February 29, 2012

Angular Momentum 2

(1) The Semiclassical Vector Model

(2) The Matrix Representation

(3) The Ladder Operators

The Semiclassical Vector Model

Classically, angular momentum is represented by a vector with a magnitude and a direction.

Classically, the angular momentum can have any value, and its z-projection can have any value.

The semiclassical vector model represents the quantum angular momentum with a vector in analogy with the classical description.

Quantum mechanically, the length of the vector and its z-projection are quantized.

It is not a perfect representation, but it is the best that anyone has concocted. Orbital angular momentum can be visualized using spherical harmonics, but spin angular momentum cannot. The semiclassical vector model represents both.

The Semiclassical Vector Model (1)

This is a useful semi-classical model of the quantum results.

Imagine L precesses around the z-axis. Hence the magnitude of **L** and the z-component L_z are constant while the x and y components can take a range of values and average to zero, just like the quantum eigenfunctions.

A given quantum number l determines the magnitude of the vector **L** via

$$L^2 = l(l+1)\hbar^2$$

$$|\boldsymbol{L}| = \sqrt{l(l+1)}\hbar$$

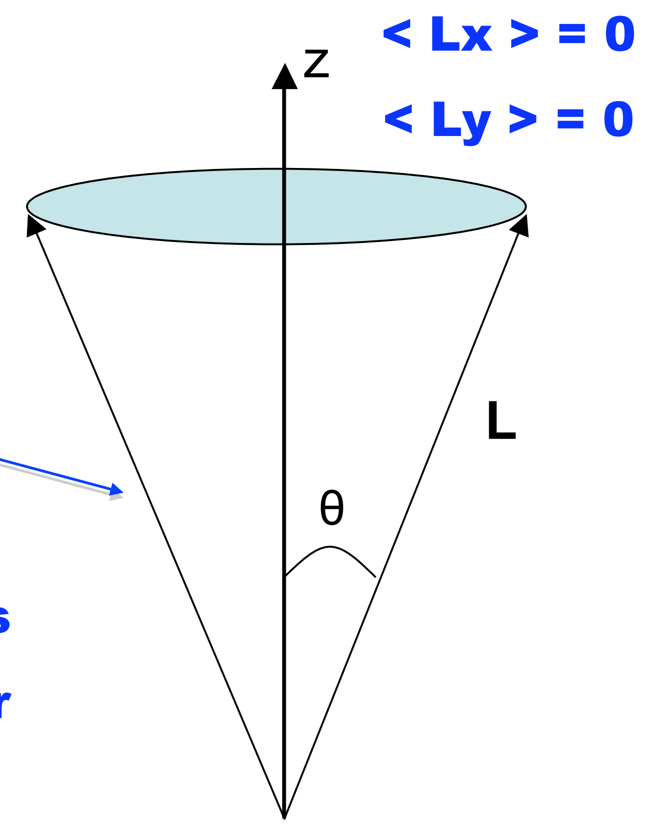
length of the vector

The z-component can have the 2l+1 values corresponding to

$$L_z = m\hbar, \quad -l \le m \le l$$

z projections of the vector

In the vector model this means that only particular special angles between the angular momentum vector and the z-axis are allowed



The Semiclassical Vector Model (2)

