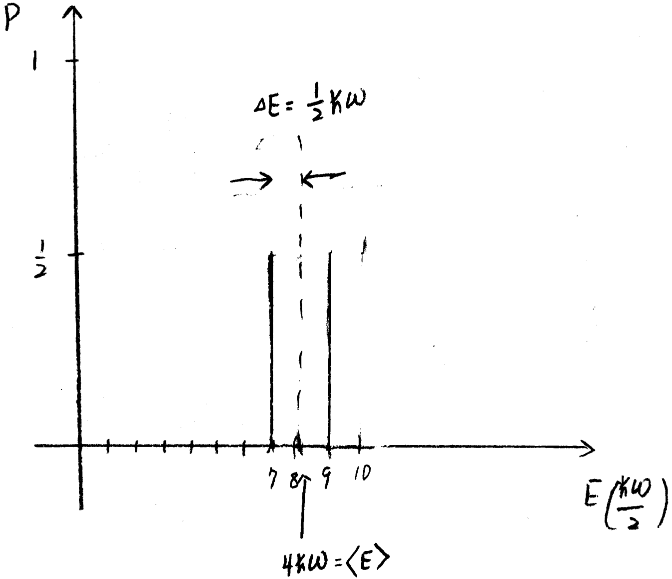
$$= \left[-\frac{11}{4}im\omega\hbar + \frac{m\omega\hbar}{2}\cos^2(\omega t)\right]^{1/2} = \boxed{\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(\cos(\omega t) - \frac{11}{2}i\right)^{1/2}}$$

$$\sqrt{2m\omega\hbar}\ (1+\cos^2\omega t)^{1/2}$$

- After checking http://falstad.com, my expressions for $\Delta p(t)$ + $\langle p(t)\rangle$ appear to agree w/ the simulations.

✓ (e)

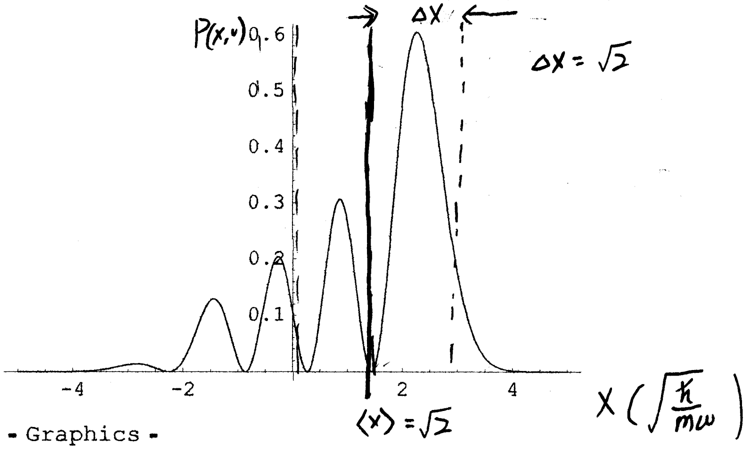
(1) P(E,0)


P
1
$\frac{1}{2}$
$\Delta E = \frac{1}{2}\hbar\omega$
7
8
9
10
$E\left(\frac{\hbar\omega}{2}\right)$
$4\hbar\omega = \langle E \rangle$

■ (2.) P(x,0): (assume $\frac{m\,\omega}{\text{hBar}}$ =1)

In[1]:= $\texttt{phi1[x\_, n\_]} = \left(\frac{1}{\pi\,(2\,\hat{}\,(2\,n))\,((n!)\,\hat{}\,2)}\right)^{0.25}\ \texttt{E}\,\hat{}\left(-\frac{x\,\hat{}\,2}{2}\right)\ \texttt{HermiteH[n, x];}$

```
In[2]:= Plot[Evaluate[0.5 (phi1[x, 3] + phi1[x, 4])^2], {x, -5, 5}]
```



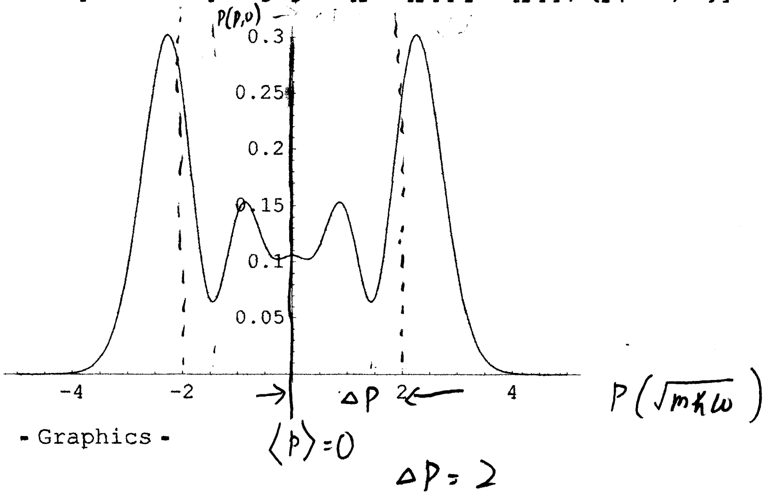
```
Out[2]= - Graphics -
```

■ (2.) P(p,0): (assume (m ω hBar) =1)

```
In[10]:= phi3[p_] = FourierTransform[phi1[x, 3], x, p];
         phi4[p_] = FourierTransform[phi1[x, 4], x, p];
```

