January 23, 2012

Mathematics Part 3

Geometry

Bra Space and Ket Space
Eigenvalues and Eigenvectors

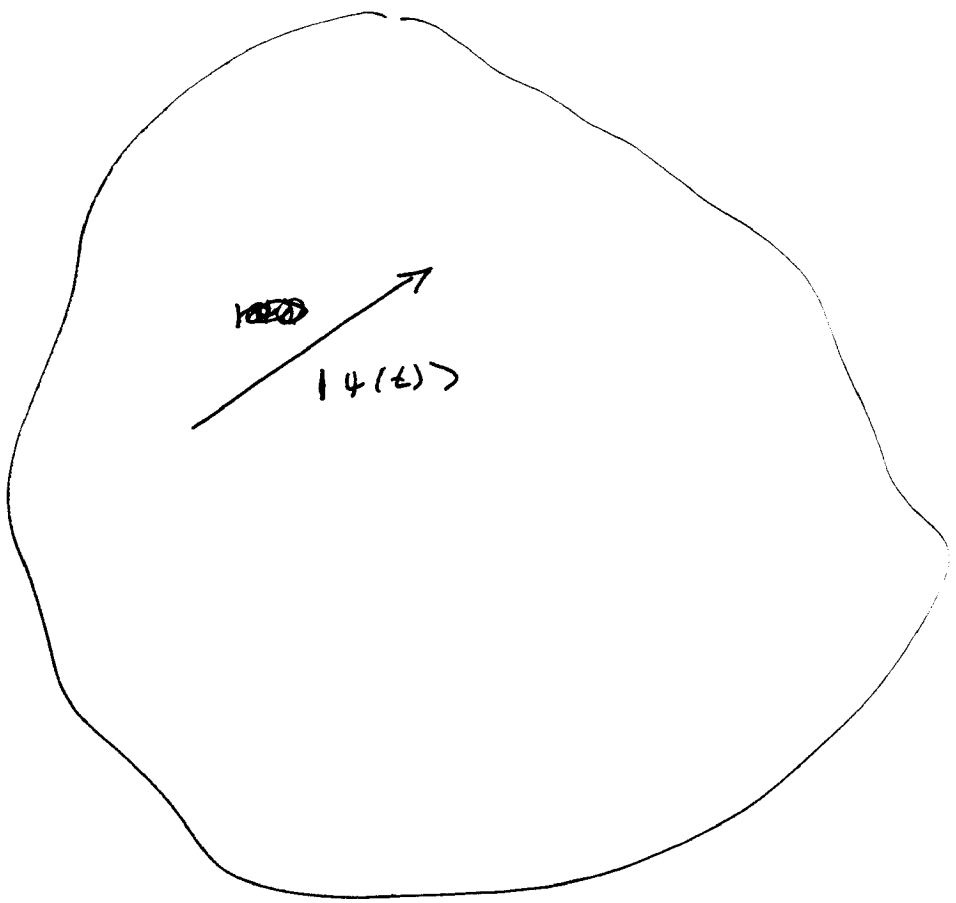
Algebra

Bra Space and Ket Space
Hermitean Operators
Anti-Hermitean Operators
Unitary Operators
Projection Operators
Degenerate Operators
Inner Products and Outer Products
Characteristic Equation
Eigenvalues and Eigenvectors
Changing Basis
Commutation Relations
The Commutator
Compatible Operators
Incompatible Operators
Partially Compatible Operators
Simultaneous Diagonalization
Complete Set of Commuting Observables (CSCOs)
Fundamental Theorem of Algebra
Abel-Ruffini Theorem

THE GEOMETRY OF QM

QM TAKES PLACE IN HILBERT SPACE,

AN INFINITE DIMENSIONAL VECTOR SPACE.

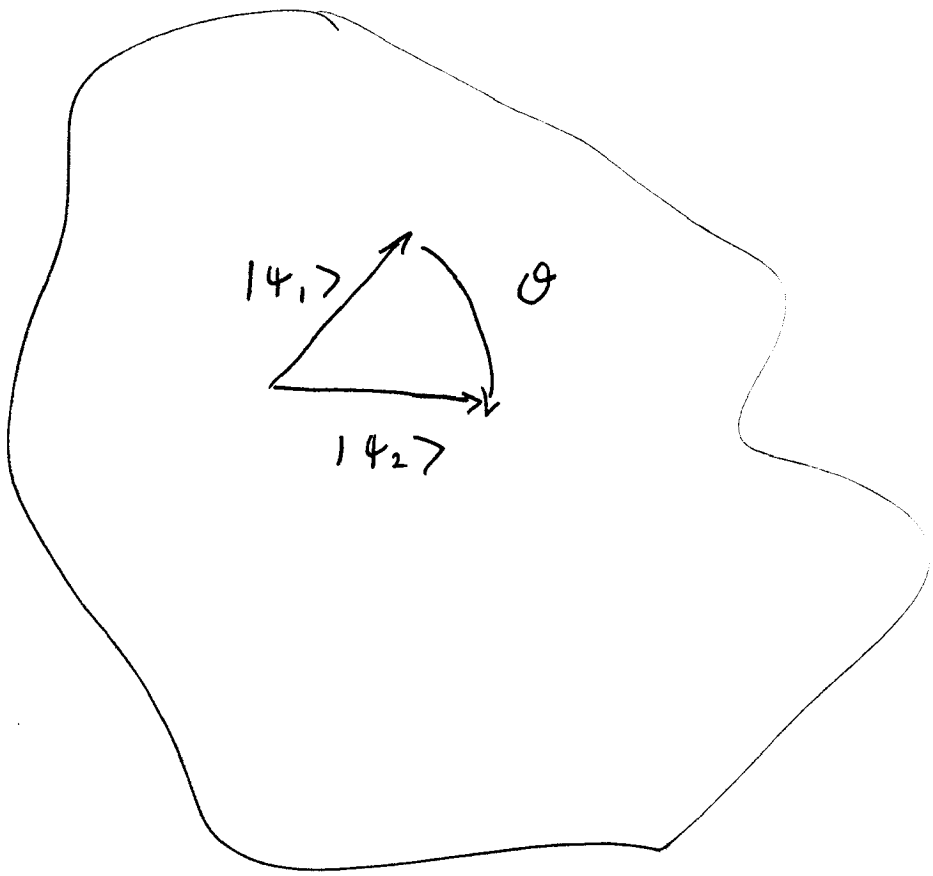


STATES OF THE SYSTEM $\Rightarrow$ VECTORS

$|\psi(t)\rangle$ = A VECTOR IN THE HILBERT SPACE

Postulate 1: The state of the system is represented by a state vector in a Hilbert space