Example: l=2

largest z-projection = +2 hbar

Magnitude of the angular momentum is

$$L^2 = l(l+1)\hbar^2 = 6\hbar^2$$

$$\boxed{|\boldsymbol{L}| = \sqrt{l(l+1)}\hbar} = \sqrt{6}\hbar$$

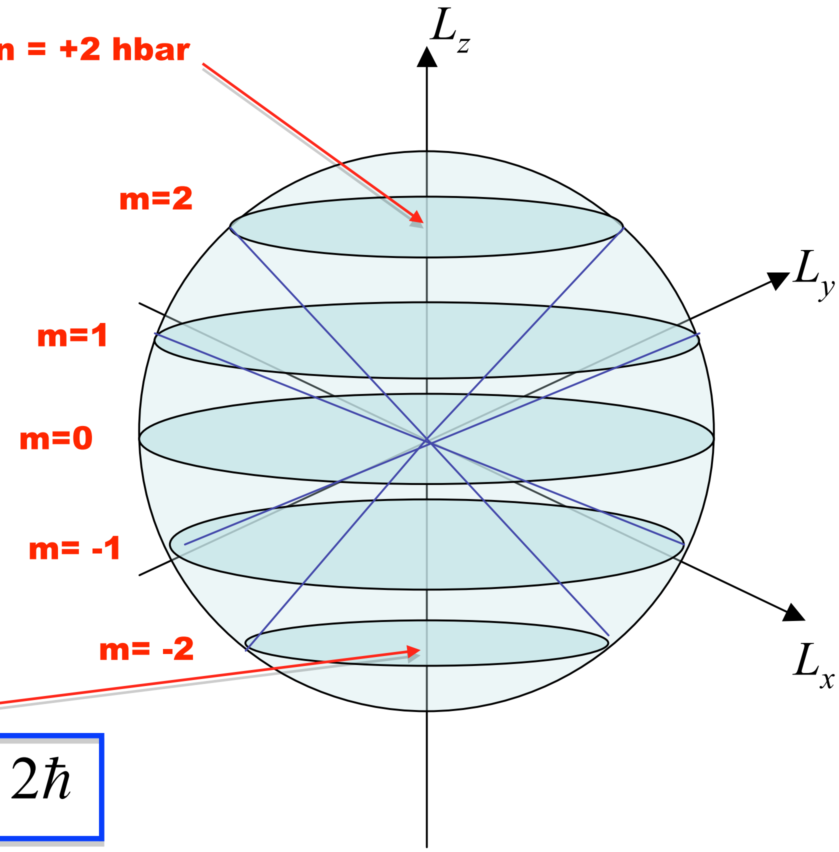
Diameter of the Sphere

Component of angular momentum
in *z*- direction can be

smallest z-projection = - 2 hbar

$$\boxed{-l \le m \le l \Rightarrow L_z = -2\hbar,\ -\hbar,\ 0,\ \hbar,\ 2\hbar}$$

