January 25, 2012

Lecture 5

Postulates
using a simple toy problem

Statistical quantities
expectation value
uncertainty

Postulate 1

EXAMPLE

The state of the system is represented by a vector in a Hilbert space Here the Hilbert space is three-dimensional

$$|\psi(t)\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

abstract state vector

state vector in the H basis

$$H = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The Hamiltonian operator in its basis

H is diagonal

$$p = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The momentum operator in the H basis

p is not diagonal

Q: IF WE MEASURE energy (momentum) AT TIME t, WHAT RESULTS WILL WE FIND AND WITH WHAT PROBABILITES WILL WE FIND THEM?

6

STEP 1: CONSTRUCT OPERATORS ✓

STEP 2: FIND ev's AND $e\vec{v}$'s OF OPERATORS

H $\quad ev = 3 \quad 4 \quad 5$

$$e\vec{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

p $\quad ev = 2 \quad 3 \quad 1$

$$e\vec{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

WHAT ENERGIES WILL WE FIND? $\quad 3, 4, 5$

WHAT MOMENTA WILL WE FIND? $\quad 2, 3, 1$

WITH WHAT PROBABILITIES WILL WE FIND THEM?

Postulate 3

After an energy measurement, the system will be in the corresponding eigenstate of H

After a momentum measurement, the system will be in the corresponding eigenstate of p

7

Measure E at t=0

$$PROB(E_i) = \frac{|\langle E_i | \psi(0) \rangle|^2}{\langle \psi(0) | \psi(0) \rangle}$$

The state vector at t=0 is

$$|\psi(0)\rangle \longrightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The eigenstates (eigenbras) of E are

$$\langle E=3| \longrightarrow (1 \quad 0 \quad 0)$$

$$\langle E=4| \longrightarrow (0 \quad 1 \quad 0)$$

$$\langle E=5| \longrightarrow (0 \quad 0 \quad 1)$$

The eigenbras must be normalized

Postulate 3

The probability that the system will be found in state A is given by the square of the inner product of A with the state of the system

SO, IF WE MEASURE THE ENERGY:

$$P(E=3) = \frac{\left| \langle \hat{E}_3^{E=3} \mid \psi(t) \rangle \right|^2}{\langle \psi(t) \mid \psi(t) \rangle}$$

Probability E=3 $= \dfrac{\left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \dfrac{1}{14}$ **Prob(E=3)**

$P(E=4) = \dfrac{\left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \dfrac{4}{14}$ **Prob(E=4)**

$P(E=5) = \dfrac{\left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \dfrac{9}{14}$ **Prob(E=5)**

The total probability must be 1

CHECK $\quad \dfrac{1}{14} + \dfrac{4}{14} + \dfrac{9}{14} = 1$ ✓

8

Measure p at t=0

$$\mathrm{PROB}(p_i) = \frac{|\langle p_i | \psi(0)\rangle|^2}{\langle \psi(0) | \psi(0)\rangle}$$

The state vector at t=0 is

$$|\psi(0)\rangle \rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The eigenstates (eigenbras) of p are

$$\langle p=2| \rightarrow (1 \quad 0 \quad 0)$$

$$\langle p=3| \rightarrow \frac{1}{\sqrt{2}}(0 \quad 1 \quad 1)$$

$$\langle p=1| \rightarrow \frac{1}{\sqrt{2}}(0 \quad 1 \quad -1)$$

The eigenbras must be normalized

Postulate 3

The probability that the system will be found in state A is given by the square of the inner product of A with the state of the system

NOW IF WE MEASURE THE MOMENTUM **(at t=0)**

$$P(p=2) = \frac{\left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{1}{14} = \frac{2}{28}$$ **Prob(p=2)**

$$P(p=3) = \frac{\left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \cancel{\frac{25}{28}} \; \frac{25}{28}$$ **Prob(p=3)**

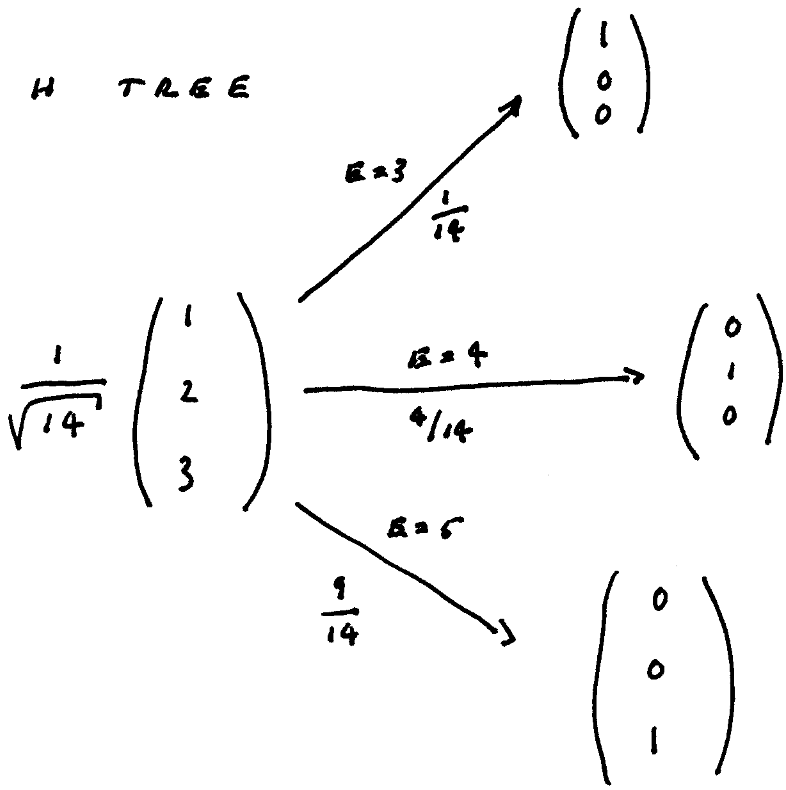
$$P(p=1) = \frac{\left| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2}{14} = \frac{1}{28}$$ **Prob(p=1)**