ALSO HAVE OPERATORS $\mathcal{O}\,|\psi_1\rangle = |\psi_2\rangle$



PHYSICAL OBSERVABLES => EIGENVALUES OF HERMITEAN OPERATORS

Postulate 2: Physical observables are represented by Hermitean operators

Postulate 3: Measurement of a physical observable will result in an eigenvalue of the operator

DYNAMICS OF THE SYSEM => $\mathcal{H}$ via TDSE

Time-Dependent Schrodinger Equation

$$\boxed{H\,|\psi(t)\rangle = i\hbar \frac{d}{dt}\,|\psi(t)\rangle}$$

TDSE

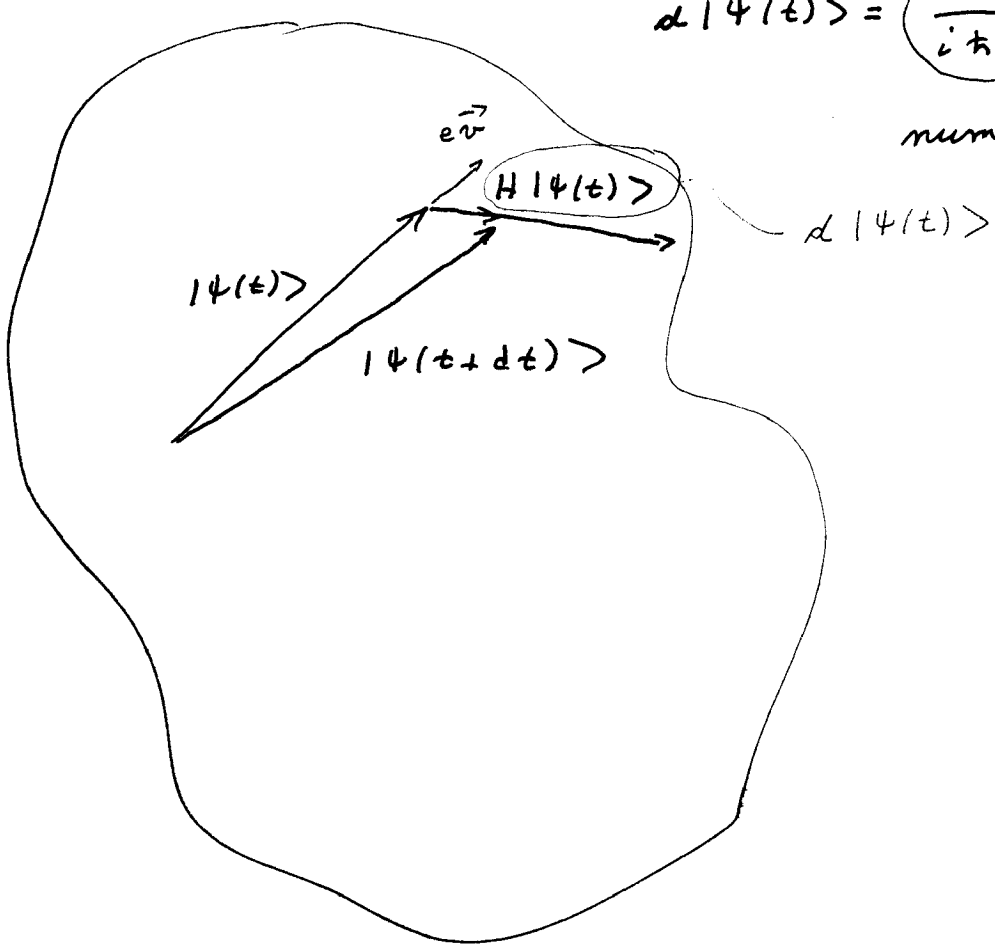
FIRST-ORDER DIFF EQ

SIMPLER THAN CM $F = m\frac{d^2x}{dt^2}$ SECOND ORDER

GEOMETRY OF THE TIME EVOLUTION

$$d\,|\psi(t)\rangle = \frac{dt}{i\hbar}\left(H\,|\psi(t)\rangle\right)$$

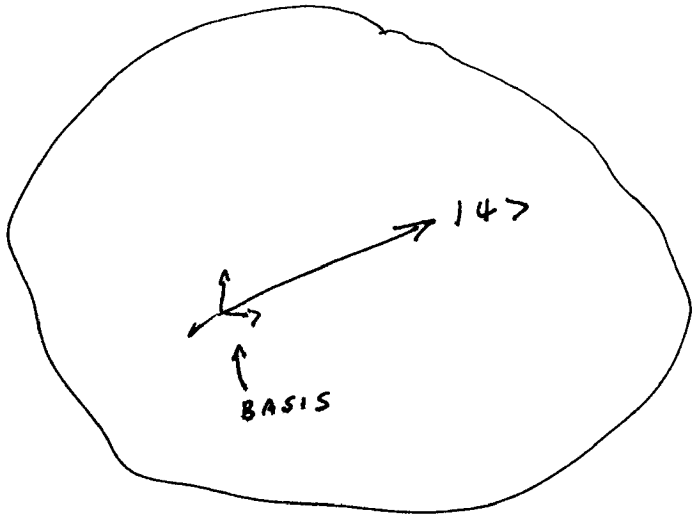
number



Postulate 4: The time evolution of the state vector is given by the time-dependent Schrodinger equation.

the vectors in the Hilbert space are "abstract vectors"
they exist/make sense without requiring any basis