$\texttt{phi[p\_]} = \frac{1}{2\,\hat{}\,0.5}\ \texttt{(phi3[p] + phi4[p]);}$

```
In[15]:= Plot[Evaluate[Conjugate[phi[p]] phi[p]], {p, -5, 5}]
```



```
Out[15]= - Graphics -
```

50 - 0 = 50

Yes, they agree

Do the integrals

c.) The possible locations for a particle are unlimited, although ~~the~~ you are very unlikely to measure a particle far outside the potential well. The probability is given by $\int_{x_1}^{x_2} \psi^*(x,0)\,\Psi(x,0)\,dx$.

$$\zeta = \sqrt{\frac{m\omega}{\hbar}}\,x$$

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \Psi(x,t)\, dx \qquad d\zeta = \sqrt{\frac{m\omega}{\hbar}}\,dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{x}{2^3 \cdot 3!} H_3^2(\zeta) e^{-\zeta^2} + \frac{x}{2^4 \cdot 4!} H_4^2(\zeta) e^{-\zeta^2} + \left[e^{\frac{i(E_3-E_4)t}{\hbar}} + e^{\frac{-i(E_3-E_4)t}{\hbar}}\right] x H_3(\zeta) H_4(\zeta) \frac{e^{-\zeta^2}\, d\zeta}{\sqrt{2^3 2^4 3!4!}}$$

$$= \frac{1}{2\sqrt{\pi}}\left[\cancel{2\sqrt{\pi}} + \int_{-\infty}^{\infty} x\cos(\omega t)\, H_3(\zeta) H_4(\zeta) \frac{e^{-\zeta^2}}{\sqrt{2^3 2^4 3!4!}}\, d\zeta\right] = \hbar$$

$$\langle x(t)\rangle = \frac{1}{\sqrt{\pi}\,2}\int_{-\infty}^{\infty} \frac{\cancel{\zeta}}{2^3 3!} H_3^2(\zeta) e^{-\zeta^2} + \frac{\cancel{\zeta}}{2^4\cdot 4!} H_4^2(\zeta)e^{-\zeta^2} + \frac{\zeta\cos(\omega t)\, H_3(\zeta)H_4(\zeta)}{\sqrt{2^3 2^4\cdot 3!\cdot 4!}}\, e^{-\zeta^2}\, d\zeta$$

$$= \frac{1}{2\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\sqrt{\frac{\hbar}{m\omega}}\left[\frac{\zeta}{2^3\cdot 3!}\left(8^2\zeta^6 - 16\cdot 12\,\zeta^4 + 12^2\zeta^2\right)^{\nearrow 0} e^{-\zeta^2} + \frac{\zeta}{2^4 4!}\left(16^2\zeta^8 - 2\cdot16\cdot48\,\zeta^6 + 2\cdot16\cdot12\,\zeta^4 + 48^2\zeta^4 - 2\cdot12\cdot48\,\zeta^2\right)^{\nearrow 0} e^{-\zeta^2}\right.\right.$$

$$\sqrt{\frac{\hbar}{m\omega}}\,\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\cos(\omega t)\left[\frac{1}{\sqrt{2^7\cdot 3!\cdot 4!}}\left(8\cdot16\,\zeta^8 - 8\cdot48\,\zeta^6 - \cancel{12\cdot16\,\zeta^5} + 12\cdot8\,\zeta^4 + \cancel{12\cdot48\,\zeta^3} - \cancel{12\cdot12\,\zeta}\right)e^{-\zeta^2}\, d\zeta\right]$$

odd terms are 0

$$= \frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!\cdot\pi}}\cdot\frac{1}{2}\cdot\sqrt{\frac{\hbar}{m\omega}}\left[8\cdot105\sqrt{\pi} + 13\cdot48\sqrt{\pi} + 6\cdot12\sqrt{\pi}\right]$$ ← UGLY

$\sqrt{\frac{2\hbar}{m\omega}}$ => Why didn't you use a and a^+?