The probabilities must sum to 1

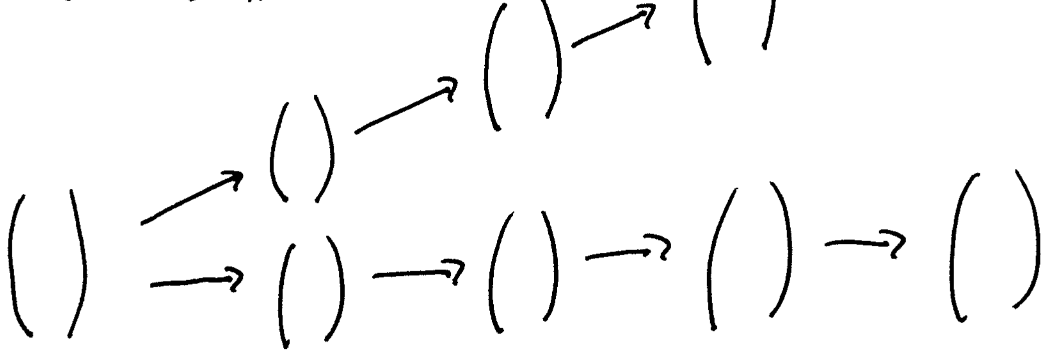
CHECK $\frac{2}{28} + \frac{25}{28} + \frac{1}{28} = 1$ ✓

System in an eigenstate of E, stays in that eigenstate.

H TREE



MEASURE E AGAIN



ALWAYS GET SAME ENERGY EV

STAY IN SAME ENERGY $E\vec{V}$

"What happens in energy, stays in energy."

P TREE

$$\begin{tikzcd}
 & \begin{pmatrix}1\\0\\0\end{pmatrix} \arrow[r, "p=2", "100\%"'] & \begin{pmatrix} \ \\ \ \\ \ \end{pmatrix} \\
\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} \arrow[ur, "p=2", "2/28"'] \arrow[r, "p=3", "25/28"'] & \begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[r, "p=3", "100\%"'] & \begin{pmatrix}0\\1\\1\end{pmatrix}
\end{tikzcd}$$

MEASURE IMMEDIATELY

$$\begin{tikzcd}
 & & \begin{pmatrix}0\\1\\1\end{pmatrix} \\
\begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[r, "\text{WAIT TIME } t", "\text{THEN MEASURE}"'] & {} \arrow[ur, "p=3"] \arrow[dr, "p=1"'] & \\
 & & \begin{pmatrix}0\\1\\-1\end{pmatrix}
\end{tikzcd}$$

System in an eigenstate of momentum, does not stay in that eigenstate at later times.

If [p, H] is not equal to zero, then the eigenstates of momentum evolve in time.

If momentum shares an eigenstate with energy, then that eigenstate does not evolve

"What happens in momentum does not necessarily stay in the same momentum"

Quantum Zeno effect: Continuous measurement prevents time evolution---an unstable system cannot decay.

REPEATED MEASUREMENTS WITH ZERO TIME DELAY

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{1}{14}]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[100\%]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$|\psi(0^-)\rangle$ $|\psi(0^+)\rangle$ $|\psi(0^{++})\rangle$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{4}{14}]{E=44} \begin{pmatrix}0\\1\\0\end{pmatrix} \xrightarrow[100\%]{E=44} \begin{pmatrix}0\\1\\0\end{pmatrix}$$

IF IN AN $e\vec{v}$ A OF H, THEN MEASURE ASSOCIATED $e\vec{v}$ OF H 100% OF THE TIME

IF IN AN $e\vec{v}$ OF P AND YOU MEASURE P_{op} AT LATER TIMES THEN IN GENERAL YOU WILL NOT STAY IN $e\vec{v}$ OF P_{op}

Wiki quotes Sudarshan and Misra (1977): "An unstable particle, if observed continuously, will never decay."

Cannot measure continuously. The projection postulate works.

FOR ZERO TIME DELAY

SAME THING FOR P

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{25}{28}]{P=3} \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \xrightarrow[100\%]{P=3} \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$|\psi(0^-)\rangle$ $\qquad\qquad$ $|\psi(0^+)\rangle$

FOR ZERO TIME DELAY

WHAT IF YOU MEASURE E, P, E, P, ...

H and p share the E=3 eigenstate.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{1}{14}]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[100\%]{p=2} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[100\%]{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

H AND P SHARE one EV $= \begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\begin{tikzcd}
 & & \begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[r, "E=4"', "50\%"] \arrow[dr, "50\%", "E=5"'] & \begin{pmatrix}0\\1\\0\end{pmatrix} \\
\begin{pmatrix}1\\2\\3\end{pmatrix} \arrow[r, "E=4", "\frac{4}{14}"'] & \begin{pmatrix}0\\1\\0\end{pmatrix} \arrow[ur, "p=3", "50\%"'] \arrow[dr, "p=1", "50\%"'] & & \begin{pmatrix}0\\0\\1\end{pmatrix} \\
 & & \begin{pmatrix}0\\1\\-1\end{pmatrix} \arrow[r, "E=4", "50\%"'] \arrow[dr, "E=5"', "50\%"'] & \begin{pmatrix}0\\1\\0\end{pmatrix} \\
 & & & \begin{pmatrix}0\\0\\1\end{pmatrix}
\end{tikzcd}$$

H AND P DO NOT SHARE ~~A~~ E=4 EV

H and p do not share the E=4 eigenstate.

H and p do not share E=5 eigenstate.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\frac{9}{14}]{E=5} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{cases} \xrightarrow[50\%]{p=3} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{cases}\xrightarrow{} E=4 \\ \xrightarrow{} E=5\end{cases} \\ \xrightarrow[50\%]{p=1} \begin{pmatrix}0\\1\\-1\end{pmatrix} \end{cases}$$

(ZERO TIME DELAY)

IF MEASURE P, E, P, E, ...

SAME TREES

Time evolution

WHAT HAPPENS IF WE WAIT t BEFORE WE MAKE THE SECOND MEASUREMENT?