THE HILBERT SPACE IS AN ABSTRACT SPACE



ABSTRACT STATE VECTOR

TO GET ~~POSITION SPACE~~ THE USUAL WAVEFCNS,

WE MUST INTRODUCE A BASIS

$|\psi\rangle$ ABSTRACT, NO BASIS, GEOMETRIC OBJECT

$\langle x|\psi\rangle = \psi(x)$ POSITION-SPACE WAVEFCN

< x |, < p |, and < E | each represent a complete set of basis vectors

$\langle p|\psi\rangle = \hat{\psi}(p)$ MOMENTUM-SPACE WAVEFCN

$\langle E|\psi\rangle = \tilde{\psi}(E)$ ENERGY-SPACE WAVEFCN

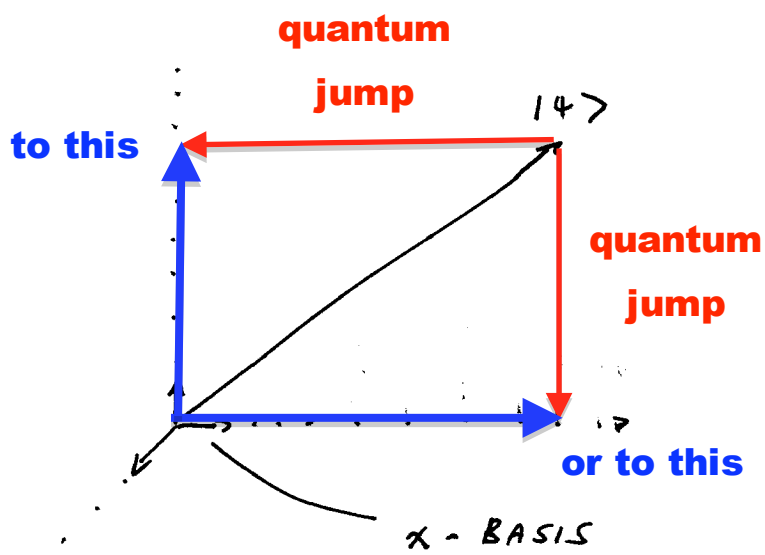
↑

inner product
in DIRAC
notation

< bra | ket > is a number defined for each value of x, p, or E
just like the dot product where v dot x-hat = v_x

GEOMETRY OF MEASUREMENT

MEASURE x



an eV x - OPERATOR

RESULT IS eV OF THE x - BASIS

STATE IS AN eV OF THE x - OPERATOR

Postulate 3: Measurement of a physical observable will result in an eigenvalue of the operator and the state of the system will correspond to one of the eigenvectors of the measured operator

LECTURE 3

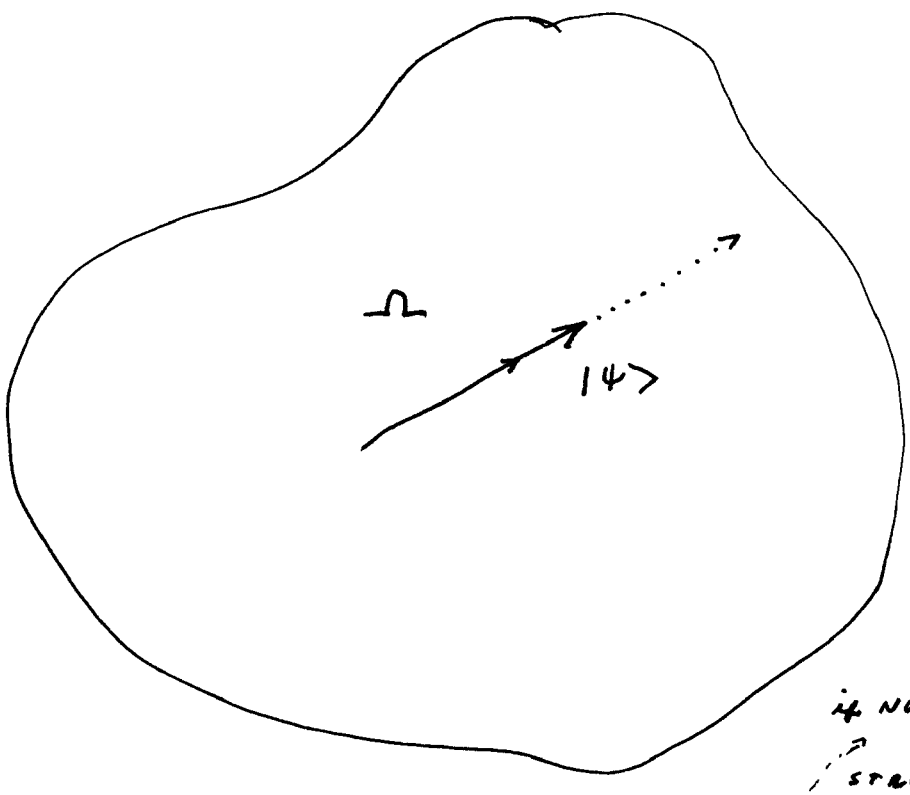
TODAY: EIGENVALUES AND EIGENVECTORS

" " EIGENFUNCTIONS

" " EIGENSTATES
BRAS
KETS

Q: WHEN DOES AN OPERATOR JUST STRETCH OR SHRINK A VECTOR WITHOUT CHANGING ITS DIRECTION?

Q: When does an operator just change the phase of a vector?



if NOT HERMITIAN

STRETCH $|\alpha|$

PHASE $e^{i\varphi}$

$$\Omega\,|\psi\rangle = \alpha\,|\psi\rangle$$

$\alpha = a + bi = |\alpha| e^{i\varphi}$

general op $\Rightarrow$ α is complex

Hermitian operator $\Rightarrow$ α is real

THE IMPORTANCE OF ev AND $e\vec{v}$:

TIME EVOLUTION IS GIVEN BY TDSE:

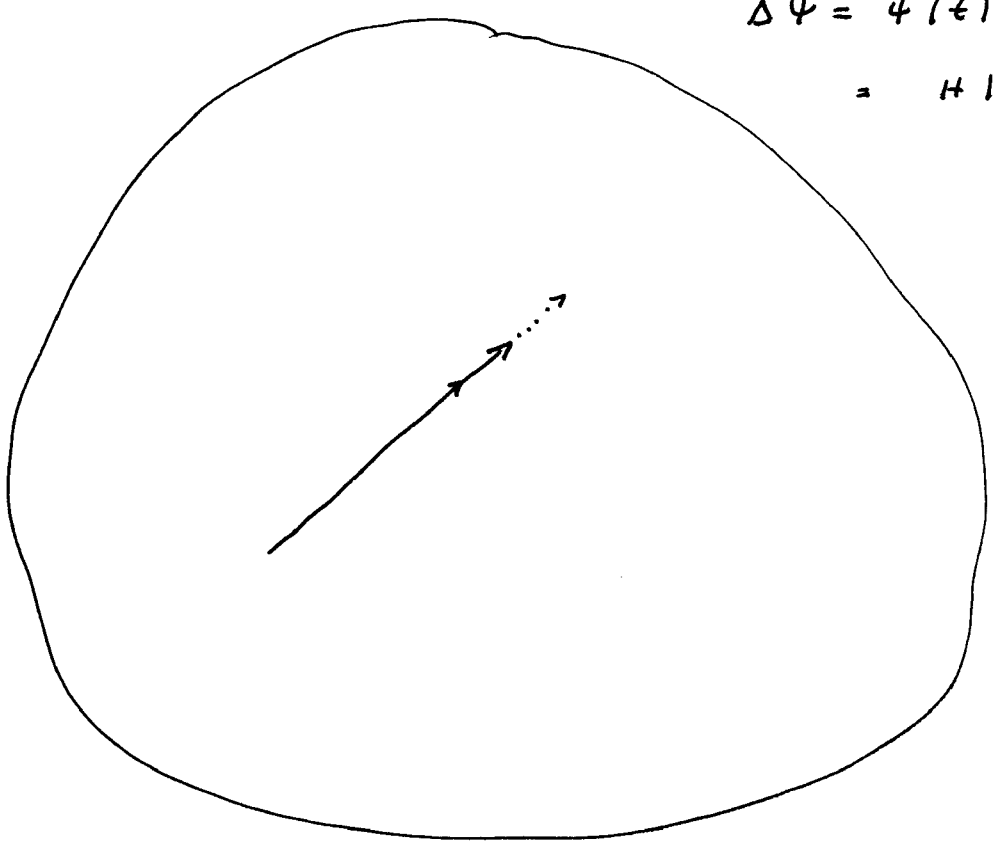
$$H\,|\psi(t)\rangle = i\hbar \frac{d}{dt}\,|\psi(t)\rangle$$