↑ ↑ ↗ PRETTY!

$$= \frac{816}{\sqrt{18432}}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t) = \frac{17}{\cancel{2\sqrt{2}}}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t)$$

$$\langle x^2(t)\rangle = \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty}\frac{\zeta^2 e^{-\zeta^2}}{2^3\cdot 3!}H_3^2(\zeta) + \frac{\zeta^2 e^{-\zeta^2}}{2^4 4!}H_4^2(\zeta) + \frac{\zeta^2 e^{-\zeta^2}}{\sqrt{2^7\cdot 3!\cdot 4!}}H_3(\zeta)H_4(\zeta)\overset{\cos(\omega t)}{\hat{}}\, d\zeta$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\left[\int_{-\infty}^{\infty}\frac{e^{-\zeta^2}}{2^3\cdot 3!}\left(8^2\zeta^8 - 16\cdot12\,\zeta^6 + 12^2\zeta^4\right) + \frac{e^{-\zeta^2}}{2^4\cdot 4!}\left(16^2\zeta^{10} - 2\cdot16\cdot48\,\zeta^8 + 2\cdot16\cdot12\,\zeta^6 + 48^2\zeta^6 - 2\cdot12\cdot48\,\zeta^4\right)d\zeta\right.$$

$$\left. + \int_{-\infty}^{\infty}\frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!}}e^{-\zeta^2}\left(\cancel{8\cdot16\,\zeta^9} - \cancel{8\cdot48\,\zeta^7} - 12\cdot16\,\zeta^6 + \cancel{12\cdot8\,\zeta^5} + 12\cdot48\,\zeta^4 - 12\cdot12\,\zeta^2\right)d\zeta\right]$$

odd terms drop out

$$= \frac{1}{2\cancel{\sqrt{\pi}}}\sqrt{\frac{\hbar}{m\omega}}\left[\overset{\frac{35}{4}\sqrt{\pi}}{\frac{2\cdot105\cdot8^2\cdot\cancel{\sqrt{\pi}}}{2^3\cdot 3!\cdot 32}} - \overset{-15/2\sqrt{\pi}}{\frac{\cancel{16}\cdot\cancel{12}\cdot\cancel{2}\cdot15}{\cancel{16}\cdot 2^{\cancel{3}}\cdot\cancel{3!}}\cancel{\sqrt{\pi}}} + \overset{9/4\sqrt{\pi}}{\frac{2\cdot12^2\cdot3}{8\cdot2^3\cdot3!}\cancel{\sqrt{\pi}}} + \frac{1}{2^4 4!}\left(\overset{\cancel{315}/16\sqrt{\pi}}{\frac{16^2\cdot2\cdot945}{64}\cancel{\sqrt{\pi}}} - \overset{-105/4\sqrt{\pi}}{\frac{4\cdot16\cdot48\cdot105}{32}\cancel{\sqrt{\pi}}} + \overset{\frac{15}{8}\sqrt{\pi}}{\frac{4\cdot16\cdot12\cdot15}{16}\sqrt{\pi}}\right.\right.$$

$$\left.\left. + \overset{45/4\sqrt{\pi}}{2\cdot\frac{48^2\cdot15}{16}\cancel{\sqrt{\pi}}} - \overset{9/4\sqrt{\pi}}{\frac{4\cdot12\cdot48\cdot3}{8}\cancel{\sqrt{\pi}}}\right) + \frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot\cancel{4!}}}\left(-2\cdot12\cdot16\cdot\frac{15}{16}\cancel{\sqrt{\pi}} + 12\cdot48\cdot2\cdot\frac{3}{8}\cancel{\sqrt{\pi}} - \frac{12^2\cdot2}{4}\cancel{\sqrt{\pi}}\right)\right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{197}{16} + \cos(\omega t)\,(0)\right) = \frac{197}{32}\sqrt{\frac{\hbar}{m\omega}}$$

$$\Delta x(t) = \sqrt{\frac{197}{32}\sqrt{\frac{\hbar}{m\omega}} - \frac{289}{8}\cdot\frac{\hbar}{m\omega}\cos^2(\omega t)}$$

Both seem to match falstad.com.

$$\sqrt{\frac{2\hbar}{m\omega}}\left(1 + \sin^2\omega t\right)^{1/2}$$

again ladder operators are more beautiful and easier!

d) If you measure p at ~~any~~ time zero the value of p could be any where between a $-\infty$ and ∞ with probability $\int_{P_1}^{P_2} \hat{\varphi}^*(p,0)\,\hat{\varphi}(p,0)\,dp$.

$$\langle p(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\; -i\hbar\frac{\partial}{\partial x}\,\psi(x,t)\,dx$$

$$= \frac{-i}{2}\hbar \int_{-\infty}^{\infty} \underbrace{\psi_3^*(x)\frac{\partial}{\partial x}\psi_3(x)}_{\text{part one}} + \underbrace{\psi_4^*(x)\frac{\partial}{\partial x}\psi_4(x)}_{\text{part two}} + \underbrace{e^{\frac{i(E_4-E_3)t}{\hbar}}\,\psi_4^*(x)\frac{\partial}{\partial x}\psi_3(x)}_{\text{part three}} + \underbrace{e^{\frac{-i(E_4-E_3)t}{\hbar}}\,\psi_3^*(x)\frac{\partial}{\partial x}\psi_4(x)}_{\text{part four}}\,dx$$