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow{E=3} \begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow[\cancel{\text{PERFORM}}\ \text{MEASURE } E]{\text{WAIT } t} \begin{pmatrix}?\\?\\?\end{pmatrix}$$

$$|\psi(0^-)\rangle \qquad\qquad |\psi(0^+)\rangle \qquad\qquad |\psi(t)\rangle$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \xrightarrow{p=3} \begin{pmatrix}0\\1\\1\end{pmatrix} \xrightarrow[\text{MEASURE } p]{\text{WAIT } t} \begin{pmatrix}?\\?\\?\end{pmatrix}$$

ev = ?

ev = ?

TDSE

Postulate 4

IV. The state variables change with time according to Hamilton's equations:

$$\dot{x} = \frac{\partial \mathscr{H}}{\partial p}$$

$$\dot{p} = -\frac{\partial \mathscr{H}}{\partial x}$$

IV. The state vector $|\psi(t)\rangle$ obeys the *Schrödinger equation*

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

where $H(X, P) = \mathscr{H}(x \to X, p \to P)$ is the quantum Hamiltonian operator and $\mathscr{H}$ is the Hamiltonian for the corresponding classical problem.

TIME EVOLUTION POSTULATE 4

(1) EXPAND $|\psi(0)\rangle$ IN ENERGY EIGEN BASIS:

| psi(0) > = a1 |E=3> + b | E=4 > + c |E=5 > in Dirac notation

$$\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{2}{\sqrt{14}}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{3}{\sqrt{14}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

In basis notation. The basis is the eigenbasis of H

$\vec{ev}$ OF H OBEY TDSE **TDSE**

$$H\,|E_i\rangle = E_i\,|E_i\rangle$$

$$H\,|E_i\rangle = i\hbar\frac{d}{dt}|E_i\rangle = E_i\,|E_i\rangle$$

$$\boxed{|E_i(t)\rangle = e^{-iEt/\hbar}\,|E_i(0)\rangle}$$

The time dependence of the energy eigenstates

TIME EVOLUTION OF $\vec{ev}$'s OF H IS VERY SIMPLE: ONLY THEIR PHASE CHANGES

(2) SIMPLY WRITE DOWN THE TIME DEPENDENCE:

$$|\psi(t)\rangle = \sum e^{-iE_it/\hbar}\, a_i\,|E_i\rangle$$

You must express the expansion vectors in the basis of H

$$= \frac{1}{\sqrt{14}}e^{-i3t/\hbar}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{2}{\sqrt{14}}e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{3}{\sqrt{14}}e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

If H is not diagonal, these expansion vectors will not be the simple unit vectors

The Stationary States

ef's
$e\vec{v}$'s OF H ARE CALLED STATIONARY STATES
es's

ONLY THEIR PHASE VARIES

$$\xrightarrow{E=3} \begin{pmatrix}1\\0\\0\end{pmatrix} e^{-i3t/\hbar} \xrightarrow[t=0]{\substack{\text{MEASURE}\\ \text{IMMEDIATELY}\\ E=3}} \begin{pmatrix}1\\0\\0\end{pmatrix} e^{-i3t/\hbar}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} e^{-i3t/\hbar} \xrightarrow[\text{THEN MEASURE}]{\text{WAIT } t'} \left(\begin{pmatrix}1\\0\\0\end{pmatrix} e^{-i3t/\hbar}\right) e^{-i3t'/\hbar}$$

ALWAYS GET $E=3$

ALWAYS IN STATE $\begin{pmatrix}1\\0\\0\end{pmatrix} e^{i\theta_0}$

TIME EVOLUTION OF $e\vec{v}$ OF H:

$$e^{i\theta_0}\begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{\text{WAIT } t} \left[\begin{pmatrix}1\\0\\0\end{pmatrix} e^{i\theta_0}\right] e^{-i3t/\hbar}$$

only their phase changes

stay in the same state

WHERE DOES THE SIMPLE TIME DEPENDENCE COME FROM?

THE EV'S OF H

OBEY TDSE $\quad H|E_i\rangle = i\hbar \frac{d}{dt}|E_i\rangle$

OBEY TISE $\quad H|E_i\rangle = E_i|E_i\rangle$

~~FROM~~

$$|E_i(t)\rangle = e^{-iE_it/\hbar}|E_i(0)\rangle$$

PHASE EVOLUTION OF ENERGY EIGENKETS

Resolution of the identity
Expansion in a comple set of states
In this case expand in the eigenstates of H

Recipe for calculating time evolution

The sum of the all of the outer products of the energy eigenstates is equal to the identity operator

TIME EVOLUTION

(1) EXPAND $|\psi(0)\rangle$ IN ENERGY EIGENKETS

$$|\psi(0)\rangle = I\,|\psi(0)\rangle$$

$$= \boxed{\left(\sum_i |E_i\rangle\langle E_i|\right)}\,|\psi(0)\rangle$$

$$= \sum_i |E_i\rangle \underbrace{\langle E_i|\psi(0)\rangle}_{a_i}$$

$$= \sum_i a_i\,|E_i\rangle$$

Once you have the expansion in terms of energy eigenstates at t=0 just insert the simple phase factors to obtain the state vector at time t

(2) ADD THE TIME DEPENDENCE

$$|\psi(t)\rangle = \sum_i a_i\,|E_i\rangle\, e^{-iE_i t/\hbar}$$

TIME-DEP
PHASE FACTORS

WORKS FOR ALL t

The time dependence of the energy eigenstates is very simple. The time dependence of state vectors that are not eigenstates of E are not quite so simple

In general

ONLY EIGENKETS OF H ARE STATIONARY

LOOKED AT

$$|P=3\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

again, resolution of the identity

| p=3 > = Identity | p=3 >

$$|P=3\rangle = \boxed{\sum_i |E_i\rangle\langle E_i|}\, P=3\rangle$$

Works for any basis

$$= |E_1\rangle\langle E_1|P=3\rangle$$

$$+ |E_2\rangle\langle E_2|P=3\rangle$$

$$+ |E_3\rangle\langle E_3|P=3\rangle$$

$$= 0\,|E_1\rangle + \frac{1}{\sqrt{2}}|E_2\rangle + \frac{1}{\sqrt{2}}|E_3\rangle$$

$$|P=3,t\rangle = 0\,|E_1\rangle e^{-iE_1t/\hbar}$$

$$+ \frac{1}{\sqrt{2}}|E_2\rangle e^{-iE_2t/\hbar}$$

$$+ \frac{1}{\sqrt{2}}|E_3\rangle e^{-iE_3t/\hbar}$$

TIME EVOLUTION OF $\vec{ev}$'s OF p

$p=2$ $\vec{ev}$ OF P_{op} $= \begin{pmatrix}1\\0\\0\end{pmatrix} =$ $E=3$ $\vec{ev}$ OF H_{op}