z-projections go in integer steps



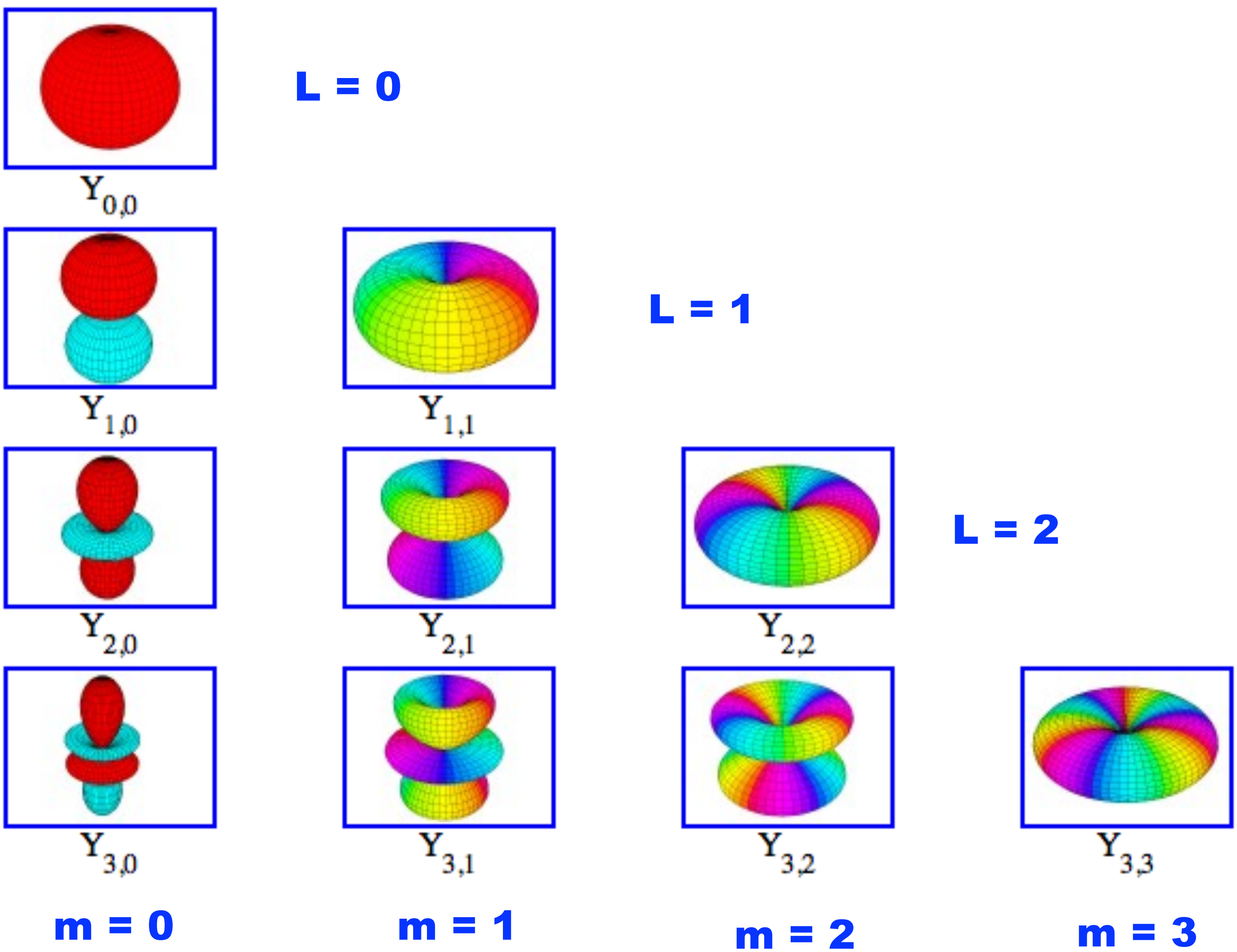
TOTAL = 2L + 1 = 2(2) + 1 = 5
FIVE CONES SHOWN ABOVE

Quantum eigenfunctions correspond to a cone of solutions for **L** in the vector model

Orbital Angular Momentum in Position Space

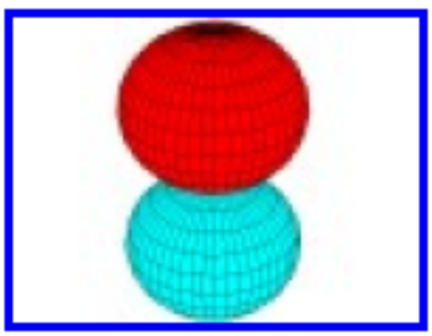
Represented by the Spherical Harmonics

Total angular momentum is an integer L = 0, 1, 2, 3, ...