$A = \frac{m\omega}{\hbar}$

$$\text{part one} = \frac{-i\hbar}{2}\int_{-\infty}^{\infty} \frac{A}{\sqrt{\pi}}\cdot\frac{1}{2^3 3!}\left[(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2^4\,3!}\,\frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\left[(24Ax^2-12)e^{-\frac{Ax^2}{2}} - Ax(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2^4\,3!}\,\frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\left[\cancel{8\cdot 24A^2x^5} - \cancel{8\cdot 12Ax^3} - \cancel{8^2A^3x^7} + \overset{2^3\,2^2\,3}{\cancel{8\cdot 12Ax^5}} - \cancel{12\cdot 24Ax^3} + \cancel{12^2x} + \cancel{12\cdot 8A^2x^5} - \cancel{12^2Ax^3}\right]dx$$

odd terms drop out

$$= \cancel{\frac{-i\hbar A^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}x^4e^{-Ax^2}dx = \frac{-i\hbar A^2}{A^{5/2}}\cdot\frac{3}{4} = \frac{-i\hbar\cdot 3}{A^{1/2}\cdot 4}} = 0$$

$$\text{part two} = \frac{-i\hbar}{2}\int_{-\infty}^{\infty}\frac{A^{1/2}}{\sqrt{\pi}}\,\frac{1}{2^4 4!}\left[(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}dx\right]$$

$$= \frac{-i\hbar A^{1/2}}{2^6\,4!\sqrt{\pi}}\int_{-\infty}^{\infty}(16A^2x^4-48Ax^2+12)e^{-Ax^2}\left[(64A^2x^3-96Ax) - Ax(16A^2x^4-48Ax^2+12)\right]dx$$

$$= \frac{-i\hbar A^{1/2}}{2^8\cdot 3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\left[\cancel{16\cdot 64A^4x^7} - \cancel{16\cdot 96A^3x^5} - \cancel{16^2A^4x^9} + \cancel{16\cdot 48A^3x^7} + \cancel{12\cdot 16A^3x^5} - \dots\right] = 0$$

$$\text{part three} = \frac{-i\hbar}{2}\,e^{\frac{i(E_4-E_3)t}{\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2^7 3!\,4!}}\,\frac{A^{\cancel{1/2}}}{\sqrt{\pi}}\left[(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(8A^{\cancel{1/2}}x^3-12\cancel{A^{1/2}}x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2\cdot 3\sqrt{2^{11}}}\,\frac{A}{\sqrt{\pi}}\,e^{i\omega t}\int_{-\infty}^{\infty}(16A^2x^4-48Ax^2+12)e^{-Ax^2}\Big[\underbrace{(3\cdot 8Ax^2-12)}_{24Ax^2-12}\underbrace{-Ax(8Ax^3-12x)}_{-8A^2x^4+12Ax^2}\Big]dx$$

$$= \frac{-i\hbar A}{2\cdot 3\cdot\sqrt{\pi}}\,e^{i\omega t}\int_{-\infty}^{\infty}e^{-Ax^2}\Big(\overset{\checkmark}{16\cdot 24A^3x^6} - \overset{\checkmark}{12\cdot 16A^2x^4} - \overset{\checkmark}{8\cdot 16\cdot A^4x^8} + \overset{\checkmark}{12\cdot 16A^3x^6} - \overset{\checkmark}{48\cdot 24A^2x^4} + \overset{\checkmark}{12\cdot 48Ax^2} + \overset{\checkmark}{8\cdot 48A^3x^6} - \overset{\checkmark}{12\cdot 48A^2x^4}$$
$$+ \overset{\checkmark}{12\cdot 24Ax^2} - \overset{\checkmark}{12^2} - \overset{\checkmark}{8\cdot 12A^2x^4} + \overset{\checkmark}{12^2Ax^2}\Big)dx$$

$$= \frac{-i\hbar A}{2^{13/2}\cdot 3\cdot\sqrt{\pi}}\,e^{i\omega t}\sqrt{\pi}\left[\frac{-8\cdot 16\cdot A^4\cdot 2\cdot 105}{32\cdot A^{9/2}} + \frac{2\cdot(16\cdot 24\cdot A^3 + 12\cdot 16A^3 + 8\cdot 48A^3)\cdot 15}{16\,a^{7/2}} + \frac{2\cdot 3(-12\cdot 16A^2 - 48\cdot 24A^2 - 12\cdot 48A^2 - 8\cdot 12A^2)}{8\,a^{5/2}}\right.$$

$$+\frac{1}{2A^{3/2}}\left(12\cdot 48A+12\cdot 24A+12^{2}A\right)+\frac{-12^{2}}{A^{1/2}}\Bigg]$$