$p=2$ IS A STATIONARY
BECAUSE $|p=2\rangle \equiv |E=3\rangle$
AND $|E=3\rangle$ IS STATIONARY

SO ITS TIME EVOLUTION IS EXACTLY THE SAME

AS THE $E=3$ $\vec{ev}$ OF H

$p=3$ $\vec{ev}$ OF p $= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$

NO EQUAL TO AN $\vec{ev}$ OF H
SO $|p=3\rangle$ IS TIME
DEPENDENT

TO FIND
ITS TIME EVOLUTION

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[0\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

EXPAND IN $\vec{ev}$'s OF H

BECAUSE WE KNOW THEIR
TIME EVOLUTION

If you are in a basis where H is not diagonal you must expand in that basis

PUT IN THEIR
TIME
DEPENDENCE

the H basis vectors

$$= \frac{1}{\sqrt{2}}\left[0\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-i3t/\hbar} + 1\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-i4t/\hbar} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-i\overset{5}{\cancel{4}}t/\hbar}\right]$$

IF $|\psi(0)\rangle = |p=3\rangle$ THEN $|\psi(t)\rangle \neq |p=3\rangle$

What does the time evolution of | p = 3 > look like?

$$| p = 3, t > = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-i 4 t/\hbar} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-i 5 t/\hbar}$$

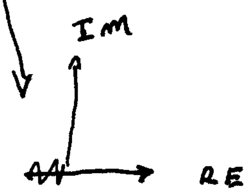
factor out exp(-i 4 t/hbar)

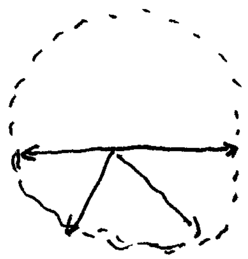
phase of (0 0 1) is different than the phase of (0 1 0)

$$= \frac{1}{\sqrt{2}} e^{-i 4 t/\hbar} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{-i (1) t/\hbar} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

over all phase factor

COEFF'S





At t = zero

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

At some specific times

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

At any time t

$$\begin{pmatrix} 0 \\ 1 \\ e^{-i t/\hbar} \end{pmatrix}$$

START IN $p = 3$ e$\vec{v}$ OF P_{op}

$\longleftarrow$ IN BETWEEN 50:50

LATER IN $p = 1$ e$\vec{v}$ OF P_{op}

$\longleftarrow$ AGAIN IN BETWEEN

BACK IN $p = 3$ e$\vec{v}$ OF P_{op}
JUST LIKE COUPLED PENDULA

IN BETWEEN, YOU ARE
IN A LINEAR COMBINATION
OF $|E=4\rangle$ AND $|E=5\rangle$

DEGENERATE OPERATORS

$$H' = \begin{pmatrix} 3 & & \\ & 4 & \\ & & 4 \end{pmatrix}$$

$$\frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

POSSIBLE ENERGIES	3	4	4
RESULTING STATE	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	?	?

PROB = $\frac{1}{14}$ TOGETHER $\frac{13}{14}$

NEED TO (BOLDLY) GO BEYOND

$$\text{PROB}(E_i) = |\langle E_i | \psi \rangle|^2 \longrightarrow \text{STATE} = |E_i\rangle$$

GENERAL CASE

$$\text{PROB}(E_i) = \langle \psi | \mathbb{P}(E_i) | \psi \rangle$$

$$= \left| \mathbb{P}(E_i) | \psi \rangle \right|^2$$

General form of Postulate 3

WHAT IS $\mathbb{P}(E_i)$

project onto the E=4 subspace

$$\mathbb{P} = |2\rangle\langle 2| + |3\rangle\langle 3|$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0\ 0\ 1)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{PROB}(E=4) = \left| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

$$= \frac{2^2 + 3^2}{14} = \frac{13}{14}$$

WHAT IS THE STATE?

$$|\psi(0^+)\rangle = \mathbb{P}\,|\psi(0^-)\rangle$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{14}} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$ NOT NORMALIZED

$$= \frac{1}{\sqrt{13}} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

MEASURE E AGAIN • ALWAYS FIND $E = 4$

STAY IN SAME STATE

Statistical Description
Expectation Value aka Mean or Average

THE STATISTICAL DESCRIPTION

DICTIONARY

THE REST OF PHYSICS		QUANTUM MECHANICS	
$\bar{x}$	MEAN AVERAGE	$\langle x \rangle$	EXPECTATION VALUE
σ	STANDARD DEVIATION	Δx	UNCERTAINTY

EXPECTATION VALUE

Two versions

(1)
$$\boxed{\langle \Omega \rangle_\psi = \sum_i P(\omega_i)\, \omega_i}$$

$$= \frac{\langle \psi | \Omega | \psi \rangle}{\langle \psi | \psi \rangle}$$

if $\langle \psi | \psi \rangle = 1$

(2)
$$\boxed{\langle \Omega \rangle_\psi = \langle \psi | \Omega | \psi \rangle}$$

WORKS IN ANY BASIS!