$$\left(\frac{dt}{i\hbar}\right)\left\{H\,|\psi(t)\rangle\right\} = d\,|\psi(t)\rangle$$

$$\frac{H\,|\psi(0)\rangle}{i\hbar}\int_0^t dt = \int_0^t d(\ )$$

$$\Delta\psi = \psi(t) - \psi(0)$$

$$= H\,|\psi(0)\rangle\,\frac{t}{i\hbar}$$



$e\vec{v}$ of H are called stationary states

for stationary states the only time evolution is the phase of $e\vec{v}$ or ef

NORMAL MODE DEMO!

2 KINDS OF EVOLUTION: HAMILTONIAN

MEASURE MEASUREMENT

TALK ABOUT THIS

(1) jump is instantaneous

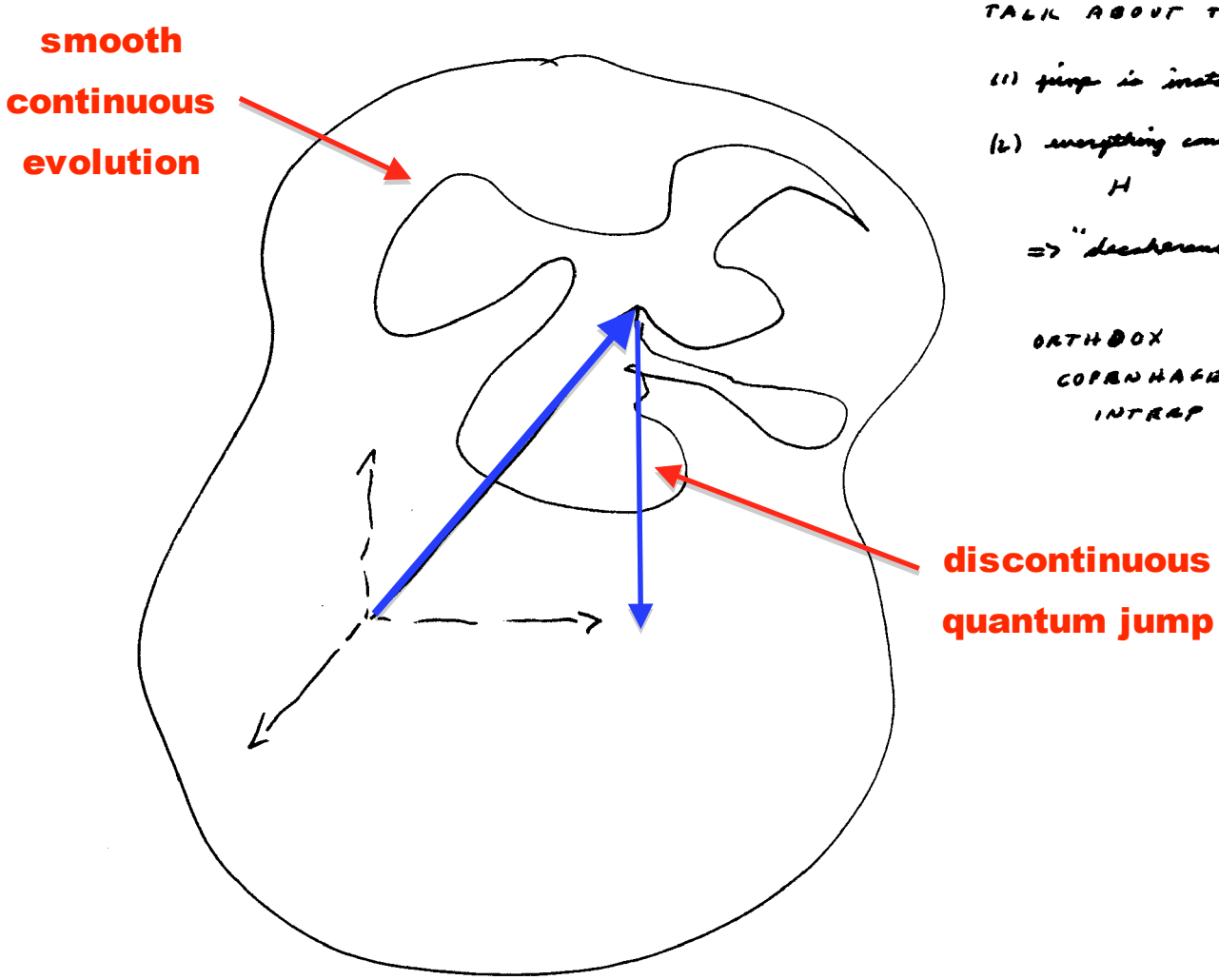
(2) everything controlled by H

=> "decoherence"

ORTHODOX

COPENHAGEN

INTERP



$\vec{v}$ jumps to e$\vec{v}$ of the thing you measure

measure E eigenstate of H energy eigenstate

measure X eigenstate of X_{op} position eigenstate

measure P eigenstate of P_{op} momentum eigenstate

general case

$$\Omega \, |v\rangle = |w\rangle$$

$|w\rangle$ is not parallel to $|v\rangle$

special case

$$\Omega \, |i\rangle = w_i \, |i\rangle$$

eigenvalue eigenvector

When Ω is Hermitian $\Rightarrow$ all w_i's are real

$\Rightarrow$ $\{|i\rangle\}$ is a basis for the Hilbert space

VERY SPECIAL CASE: IDENTITY OPERATOR

$$I \, |\psi\rangle = (+1) \, |\psi\rangle$$

every vector is an $e\vec{v}$ with $ev = +1$.

consider

EXAMPLE #1:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

nondegenerate case

unique ev $\Rightarrow$ unique ev $\vec{v}$

$\Rightarrow$ unique ~~ev~~ eigenbasis

EXAMPLE #2

or $$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

degenerate case

degenerate eigenvalues $\Rightarrow$ many eigenbases

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

This is very important for the physics!!!