$$=\frac{-i\hbar A^{1/2}}{2^{13/2}\cdot 3}e^{i\omega t}\left[-840+1800\overset{-192}{+}-1312+504-144\right]=i\hbar A^{1/2}\cdot\frac{\sqrt{2}}{2}e^{i\omega t}=\frac{i\hbar A^{1/2}}{\sqrt{2}}e^{i\omega t}=i\sqrt{\frac{m\omega\hbar}{2}}e^{i\omega t}$$

part four $= -\dfrac{i\hbar}{2}\dfrac{e^{-i\omega t}A^{1/2}}{\sqrt{2^{11}\cdot 3^{2}\sqrt{\pi}}}\displaystyle\int_{-\infty}^{\infty}(8A^{3/2}x^{3}-12A^{1/2}x)e^{-\frac{Ax^{2}}{2}}\frac{\partial}{\partial x}(16A^{2}x^{4}-48Ax^{2}+12)e^{-\frac{Ax^{2}}{2}}dx$

$$=\frac{-i\hbar A^{1/2}}{2^{13/2}\cdot 3\sqrt{\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}(8A^{3/2}x^{3}-12A^{1/2}x)e^{-Ax^{2}}\left[(16\cdot 4A^{2}x^{3}-48\cdot 2Ax)-Ax(16A^{2}x^{4}-48Ax^{2}+12)\right]dx$$

$$=\frac{-i\hbar A^{1/2}}{2^{13/2}\cdot 3\sqrt{\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-Ax^{2}}\Big[\overset{\checkmark}{8\cdot 16\cdot 4A^{7/2}x^{6}}-\overset{\checkmark}{8\cdot 48\cdot 2A^{5/2}x^{4}}-\overset{\checkmark}{8\cdot 16\cdot A^{9/2}x^{8}}+\overset{\checkmark}{8\cdot 48\cdot A^{7/2}x^{6}}-\overset{\checkmark}{12\cdot 8\cdot A^{5/2}x^{4}}-\overset{\checkmark}{12\cdot 16\cdot 4A^{5/2}x^{4}}$$
$$+\overset{\checkmark}{12\cdot 48\cdot 2A^{3/2}x^{2}}+\overset{\checkmark}{12\cdot 16A^{7/2}x^{6}}-\overset{\checkmark}{12\cdot 48A^{5/2}x^{4}}+\overset{\checkmark}{12^{2}A^{3/2}x^{2}}\Big]dx$$

$$=\frac{-i\hbar A^{1/2}}{2^{13/2}3\sqrt{\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-Ax^{2}}\left[-128A^{9/2}x^{8}+1088A^{7/2}x^{6}-2208A^{5/2}x^{4}+1296A^{3/2}x^{2}\right]dx$$

$$=\frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\cdot 3\cdot\cancel{\sqrt{\pi}}}2\cdot\cancel{\sqrt{\pi}}\left[-420+1020-828+324\right]=-i\sqrt{\frac{m\omega\hbar}{2}}e^{-i\omega t}$$

Again LADDER OP's!

$$\langle p(t)\rangle=\sqrt{\frac{m\omega\hbar}{2}}\left[ie^{i\omega t}-ie^{-i\omega t}\right]=\sqrt{2m\omega\hbar}\left[-\frac{e^{i\omega t}+e^{-i\omega t}}{2i}\right]=-\sqrt{2m\omega\hbar}\sin(\omega t)\checkmark$$

$$\langle p^{2}(t)\rangle=\int_{-\infty}^{\infty}-\hbar^{2}\Psi^{*}(x,t)\frac{\partial^{2}}{\partial x^{2}}\Psi(x,t)\,dx$$

$$=\frac{-\hbar^{2}}{2}\int_{-\infty}^{\infty}\left(\Psi_{3}^{*}(x,t)+\Psi_{4}^{*}(x,t)\right)\frac{\partial^{2}}{\partial x^{2}}\left(\Psi_{3}(x,t)+\Psi_{4}(x,t)\right)dx$$

$$=\frac{-\hbar^{2}}{2}\int_{-\infty}^{\infty}\Psi_{3}^{*}(x,t)\frac{\partial^{2}}{\partial x^{2}}\Psi_{3}(x,t)+\Psi_{4}^{*}(x,t)\frac{\partial^{2}}{\partial x^{2}}\Psi(x,t)+\Psi_{3}^{*}(x,t)\frac{\partial^{2}}{\partial x^{2}}\Psi_{4}(x,t)+\Psi_{4}^{*}(x,t)\frac{\partial^{2}}{\partial x^{2}}\Psi_{3}(x,t)\,dx$$

$$\int_{-\infty}^{\infty}\Psi_{3}^{*}(x,t)\frac{\partial^{2}}{\partial x^{2}}\Psi_{3}(x,t)dx=\int_{-\infty}^{\infty}\frac{A^{1/2}}{2^{3}3!\sqrt{\pi}}(8A^{3/2}x^{3}-12A^{1/2}x)e^{-\frac{Ax^{2}}{2}}\frac{\partial^{2}}{\partial x^{2}}(8A^{3/2}x^{3}-12A^{1/2}x)e^{-\frac{Ax^{2}}{2}}dx$$