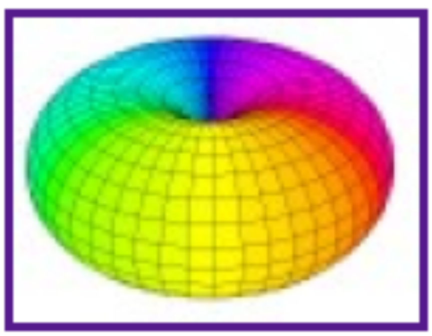


The z-projection is an integer starting at m = -L

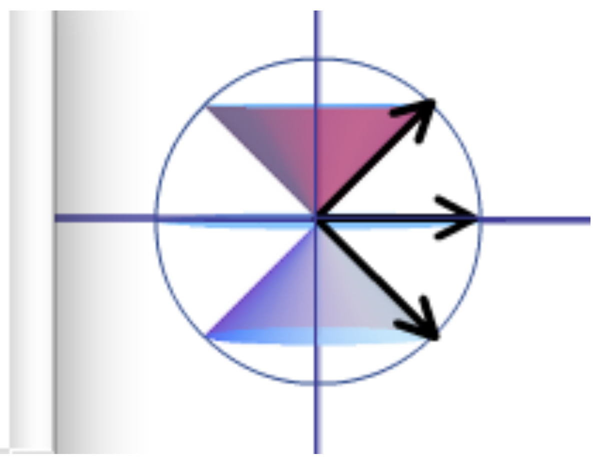
and increasing in integer steps up to m = +L