We will want to find a complete set of commuting operators. CSCO

E, L^2, L_z for the hydrogen atom

$|n, l, m\rangle$

in CM find all of the CONSERVED QUANTITIES

in QM " " " " COMMUTING OBSERVABLES

commute with H

$H\Omega \equiv \Omega H$

commutator [A, B] = AB - BA

share a common eigenbasis

Simultaneously Diagonalizable:

in the common shared basis

they are both diagonal

CM: find all conserved quantities

QM: find a complete set of commuting observables

(2) PROJECTION OPERATORS

$$\mathbb{P}_i \, |i\rangle = \left(|i\rangle\langle i| \right) |i\rangle = (+1)\, |i\rangle$$

$|i\rangle$ is eigenvector with eigenvalue $+1$

$$\mathbb{P}_i \, |j\rangle = \left(|j\rangle\langle j| \right) |i\rangle = (0)\, |j\rangle$$

$|j\rangle$ are eigenvector with eigenvalue 0

Diagonalizing the Hamiltonian:

FINDING ev and $e\vec{v}$ OF Ω

(1) $\det(\Omega - \omega I) = 0 \Rightarrow$ CHARACTERISTIC EQN (CE)

(2) SOLVE CE TO GET ev $= \{\omega_i\}$

(3) $\Omega \, |\omega_i\rangle = \omega_i \, |\omega_i\rangle$ SOLVE FOR $e\vec{v} = \{|\omega_i\rangle\}$

Hamiltonian is not a function of time => energy is conserved

The Hamiltonian generates the time evolution via TDSE

2

EXAMPLE 3

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & 2-\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = 0$$

$$(1-\omega)(2-\omega)(1-\omega) - 0 + (1)\left[0 - (2-\omega)(1)\right] = 0$$

$$(2 - 3\omega + \omega^2)(1-\omega) \qquad -2 + \omega = 0$$

$$\cancel{2} - 3\omega + \omega^2 - \cancel{2}\omega + 3\omega^2 - \omega^3 \quad \cancel{-2} + \cancel{\omega} = 0$$

$$-4\omega + 4\omega^2 - \omega^3 = 0 \qquad \text{CUBIC EQN}$$

$$\omega^3 - 4\omega^2 + 4\omega = 0$$

5

$$\omega(\omega^2 - 4\omega + 4) = 0 \qquad \text{"CE"}$$

$$\omega(\omega - 2)(\omega - 2) = 0$$

SO EV'S ARE 0, 2, 2

FIND THE $\vec{e}$'s

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a + c \\ 2b \\ a + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad b = 0$

$\Rightarrow \quad c = -a \qquad$ CHOOSE $a = +1,\ c = -1$

$$|\omega = 0\rangle = A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

NORMALIZATION CONSTANT

NORMALIZE

$$\langle \omega = 0 \mid \omega = 0 \rangle = 1$$

$$A^* \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^* \quad A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1$$

$$AA^* \quad (1 + 1) = 1$$

$$2 \, |A|^2 = 1$$

$$A = \frac{1}{\sqrt{2}}$$

"PHASE CONVENTION"

conventional to choose real positive value

$$|\omega = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

TO FIND THE OTHER TWO e$\vec{v}$'s

ASSOCIATED WITH $\omega = 2$ degeneracy => no unique pair of e$\vec{v}$'s

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a+c \\ 2b \\ a+c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

Solve this set of coupled algebraic equations

$$\left.\begin{aligned} a+c &= 2a \\ a+c &= 2c \end{aligned}\right\} \Rightarrow \begin{aligned} 2a &= 2c \\ a &= c \end{aligned}$$

choose $a = c = 1$

$$2b = 2b \quad \Big\} \Rightarrow b = \text{anything}$$

no unique solution because eigenvalues are degenerate

INFINITELY MANY $e\vec{v}$'s

$$\begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}$$

~~SO WE CAN~~ LET'S CHOOSE ONE ARBITRARILY $\Rightarrow$ $b = 0$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

THEN CONSTRUCT SECOND $\perp$ TO FIRST

$$\begin{pmatrix} a & b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

any linear combination is an $e\vec{v}$ with ev = 2

$a = 0$ [redacted]

$b =$ anything

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

VERY IMPORTANT STEP!!!

CHECK YOUR ANSWER

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \checkmark$$

$$\boxed{\text{WHEW!!!}}$$

Eigenvalue and Eigenvector Calculators

http://www.bluebit.gr/matrix-calculator/calculate.aspx

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm

http://wims.unice.fr/wims/wims.cgi?session=6S051ABAFA.2&+lang=en&+module=tool%2Flinear%2Fmatrix.en

Geometry of Eigenvalues and Eigenvectors

http://web.mit.edu/18.06/www/Demos/eigen-applet-all/eigen_sound_all.html

Fundamental Theorem of Algebra

http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

Every polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity

The Abel-Ruffini Theorem (1824)
aka, Abel's Impossibility Theorem

http://en.wikipedia.org/wiki/Abel-Ruffini_theorem

There is no general algebraic solution—that is, solution in radicals—to polynomial equations of degree five or higher

LECTURE 4 JUNE 26, 2007

YESTERDAY ev's AND $e\vec{v}$'s

$$\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$e\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad ev = 0$$

$$e\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad ev = 2$$

$$e\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad ev = 2$$

$\{ e\vec{v} \}$ ORTHONORMAL BASIS

MUTUALLY PERP

NORMALIZED

CAN MAKE ANY 3d VECTOR

TODAY : MACHINERY TO CHANGE BASIS

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$$\Omega \longrightarrow U^{+}\Omega U$$

$U =$ UNITARY OP BUILT OUT OF ORTHONORMAL BASIS OF IT'S $\vec{e}\vec{v}$'S

$$\begin{array}{ccc} \omega = 2 & \omega = 2 & \omega = 0 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{array}$$

$$U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

CONVENTION

HIGHEST . . . . . LOWEST

ev ev

WHY DOES THIS WORK?