$$=\frac{A^{1/2}}{2^{3}3!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^{2}}\left[(8A^{3/2}x^{3}-12A^{1/2}x)\left[8A^{7/2}x^{5}-68A^{5/2}x^{3}+84A^{3/2}x\right]\right]dx$$

$$=\frac{A^{1/2}}{2^{3}3!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^{2}}\left[64A^{8}x^{8}-640A^{4}x^{6}+1488A^{3}x^{4}-1008A^{2}x^{2}\right]dx$$

$$= \frac{A^{1/2}\, 2 \cdot \cancel{\sqrt{\pi}}}{2^3\, 3!\, \cancel{\sqrt{\pi}}} \left[A^{1/2} 210 - A^{1/2} 600 + A^{1/2} 558 - A^{1/2} 252 \right] = A \left[-\tfrac{7}{2} \right]$$

$$\int_{-\infty}^{\infty} \psi_4^*(x,t) \frac{\partial^2}{\partial x^2} \psi_4(x,t) = \frac{A^{1/2}}{2^4 4!} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \left(16A^2x^4 - 48Ax^2 + 12\right) \frac{\partial^2}{\partial x^2} \left(16A^2x^4 - 48Ax^2 + 12\right) dx$$

$$= \frac{A^{1/2}}{2^4 4!\, \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-Ax^2} \left[256 A^6 x^{10} - 3840 A^5 x^8 + 18624 A^4 x^6 - 31680 A^3 x^4 + 12096 A^2 x^2 - 1296 A \right] dx$$

$$= \frac{A}{2^3 4!} \left[3780 - 12600 + 17460 - 11880 + 3024 - 648 \right] = A \left(\frac{-9}{2} \right)$$

$$\int_{-\infty}^{\infty} \psi_3^*(x,t) \frac{\partial^2}{\partial x^2} \psi_4(x,t)\, dx = \frac{A^{1/2}}{\sqrt{2^7\, 3!\, 4!\, \pi}} e^{-i\omega t} \int_{-\infty}^{\infty} \left(8A^{3/2}x^3 - 12A^{1/2}x\right) e^{-\frac{Ax^2}{2}} \frac{\partial^2}{\partial x^2} \left(16A^2x^4 - 48Ax^2 + 12\right) e^{-\frac{Ax^2}{2}} dx$$

$$= \frac{A^{1/2} e^{-i\omega t}}{\sqrt{2^7\, 3!\, 4!\, \pi}} \int_{-\infty}^{\infty} \left(8A^{3/2}x^3 - 12A^{1/2}x\right)\left(16A^4x^6 + -192A^3x^4 + 576A^2x^2 - 108A\right) dx = 0$$

odd terms drop out

Likewise $\int_{-\infty}^{\infty} \psi_4^*(x,t) \frac{\partial^3}{\partial x^2} \left(\psi_3(x,t)\right) dx = 0$

AGAIN LADDER OP'S!

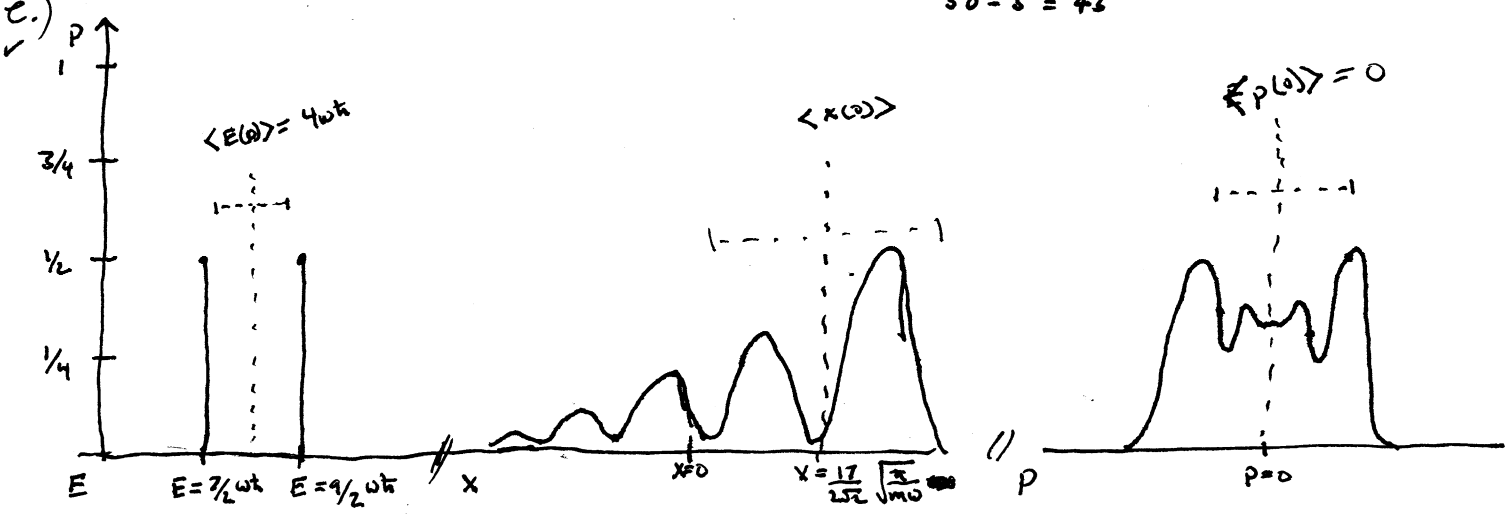
$$\langle p^2(t) \rangle = -\frac{\hbar^2}{2} \left[-8A \right] = 4m\omega\hbar$$