SHOW THE TWO DEF'S AGREE

$$\langle \Omega \rangle_\psi = \sum_i P(\omega_i)\, \omega_i$$

$$= \sum |\langle i | \psi \rangle|^2\, \omega_i$$

$$= \sum_i \langle \psi | i \rangle \langle i | \psi \rangle\, \omega_i$$

just a number

$$= \sum_i \langle \psi | \Omega | \underbrace{i \rangle \langle i}_{I} | \psi \rangle = \langle \psi | \Omega | \psi \rangle$$

BACK TO OUR TOY PROBLEM

Version 1

$$H = \begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$P(E=3) = |\langle i|\psi\rangle|^2 = \left| (1\ 0\ 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

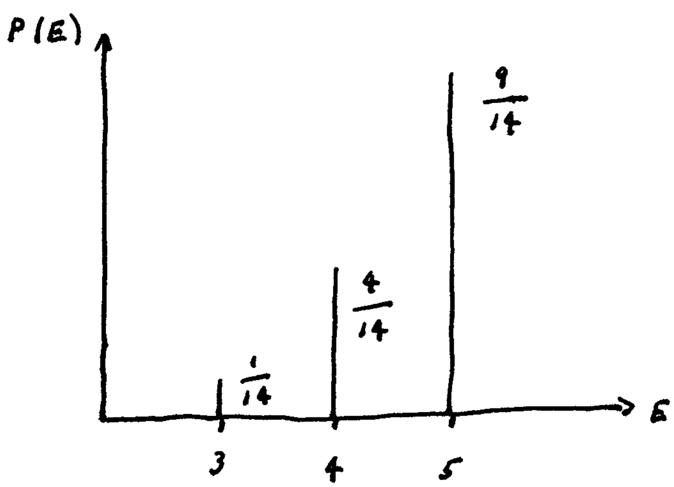
$$P(E=3) = \frac{1}{14}$$

$$P(E=4) = |\langle i|\psi\rangle|^2 = \left| (0\ 1\ 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

$$P(E=4) = \frac{4}{14}$$

$$P(E=5) = |\langle i|\psi\rangle|^2 = \left| (0\ 0\ 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

$$P(E=5) = \frac{9}{14}$$



Now compute < H > using method 1

FIRST WAY $\langle H \rangle$

$$\langle \Omega \rangle_\psi = \sum_i P(\omega_i)\,\omega_i$$

$$= \sum_i |\langle \omega_i | \psi \rangle|^2 \omega_i$$

$$\langle H \rangle = \langle E \rangle = \frac{1}{14}(3) + \frac{4}{14}(4) + \frac{9}{14}(5)$$

$$= \frac{64}{14} \approx 4.57$$

SECOND WAY **Version 2**

$$\langle \psi | H | \psi \rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 8 \\ 15 \end{pmatrix}$$

$$= \frac{64}{14}$$

$\Rightarrow$ SAME RESULT

BUT $\langle \psi | H | \psi \rangle$ DOES NOT DEPEND ON THE BASIS

CAN CALCULATE $\langle \psi | H | \psi \rangle$ WITHOUT ev's AND e$\vec{v}$'s!

CANNOT CALCULATE $\sum_i P(\omega_i)\, \omega_i$ w/o ev's AND e$\vec{v}$'s

Do not need to know the eigenvalues or the eigenstates of the operator

The eigenstates of H are stationary, so < H(t)> = < H(0) >

EXPECTATION VALUE OF ENERGY VERSUS TIME

$$\langle E \rangle_{\psi(0)} = \langle \psi(0) | H | \psi(0) \rangle$$

$$\langle E(0) \rangle = \frac{64}{14}$$

$$\langle E \rangle_{\psi(t)} = \langle E(t) \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

$$= \frac{1}{\sqrt{14}} \left(e^{+i3t/\hbar}, 2e^{+4it/\hbar}, 3e^{+i5t/\hbar} \right) \times$$

$$\begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix} \begin{pmatrix} e^{-i3t/\hbar} \\ 2e^{-i4t/\hbar} \\ 3e^{-i5t/\hbar} \end{pmatrix}$$

$$\langle E(t) \rangle = \langle E(0) \rangle$$ **< H(t)> = < H(0) >**

$\Rightarrow$ ENERGY IS CONSERVED!

Calculate the commutator:

GENERAL CASE: IF $[\Omega, H] = 0$

THEN $\langle \Omega(0) \rangle = \langle \Omega(t) \rangle$

$\Rightarrow$ Ω IS CONSERVED

$$[H, H] = 0$$

$$\boxed{\text{HAMILTONIAN GENERATES TIME TRANSLATION!}}$$

WE ALSO SAID

$$|\psi(t)\rangle = \sum_j |E_j\rangle\langle E_j|\psi(0)\rangle e^{-iE_jt/\hbar}$$

$$= \underbrace{\boxed{\sum_j e^{-iE_jt/\hbar} |E_j\rangle\langle E_j|}}_{U(t)} \quad |\psi(0)\rangle$$

This unitary operator is called the propagator.

It generates the time evolution of every state.

$U(t)$ IS UNITARY $\Rightarrow$ STATES STAY NORMALIZED!

$$\langle\psi(0)|\psi(0)\rangle = 1$$

$$\uparrow$$
$$I$$

$$1 = \langle\psi(0)|U(t)\,U^{\dagger}(t)|\psi(0)\rangle$$

$$1 = \langle\psi(t)|\psi(t)\rangle$$

It is always diagonal in the energy basis

IN DIAGONAL REPRESENTATION

$$U(t) = \begin{pmatrix} e^{-iE_1 t/\hbar} & & 0 \\ & e^{-iE_2 t/\hbar} & \\ 0 & & e^{-iE_3 t/\hbar} \end{pmatrix}$$

general unitary form $\quad e^{i\theta}$ on the diagonal

In general U(t) = exp(- i H t/hbar)

TIME-EVOLUTION OF THE STATIONARY STATES

$$U(t) = \begin{pmatrix} e^{-i3t/\hbar} & & \\ & e^{-i4t/\hbar} & \\ & & e^{-i5t/\hbar} \end{pmatrix}$$