THM: THE EV'S OF ANY HERMITEAN OPERATOR ARE REAL

$\Rightarrow$ ANY HERMITEAN OPERATOR HAS (AT LEAST ONE) SET OF ORTHONORMAL EV'S THAT ARE A BASIS

IN THIS BASIS, Ω IS DIAGONAL WITH ITS ω_i'S ON THE DIAGONAL

BUT HOW DOES IT WORK?

$\langle w | v \rangle$ inner product number $= |\langle w|| |v\rangle| \cos\theta$

CHANGE BASIS

$\langle w' | v' \rangle$ = SAME NUMBER

$$\langle w | v \rangle = \langle w | I | v \rangle$$
$$= \langle w | U U^{+} | v \rangle$$
$$= \langle w' | v' \rangle$$

$U U^{+} = I$ identity operator

CHANGE BASIS FOR OPERATORS

$$\langle w | \Omega | v \rangle = \langle w | I \, \Omega \, I | v \rangle$$

$$\langle w | \, U U^{+} \, \Omega \, U U^{+} | v \rangle$$

$$\left(\langle w | U \right) \left(U^{+} \Omega U \right) \left(U^{+} | v \rangle \right)$$

$$\langle w' | \quad \Omega' \quad | v' \rangle$$

$$U = \left(\begin{pmatrix} e^{\vec{v}_1} \end{pmatrix} \begin{pmatrix} e^{\vec{v}_2} \end{pmatrix} \begin{pmatrix} e^{\vec{v}_3} \end{pmatrix} \right)$$

$$U^{+} = (U^{T})^{*}$$

$$U^{+} = \begin{bmatrix} \left(e^{\vec{v}_1} \right)^{*} \\ \left(e^{\vec{v}_2} \right)^{*} \\ \left(e^{\vec{v}_3} \right)^{*} \end{bmatrix}$$

APPLY U^+ TO THE $e\vec{v}$'s OF Ω

$$\begin{bmatrix} \overset{*}{\overset{\frown}{e\vec{v}_1}} \\ \overset{*}{\overset{\frown}{e\vec{v}_2}} \\ \overset{*}{\overset{\frown}{e\vec{v}_3}} \end{bmatrix} \begin{pmatrix} \\ e\vec{v}_1 \\ \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \\ e\vec{v}_2 \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \\ e\vec{v}_3 \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

SO U^+ CHANGES THE KET BASIS

$\Rightarrow$ U " " BRA BASIS

$$U^+ \Omega U = \Omega$$

DOES IT WORK? CALCULATE $U^{+}\Lambda U$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 2\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

it works

SAY:

FIND ev's AND $\vec{ev}$'s OF H

DIAGONALIZE THE H

THM 10: FOR EVERY HERMITEAN OPERATOR, THERE EXISTS A BASIS OF ITS ORTHONORMAL EIGENVECTORS

x, p, H are Hermitian

$\Rightarrow$ unique expansion in x
p
E

ANY COMMON EIGENVECTORS?

THM 13: IF Ω and Λ are commuting Hermitian operators then there exists a common basis of eigenvectors that diagonalizes them both.

COMMUTATOR

$$[\Omega, \Lambda] = 0 = \Omega\Lambda - \Lambda\Omega$$

THREE CASES:

(1) COMPATIBLE: SHARE ALL $\vec{ev}$

(2) INCOMPATIBLE: SHARE NO $\vec{ev}$

(3) MIXED: SHARE SOME $\vec{ev}$ BUT NOT ALL

$$\Omega \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$ev = 1 \quad e\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad ev = 2 \quad e\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$ev = 3 \quad e\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Lambda \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$ev = 1 \quad e\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

many different

$ev = 2 \quad e\vec{v} =$

If they commute, then they will both be diagonal in some basis

If they are both diagonal in some basis, then they commute

$$\Omega\Lambda = \begin{pmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 3\cdot 2 \end{pmatrix}$$

$$\Lambda\Omega = \begin{pmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 2\cdot 3 \end{pmatrix}$$

$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = 0 \quad \Rightarrow$ THERE EXISTS A UNIQUE SET OF $e\vec{v}$'S THAT DIAGONALIZE BOTH

CSCO $\Rightarrow$ unique eigenbasis **| n, l, m >**

Operators that commute with the Hamiltonian are the quantum analogs to conserved quantities in classical mechanics

Lecture 3

GEOMETRY OF HILBERT SPACES

⇒ ALGEBRA OF HILBERT SPACES

DIRAC NOTATION

$$\langle \quad | \quad | \quad \rangle$$

BRA KET

BRACKET

	3d	HILBERT	
VECTORS	$\vec{v}$, $\vec{w}$ BOLD	$\lvert v \rangle$	TWO FLAVORS
		$\langle w \rvert$	no arrow not bold
ADDITION	$\vec{v} + \vec{w}$	$\lvert v \rangle + \lvert w \rangle$	
SUBTRACTION	$\vec{v} - \vec{w}$	$\lvert v \rangle - \lvert w \rangle$	
MULTIPLY BY A SCALER	$a\vec{v}$	$a \lvert v \rangle$	
DOT PRODUCT INNER PRODUCT	$\vec{v} \cdot \vec{w}$ real number	$\langle v \rvert w \rangle$ complex number	
OUTER PRODUCT		$\lvert v \rangle \langle w \rvert$ MATRIX OPERATOR	

DIRAC NOTATION

ABSTRACT VECTORS

(BASIS DEPENDENT REPRESENTATION)

$\vec{v}$ $\quad$ $\mathbb{V}$

$|v\rangle$

no arrow

no bold

$$\vec{v} = \sum_i \alpha_i \hat{e}_i$$

$$\vec{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix}$$

ADDITION

$\vec{v} + \vec{w}$

$|v\rangle + |w\rangle$

$$\vec{v} = \sum \alpha_i \hat{e}_i$$

$$\vec{w} = \sum \beta_i \hat{e}_i$$

$$\vec{v} + \vec{w} = \sum (\alpha_i + \beta_i) \hat{e}_i$$

$$\vec{v} + \vec{w} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \vdots \\ \alpha_m \beta_m \end{pmatrix}$$

INNER PRODUCT

$$\langle v | w \rangle = \alpha_1^* \beta_1 + \alpha_2^* \beta_2 + \cdots + \alpha_m^* \beta_m$$

$$= (\alpha_1^*, \alpha_2^*, \alpha_3^*, \ldots, \alpha_m^*) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}$$