$$\Delta p(t) = \sqrt{\langle p^2(t) \rangle - \langle p(t) \rangle^2} = \sqrt{4m\omega\hbar - 2m\omega\hbar \sin^2(\omega t)} = \sqrt{2m\omega\hbar \left(2 - \sin^2(\omega t)\right)}$$

The expectation of p is 90° out of phase with $\langle x(t) \rangle$ exactly as the simulation on falsted.com shows. The uncertainty seems to match as well.

50 - 5 = 45

e.) ✓



Dirac Notation

2. Quantitative aspects of SHO

✓ (a)

(1) we have $\langle\psi(0)|\psi(0)\rangle = N^*N(\langle 4| + \langle 3|)(|3\rangle + |4\rangle) = 2N^2 = 1$

$\Rightarrow N$ can be $\frac{1}{\sqrt{2}}$

(2) the normalized wave function at $t=0$ is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle) \quad \text{———} \quad ①$$

(3) Assuming the ground state is $|0\rangle$, we have the time-dependent state vector

$|\psi(t)\rangle$ be : $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|3\rangle e^{-\frac{iE_3 t}{\hbar}} + |4\rangle e^{-\frac{iE_4 t}{\hbar}}\right)$, where ——— ②

$$E_3 = (3+\tfrac{1}{2})\hbar\omega = \tfrac{7}{2}\hbar\omega$$
$$E_4 = (4+\tfrac{1}{2})\hbar\omega = \tfrac{9}{2}\hbar\omega \quad \#$$

(4) Using $\langle x|\psi(t)\rangle = \psi(x,t)$ we have

$$\psi(x,t) = \langle x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\langle x|3\rangle e^{-\frac{iE_3 t}{\hbar}} + \langle x|4\rangle e^{-\frac{iE_4 t}{\hbar}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\psi_3(x) e^{-\frac{iE_3 t}{\hbar}} + \psi_4(x) e^{-\frac{iE_4 t}{\hbar}}\right) \#$$

✓ (b) (1) Since $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$ contains only eigenstates corresponding to E_3 and E_4,

the possible energy measurements are only :

$E_3 = \frac{7}{2}\hbar\omega$, with probability $|\langle 3|\psi(0)\rangle|^2 = \frac{1}{2}$

$E_4 = \frac{9}{2}\hbar\omega$, - - - - $|\langle 4|\psi(0)\rangle|^2 = \frac{1}{2}$

(2) $\langle E(t) \rangle = \langle \psi | H | \psi \rangle = \langle \psi(t) | \left(\frac{1}{\sqrt{2}} E_3 |3\rangle e^{-\frac{iE_3 t}{\hbar}} + \frac{1}{\sqrt{2}} E_4 |4\rangle e^{-\frac{iE_4 t}{\hbar}} \right)$

$= ($using $e^{ix} \cdot e^{-ix} = 1$ and $\langle n | m \rangle = \delta_{nm}$ $\quad \frac{1}{2} E_3 + \frac{1}{2} E_4 = \frac{1}{2}\left(\frac{7}{2} + \frac{9}{2} \right) \hbar\omega$

$= 4\hbar\omega \ \ast$

(3) $\langle \Delta E(t)^2 = \langle H^2 \rangle - \langle H \rangle^2 = \langle \psi(t) | \left(\frac{1}{\sqrt{2}} E_3^2 |3\rangle e^{-\frac{iE_3 t}{\hbar}} + \frac{1}{\sqrt{2}} E_4^2 e^{-\frac{iE_4 t}{\hbar}} |4\rangle \right) - 16 \hbar^2 \omega^2$

$= \frac{1}{2} E_3^2 + \frac{1}{2} E_4^2 - 16 (\hbar\omega)^2 = \frac{1}{4} \hbar^2 \omega^2$

$\Delta E(t) = \frac{1}{2} \hbar\omega \quad \ast$

(4) Since the energy E is the eigenvalue of the stationary states of the system, we expect that $\langle E \rangle$ and ΔE will not change with t. Our calculation in (2),(3) verify this.