$t=0$ | $t=t$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} e^{-i3t/\hbar} \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ e^{-i4t/\hbar} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ e^{-i5t/\hbar} \end{pmatrix}$$

start in $e\vec{s}$ of H, stay in $e\vec{s}$ of H

STATIONARY STATES $\Longleftrightarrow$ ENERGY CONSERVATION

HOWEVER,

e$\vec{s}$ OF Ω WHERE $[H, \Omega] \neq 0$ ARE NOT STATIONARY

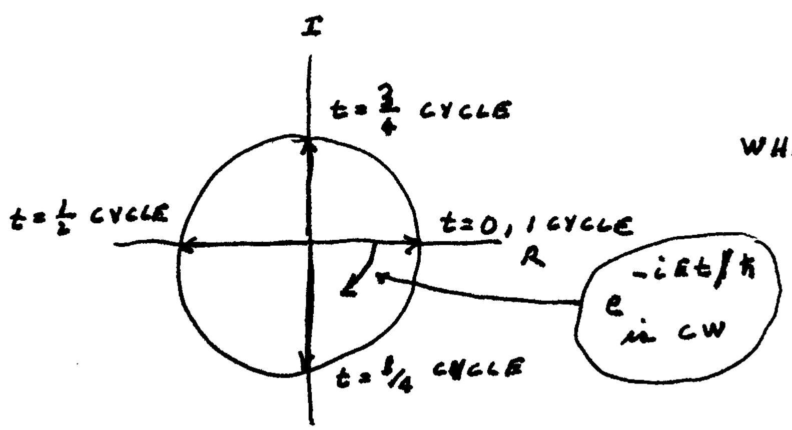
$t=0$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$p = 3$

$t = t$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{-i4t/\hbar} \\ e^{-i5t/\hbar} \end{pmatrix}$$

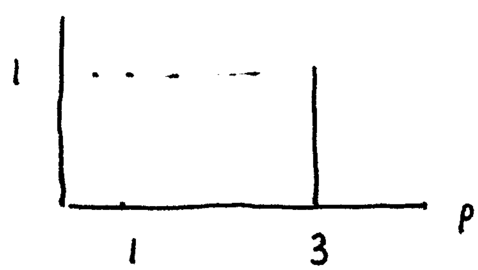


WHEN $\theta_5 - \theta_4 = \pi$

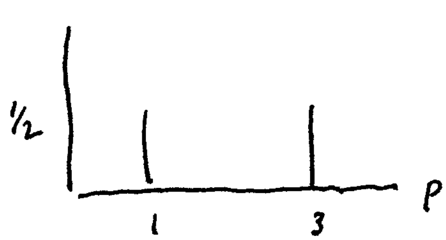
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

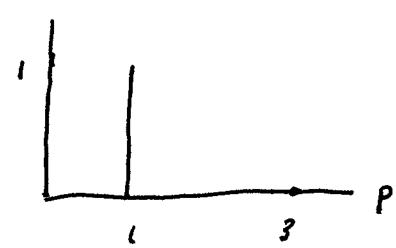
$p = 1$

PROB (P) AT $t = 0$



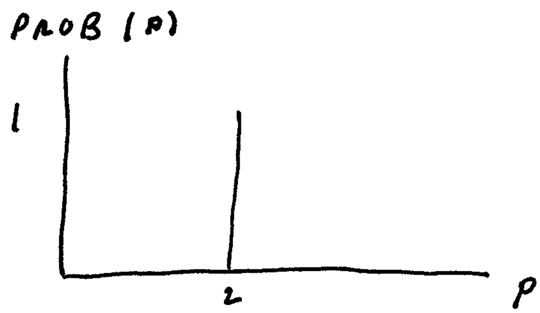
$\langle P \rangle$ DEPENDS ON TIME!





WHAT ABOUT $p=2$ STATE

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i3t/\hbar} \\ 0 \\ 0 \end{pmatrix}$$



MEASURE $p=2$ INDEPENDENT OF TIME

Statistical Description

Uncertainty aka Standard Deviation

$\bar{x}$ $\quad\quad$ $\langle x \rangle$

σ_x $\quad\quad$ Δx

EXPECTATION VALUE

$$\langle x \rangle_\psi = \langle \psi | x_{op} | \psi \rangle$$

$$= \sum_i P(x_i)\, x_i \quad\quad \text{DISCRETE}$$

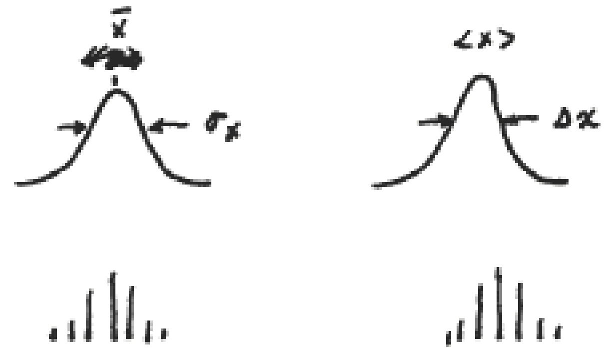
$$= \int_{x_i - \epsilon}^{x_i + \epsilon} dx\; P(x)\, x \quad\quad \text{CONTINUOUS}$$

MORE LATER ABOUT CONTINUOUS

TODAY: UNCERTAINTY

QM ANALOG OF STANDARD DEVIATION

CONTINUOUS



PROB DENSITIES

DISCRETE

UNCERTAINTY

$$\Delta\Omega = \langle (\Omega - \langle\Omega\rangle)^2 \rangle^{1/2}$$

S M R

R = ROOT

M = MEAN

S = SQUARE

AGAIN, TWO FORMS

$$(\Delta\Omega)^2 = \sum_i P(\omega_i)\left[\omega_i - \langle\Omega\rangle\right]^2$$

$$(\Delta\Omega)_\psi = \sqrt{\langle\psi|\boxed{\Omega - \langle\Omega\rangle}|\psi\rangle}$$