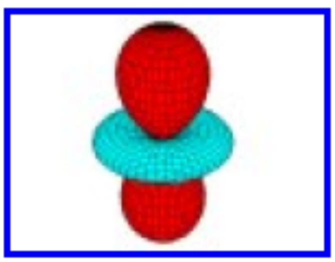


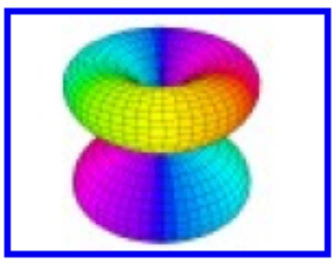
$Y_{1,0}$

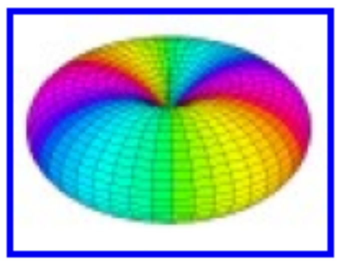


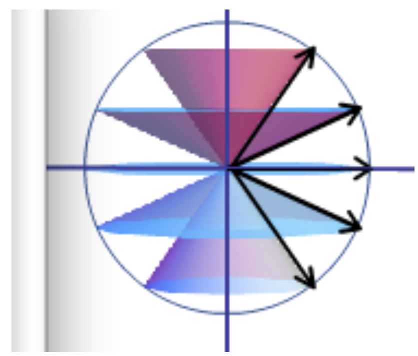
$Y_{1,1}$

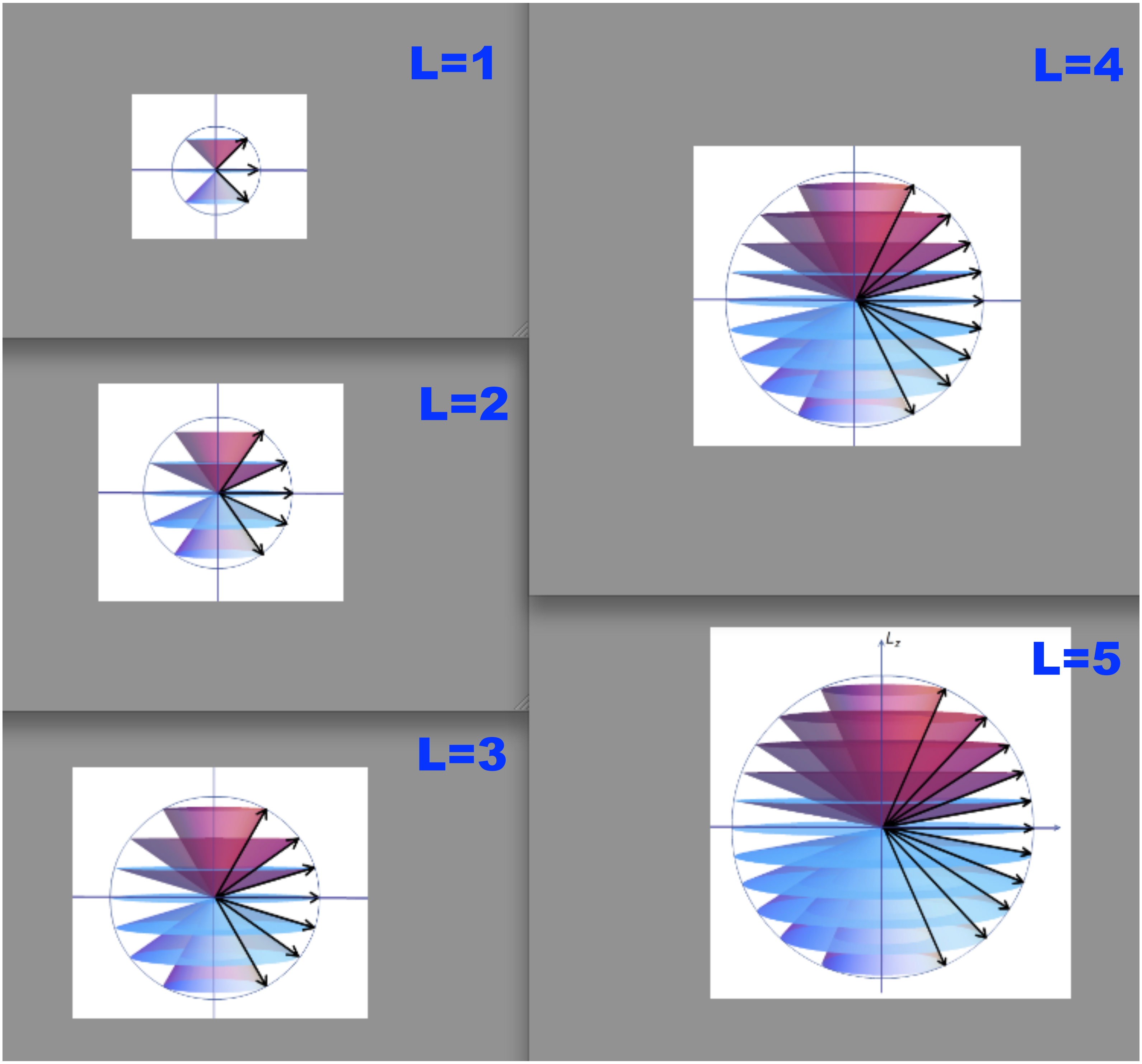



$Y_{2,0}$


$Y_{2,1}$

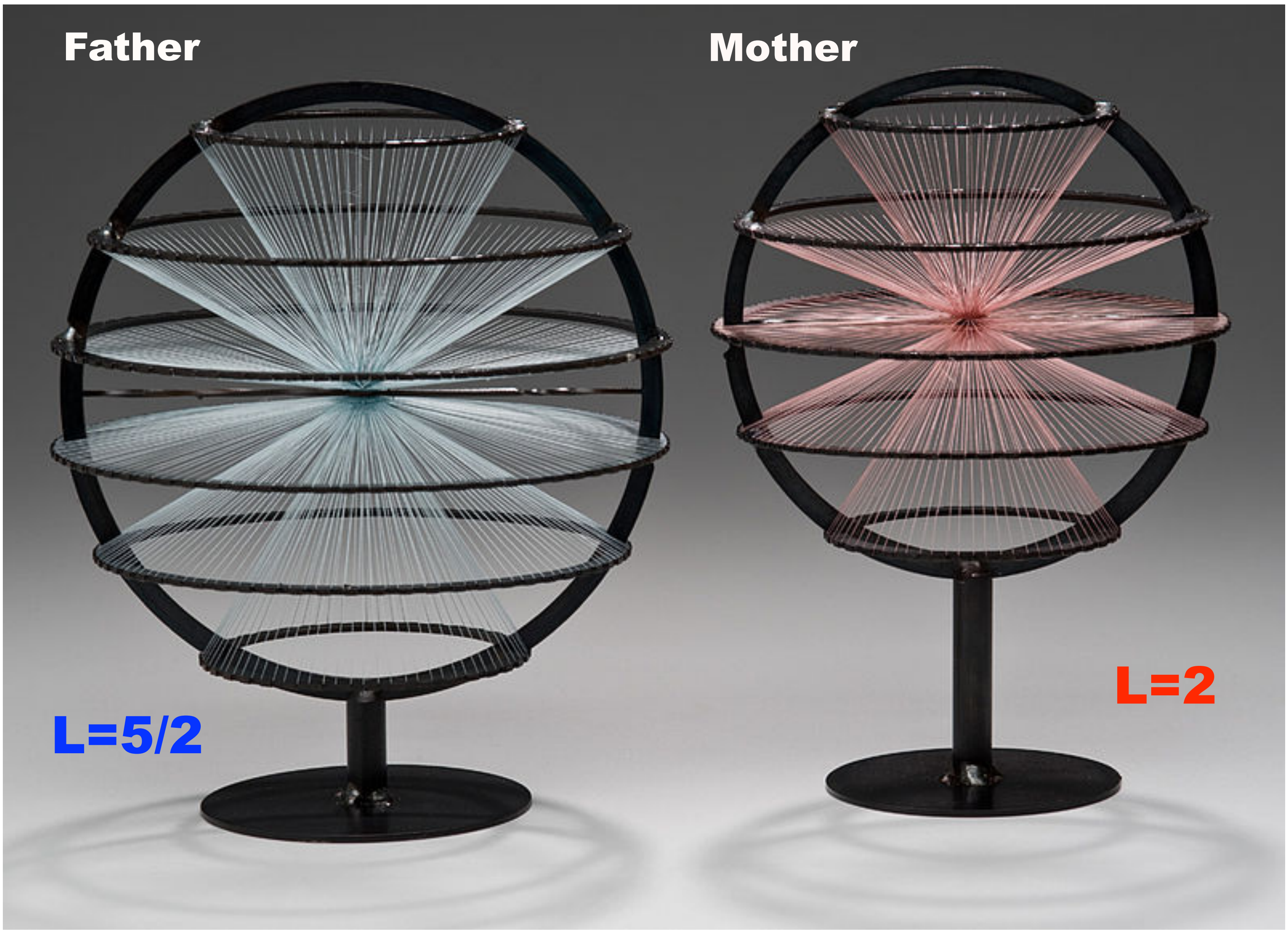

$Y_{2,2}$





http://www.st-andrews.ac.uk/~qmanim/embed_item_3.php?anim_id=47

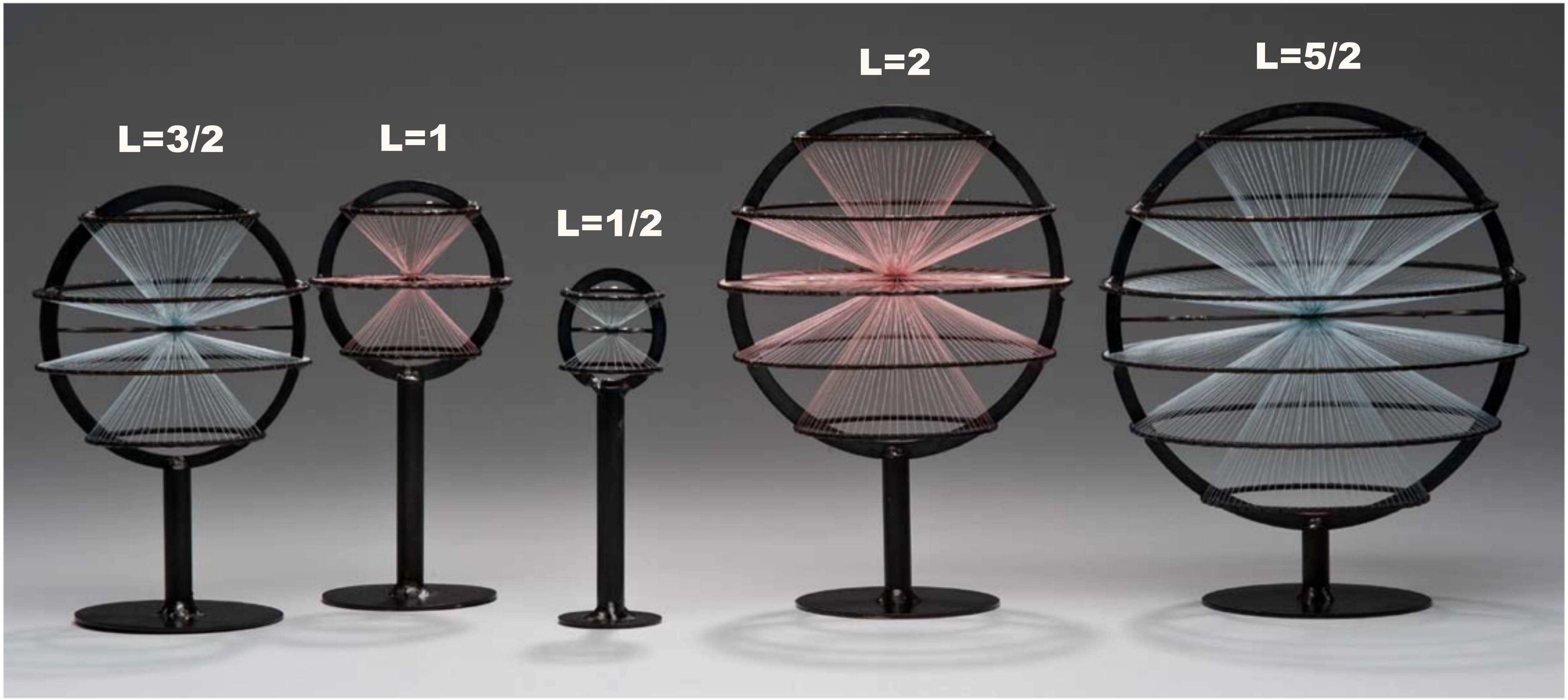
Father and Mother of the series Spin Family (2009)
by physicist-turned-sculptor Julian Voss-Andreae.