✓ (c)

(1) Since the potential energy is not infinite (unless at $x \to \pm\infty$), the possible measurement of position can be $x \in [-\infty, \infty]$, with probability density be

$$|\langle x|\psi(0)\rangle|^2 = \frac{1}{2}\left(\psi_3(x) + \psi_4(x)\right)^2$$, where

$\psi_n(x)$ is the stationary energy eigenstate in position space

(2) using $X = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a + a^{+})$, where a, a^{+} are the lowering/ raising operators,

we have

$$\langle x(t)\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\left(\cancel{\frac{1}{2}}\right)\left(\cancel{2}e^{\frac{it}{\hbar}(E_3 - E_4)} + \cancel{2}e^{\frac{it}{\hbar}(E_4 - E_3)}\right)$$

$$= \left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\cos\left(\frac{t}{\hbar}(E_4 - E_3)\right) = \left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\cos(\omega t)$$

(3) We have $\langle X^2 \rangle = \frac{\hbar}{2m\omega} \langle \psi(t) | (a+a^{+})^2 | \psi(t) \rangle = \frac{\hbar}{2m\omega} \langle (a+a^{+})\psi(t) | (a+a^{+})\psi(t) \rangle$

$$= \frac{\hbar}{2m\omega} \left| \frac{1}{\sqrt{2}} e^{\frac{-iE_3 t}{\hbar}} (\sqrt{3}|2\rangle + 2|4\rangle) + \frac{1}{\sqrt{2}} e^{\frac{-iE_4 t}{\hbar}} (2|3\rangle + \sqrt{5}|5\rangle) \right|^2$$

$$= \frac{\hbar}{4m\omega} (7 + 9) = \frac{4\hbar}{m\omega}$$

Thus, $\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{4\hbar}{m\omega} - \frac{2\hbar}{m\omega} \cos^2(\omega t)$

$$= \frac{2\hbar}{m\omega} (1 + \sin^2 \omega t)$$

$\Rightarrow \Delta X = \sqrt{\frac{2\hbar}{m\omega}} (1 + \sin^2 \omega t)^{\frac{1}{2}}$ #

(4) The simulations do agree with my calculation (both ⟨x⟩, Δx are periodic, with the frequency of Δx is twice as frequency of ⟨x⟩.

✓ (d)

(1) From the symmetric of X, P in the Hamiltonian of H we know that $\psi_n(P) = \langle p | n \rangle$ is only a translation of $\psi_n(x)$ to $\psi_n\left(\frac{P}{m\omega}\right)$. Thus, from (C) we know that the possible measurement of momentum can be $P \in [-\infty, \infty]$, with the probability density be $|\langle P | \psi(0) \rangle|^2 = \frac{1}{2}\left(\hat{\psi}_3\left(\frac{P}{m\omega}\right) + \hat{\psi}_4\left(\frac{P}{m\omega}\right)\right)^2$, where

$\psi_n(x)$ is the nth eigenstates in position space

(2) using $P = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}(a^{\dagger} - a)$, we have

$$\langle P(t) \rangle = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} \langle \psi(0) | (a^{\dagger} - a) | \psi(0) \rangle$$

$$= i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}\left(e^{i(E_4 - E_3)\frac{t}{\hbar}} - e^{-i(E_4 - E_3)\frac{t}{\hbar}}\right)$$

$$= 2i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} i \sin(\omega t) = -2\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} \sin(\omega t) \divideontimes$$

(3) for $\langle P^2\rangle$, Using $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2$ we have

$$\langle H\rangle = 4\hbar\omega = \frac{1}{2m}\langle P^2\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle$$

$$= \frac{1}{2m}\langle P^2\rangle + \frac{1}{2}m\omega^2\frac{4\hbar}{m\omega} = \frac{1}{2m}\langle P^2\rangle + 2\hbar\omega$$

$$\Rightarrow \langle P^2\rangle = 2m(4\hbar\omega - 2\hbar\omega) = 4m\hbar\omega$$

Thus, we have $\Delta P^2 = \langle P^2\rangle - \langle P\rangle^2 = 4m\hbar\omega - 2m\hbar\omega\sin^2\omega t$

$$= 2m\hbar\omega(1+\cos^2\omega t)$$

$$\Rightarrow \Delta P = \sqrt{2m\hbar\omega}\,(1+\cos^2\omega t)^{\frac{1}{2}}$$

(4) The simulation agrees with my expressions of $\langle P(t)\rangle$ and $\Delta p(t)$

(both $\langle P\rangle$, ΔP are periodic, with frequency $\omega_{(\Delta P)} = 2\,\omega_{(\langle P\rangle)}$.

There is a $\frac{\pi}{2}$ phase difference between $\langle P\rangle$ and $\langle x\rangle$, which is consistent to the $\sin\omega t$ and $\cos\omega t$ expressions