DIRAC NOTATION

ASSOCIATE COLUMN VECTORS WITH $|v\rangle$

KET VECTORS

ASSOCIATE COMPLEX-CONJUGATED ROW VECTORS WITH $\langle v|$

BRA VECTORS

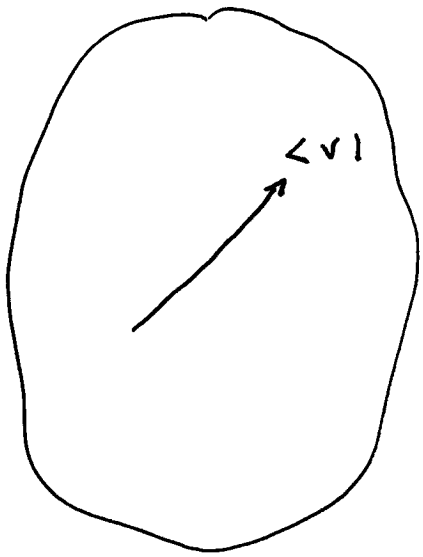
abstract vectors

KET SPACE $\quad |v\rangle \xrightarrow[\text{in a basis}]{} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix}$ **representation in a specific basis**

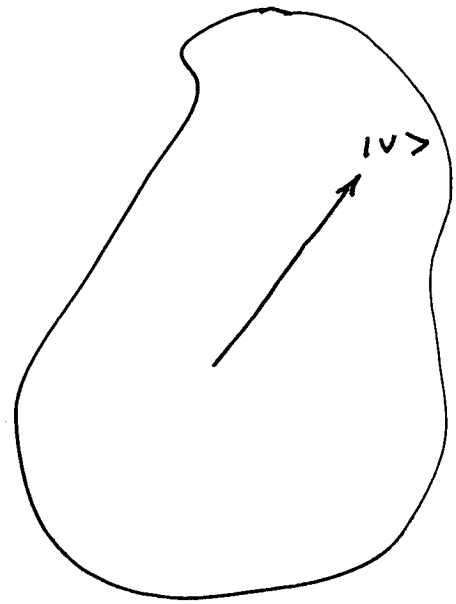
BRA SPACE $\quad \langle v| \xrightarrow[\text{in a basis}]{} (\alpha_1^*, \alpha_2^*, \alpha_3^*, \ldots, \alpha_m^*)$

PAIR OF
ISOMORPHIC DUAL SPACES

BRA SPACE



KET SPACE



THEY HAVE
EXACTLY THE SAME GEOMETRY, ALGEBRA, ...

DIRAC'S PUN BRACKET

$$\langle v | w \rangle$$

BRA - KET

OUTER PRODUCT

$|v\rangle\langle w|$ is an operator

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix} \begin{pmatrix} \beta_1^* & \beta_2^* & \beta_3^* & \cdots \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1^* & \alpha_1\beta_2^* & \alpha_1\beta_3^* & \cdots \\ \alpha_2\beta_1^* & \alpha_2\beta_2^* & \alpha_2\beta_3^* & \cdots \\ \vdots & & & \end{pmatrix}$$

ORDER MATTERS!

$$\underbrace{|a\rangle}_{\text{vector}} \underbrace{\langle b|c\rangle}_{\text{inner product = scaler}}$$

$$\underbrace{|a\rangle\langle b|}_{\text{outer product}} \underbrace{c\rangle}_{\text{vector}}$$

$\shortparallel$

~~vector~~

MATRIX

ORTHONORMAL BASIS VECTORS

$\hat{e}_i$

$\hat{x}\ \hat{y}\ \hat{z}$

$\hat{i}\ \hat{j}\ \hat{k}$

$\hat{r}\ \hat{\theta}\ \hat{\varphi}$

$$|e_i\rangle = |i\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ place}$$

$$\langle e_i| = \langle i| = (0\ 0\ 0\ \ldots\ 1\ \ldots\ 0\ 0\ 0\ \ldots\ 0)$$

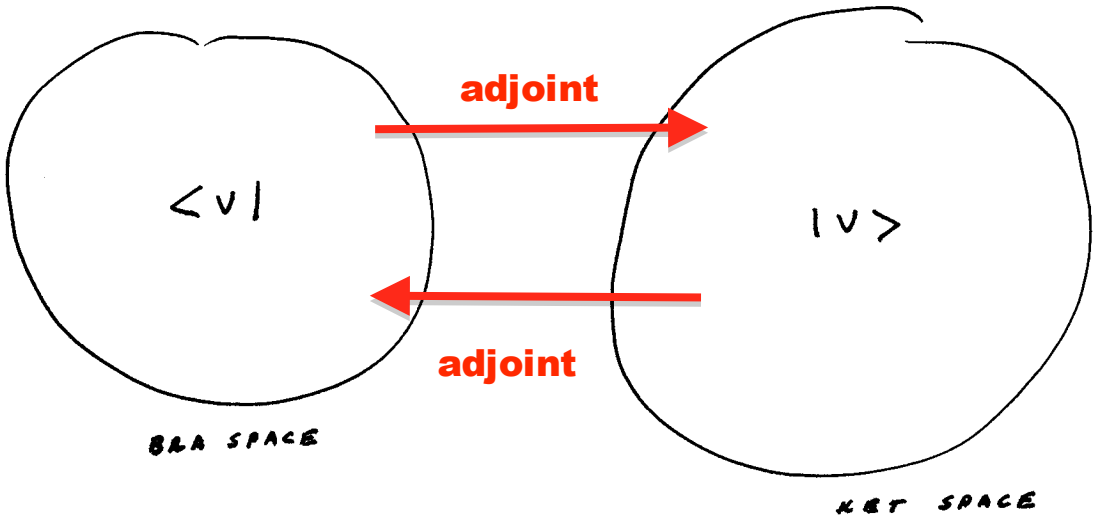
$\uparrow$ i^{th} place

EXPANSION IN AN ORTHONORMAL BASIS

$$|v\rangle = \sum v_i\, |i\rangle$$ **ket space**

$$\langle v| = \sum v_i^*\, \langle i|$$ **bra space**

WANT GEOMETRY EXACTLY THE SAME



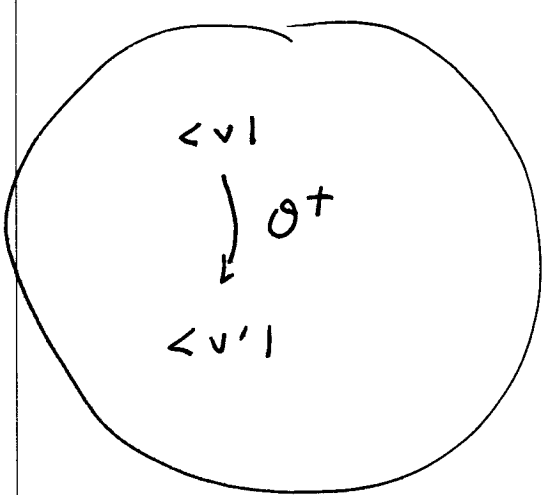
THIS DETERMINES WHAT HAPPENS TO

SCALARS $\quad \alpha = a + bi \xrightarrow{\text{COMPLEX CONJUGATION}} \alpha^* = a - bi$