Oops, this quantity must be squared as shown above

Method 1: Need to know eigenvalues and the probabilities

FIRST WAY $\Delta\Omega$

$$(\Delta H)^2 = \sum_i p(E_i)\left[E_i - \langle E\rangle\right]^2$$

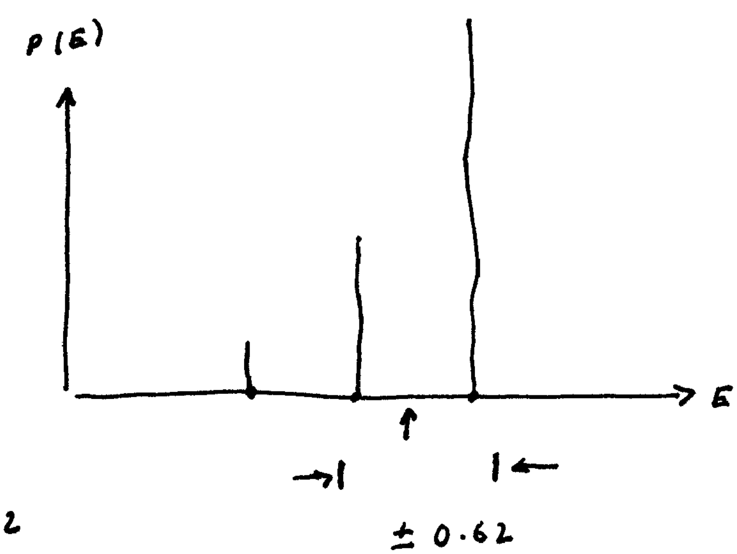
$$= \frac{1}{14}\left[3 - 4.57\right]^2$$

$$+ \frac{4}{14}\left[4 - 4.57\right]^2$$

$$+ \frac{9}{14}\left[5 - 4.57\right]^2$$

$$(\Delta H)^2 = \frac{5.43}{14} = 0.39$$

$$\Delta H = 0.62$$



$E = 4.57 \pm 0.62$

Method 2: In general this matrix is not diagonal

SECOND WAY $\Delta\Omega$

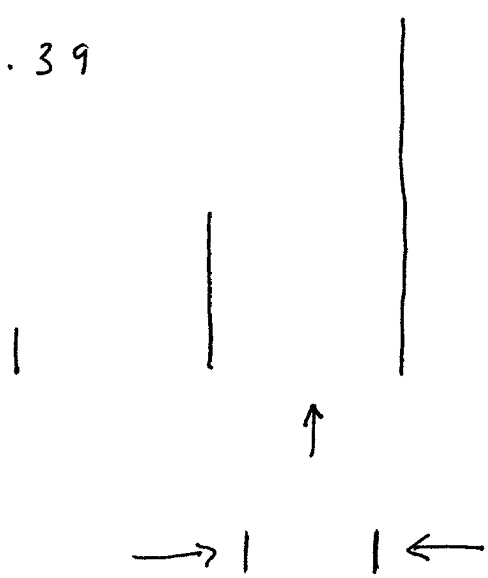
$$\Delta\Omega^2 = \frac{1}{\sqrt{14}}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\left[\begin{pmatrix}3 & & \\ & 4 & \\ & & 5\end{pmatrix} - \begin{pmatrix}4.57 & & \\ & 4.57 & \\ & & 4.57\end{pmatrix}\right]^2 \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}-1.57 & & \\ & -0.57 & \\ & & +0.43\end{pmatrix}^2\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$= \frac{1}{14}\left[(1)(-1.57)^2 + 4(-0.57)^2 + 9(0.43)^2\right]$$

$$\Delta\Omega^2 = \frac{5.43}{14} = 0.39$$

$$\Delta\Omega = 0.62$$



$\pm\ 0.62$

18

THIRD ~~SECOND~~ WAY $\Delta\Omega$ (SECOND WAY)$'$?

Method 2'

USEFUL ALGEBRA

square Omega first

$$\Delta\Omega^2 = \langle (\Omega - \langle\Omega\rangle I)^2 \rangle = \langle (\Omega^2 - 2\langle\Omega\rangle\Omega^I + \langle\Omega\rangle^2 I)\rangle$$

$$= \langle\Omega^2\rangle - 2\langle\Omega\rangle\langle\Omega\rangle + \langle\Omega\rangle^2$$

$$= \langle\Omega^2\rangle - \langle\Omega\rangle^2$$

$$\left[\frac{1}{\sqrt{14}}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}9 & & \\ & 16 & \\ & & 25\end{pmatrix}\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}\right]$$

$$-\left[\frac{1}{\sqrt{14}}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}3 & & \\ & 4 & \\ & & 5\end{pmatrix}\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}\right]^2 = \frac{5.43}{14} = 0.39$$

$$\Delta\Omega = \sqrt{(\Delta\Omega)^2} = 0.69$$

NOTE THAT

$$\langle \Omega \rangle_\psi = \frac{\langle \psi | \Omega | \psi \rangle}{\langle \psi | \psi \rangle}$$

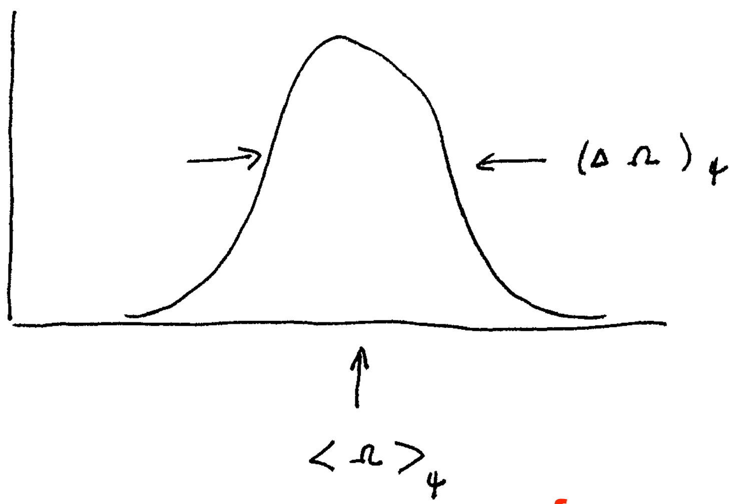
AND

$$(\Delta \Omega)_\psi = \sqrt{\frac{\langle \psi | (\Omega - \langle \Omega \rangle)^2 | \psi \rangle}{\langle \psi | \psi \rangle}}$$

DO NOT REQUIRE US TO KNOW THE

ev's OR $e\vec{v}$'s OF Ω !