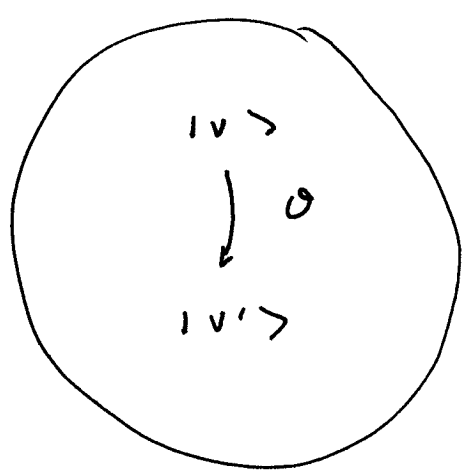
VECTORS $\quad |v\rangle \xrightarrow{\text{ADJOINT}} \langle v|$ $\quad$ TRANSPOSE CONJUGATE

$\langle v| \longrightarrow |v\rangle$

OPERATORS $\quad \mathcal{O} \xrightarrow{\text{ADJOINT}} \mathcal{O}^+$



BRA SPACE



KET SPACE

$\langle v'| = \langle \mathcal{O} v| = \langle v| \mathcal{O}^+$

left action

$|v'\rangle = |\mathcal{O} v\rangle = \mathcal{O}|v\rangle$

right action

ALGEBRA

$$|v\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \end{pmatrix}$$

is Ω Hermitean?

NO! $\Omega \neq \Omega^{+} = (\Omega^{T})^{*}$

$$\Omega\,|v\rangle = |w\rangle \qquad |w\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\langle v|\,\Omega = (1\ 0\ 0\ 0 \cdots) \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\langle v|\,\Omega \neq \langle w|$$

transpose & conjugate

$$\langle v|\,\Omega^{+} = (1\ 0\ 0\ 0 \cdots) \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \end{pmatrix} = (0\ 1\ 0\ 0 \cdots)$$

$$\langle v|\,\Omega^{+} = \langle w|$$

When $\Omega = \Omega^{+}$, left action and right action are the same!

Hermitean operators => left action = right action

ANALOGY BETWEEN OPERATORS AND COMPLEX NUMBERS

COMPLEX CONJUGATION

$$\alpha = a + bi$$

$$\alpha^* = a - bi$$

ADJOINT OPERATION

$$\Omega = H + A$$

$$\Omega^\dagger = H^\dagger + A^\dagger$$

$$= H - A$$

H IS HERMITEAN

A IS ANTI-HERMITEAN

REAL COMPONENT NUMBER $\quad \mathrm{Re}\,\alpha \qquad \mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \alpha^*)$

IMAGINARY COMPONENT NUMBER $\quad \mathrm{Im}(\alpha) = \frac{1}{2}(\alpha - \alpha^*)$

HERMITEAN OPERATOR $\quad H = H^\dagger \qquad H = \frac{1}{2}(\Omega + \Omega^\dagger)$

ANTI-HERMITEAN OPERATOR $\quad A = \frac{1}{2}(\Omega - \Omega^\dagger)$

$$A = -A^\dagger$$

COMPLEX NUMBER ON THE UNIT CIRCLE

$$\beta = e^{i\varphi}$$

$$|\beta| = 1$$

UNITARY OPERATOR

$$UU^\dagger = U^\dagger U = I$$

$$U = \begin{pmatrix} e^{i\varphi_1} & & \\ & e^{i\varphi_2} & \\ & & \ddots \end{pmatrix}$$

EXAMPLE

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

IS Ω HERMITEAN?

DOES $\Omega = \Omega^{+}$ YES

ev and $e\vec{v}$ OF Ω

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

NORMALIZE

$$N \begin{pmatrix} 1 & 1 \end{pmatrix}^{*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \qquad 2N = 1 \qquad N = \frac{1}{\sqrt{2}}$$

$$ev = 1 \qquad e\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$ev = -1 \qquad e\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

PROJECTION ONTO THE $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ SUBSPACE

$|i\rangle\langle i|$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}^* = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

ONTO THE $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ SUBSPACE

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}^* = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

complete basis if sum of the projection operators is the identity

COMPLETE $\Rightarrow$ $\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{I}$

identity

$$\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

complete basis

PROJECT $|i\rangle\langle i|A\rangle$ OUTER PRODUCT FIRST

$$\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a+b\\a+b\end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}(a+b)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a-b\\-a+b\end{pmatrix} \rightarrow \cancel{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}}$$

$$\frac{1}{\sqrt{2}}(a-b)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

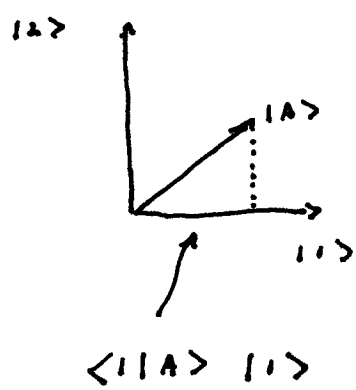
PROJECT $|i\rangle\langle i|A\rangle$ INNER PRODUCT FIRST

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}^{*}\begin{pmatrix}a\\b\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(a+b)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}^{*}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(a-b)$$



Eigenvalue and Eigenvector Calculators

http://www.bluebit.gr/matrix-calculator/calculate.aspx

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm

http://wims.unice.fr/wims/wims.cgi?session=6S051ABAFA.2&+lang=en&+module=tool%2Flinear%2Fmatrix.en

Geometry of Eigenvalues and Eigenvectors

http://web.mit.edu/18.06/www/Demos/eigen-applet-all/eigen_sound_all.html

Fundamental Theorem of Algebra

http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

Every polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity

The Abel-Ruffini Theorem (1824)
aka, Abel's Impossibility Theorem

http://en.wikipedia.org/wiki/Abel-Ruffini_theorem

There is no general algebraic solution—that is, solution in radicals—to polynomial equations of degree five or higher