DISCRETE $\longrightarrow$ CONTINUOUS



for example Omega might be the second derivative wrt x

CONTINUOUS

$$\langle\psi|\Omega|\psi\rangle \longrightarrow \int \psi^*(x)\,\Omega\,\psi(x)\,dx$$

$$(\Delta\Omega)^2_\psi = \int P(\omega)\,[\omega - \langle\Omega\rangle]^2\,d\omega$$

USEFUL ALGEBRA

$$\langle(\Omega - \langle\Omega\rangle I)^2\rangle = \langle\Omega^2 - 2\langle\Omega\rangle\Omega + \langle\Omega\rangle^2 I\rangle$$

$$= \langle\Omega^2\rangle - \langle\Omega\rangle^2$$

CONTINUOUS CASE

$$\langle \Omega^2 \rangle - \langle \Omega \rangle^2$$

$$\int \psi^*(x)\, \Omega^2\, \psi(x)\, dx - \left(\int \psi^*(x)\, \Omega\, \psi(x)\, dx \right)^2$$

If Omega is simple (for example, if it is some power of a derivative or multiplication by some power of x) then these integrals can often be done without knowing the eigenvalues or eigenvectors of Omega

Two thousand five hundred and two years ago, Zeno of Elea wrote a book about 40 paradoxes dealing with the continuum, now long lost.

The quantum Zeno effect (QZE) comes from Zeno's arrow paradox:

Since an arrow in flight is not seen to move during any single instant, it cannot possibly move at all.

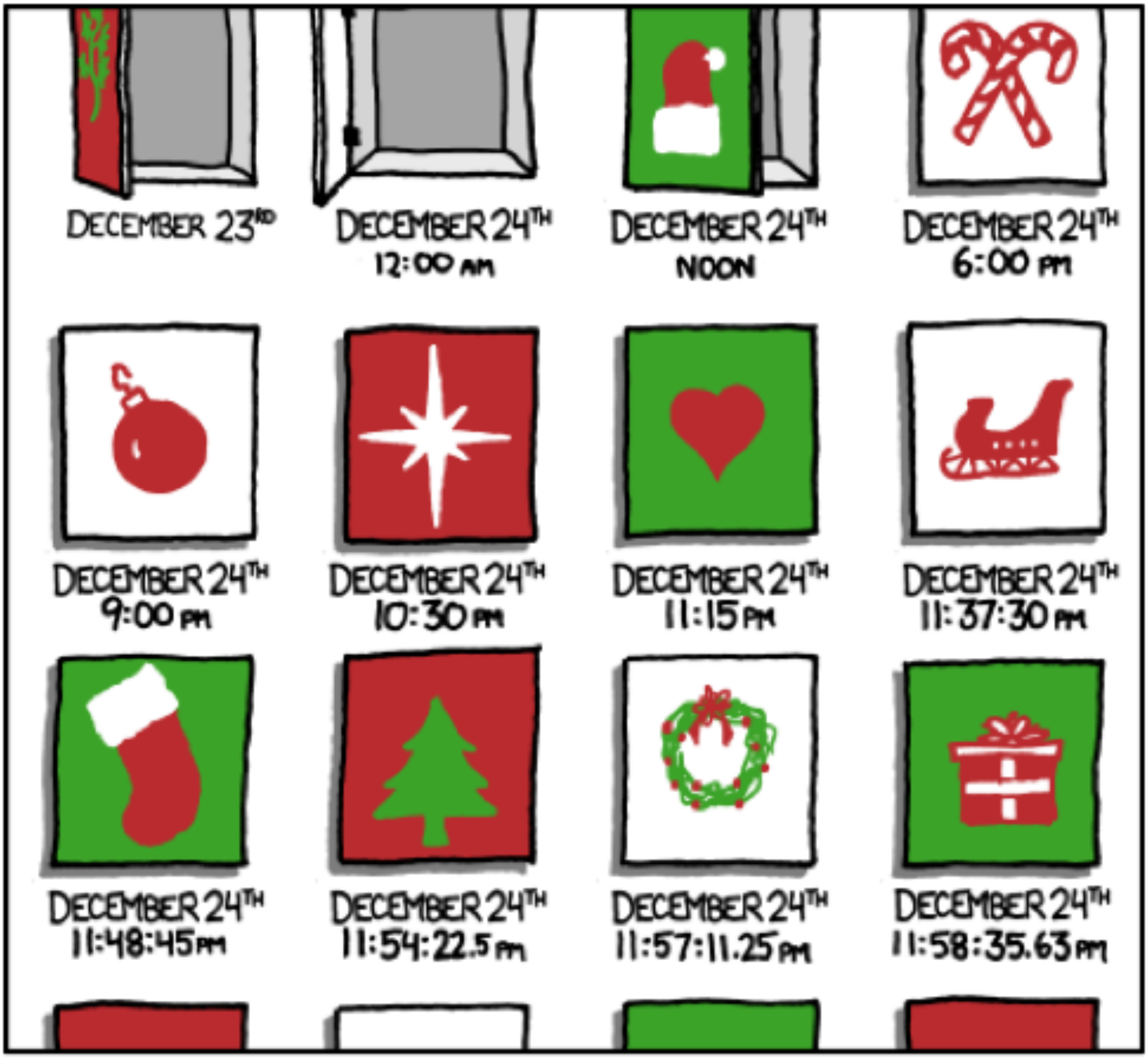
QZE: One can nearly "freeze" the evolution of the system by measuring it frequently enough.

The Projection Postulate and the Quantum Zeno Effect

The projection postulate has been used to predict a slow-down of the time evolution of the state of a system under rapidly repeated measurements, and ultimately a freezing of the state. To test this so-called quantum Zeno effect an experiment was performed by Itano et al. (Phys. Rev. A 41, 2295 (1990)) in which an atomic-level measurement was realized by means of a short laser pulse. The relevance of the results has given rise to controversies in the literature. In particular the projection postulate and its applicability in this experiment have been cast into doubt. In this paper we show analytically that for a wide range of parameters such a short laser pulse acts as an effective level measurement to which the usual projection postulate applies with high accuracy. The corrections to the ideal reductions and their accumulation over n pulses are calculated. Our conclusion is that the projection postulate is an excellent pragmatic tool for a quick and simple understanding of the slow-down of time evolution in experiments of this type. However, corrections have to be included, and an actual freezing does not seem possible because of the finite duration of measurements.

http://arxiv.org/abs/quant-ph/9512012
For many more, google: quantum zeno effect site:arxiv.org

"a watched pot never boils"



ZENO'S ADVENT CALENDAR