The two pictured objects illustrate the geometry of a spin 5/2 object (blue 'male' on the left) and spin 2 object (pink 'female' on the right).

Spin Family, on display in the art exhibition "Quantum Objects" playfully equates fermions with the male and bosons with the female gender, depicting the first spin 1/2, 1, 3/2, 2, and 5/2 objects as a family of five

The Angular Momentum Family

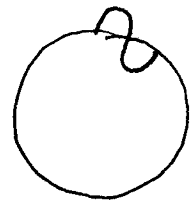


Why is orbital angular momentum quantized in integer steps?

WHY MUST ℓ BE AN INTEGER?

BOHR QUANTIZATION RULE:

$$L = n\hbar$$



Bohr said there must be an integer number of deBroglie wave around the circular orbit

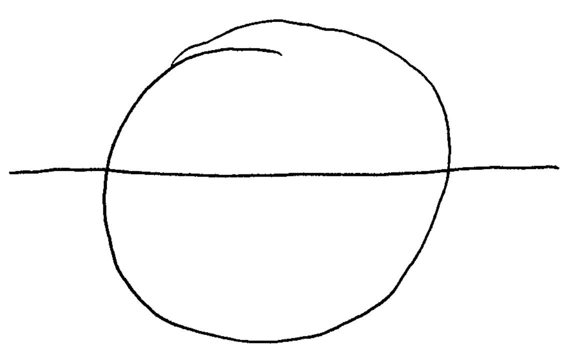
de Broglie

$$2\pi r = n\lambda = n\frac{h}{mv}$$

$$mvr = n\hbar$$

He incorrectly said that the angular momentum of the ground state was 1 (a p state) instead of zero (an s state) but his equation produced the correct energy levels

WHAT WE ARE DOING NOW IS 3d GENERALIZATION OF BOHR



The solutions to the Schrodinger equation have an integer number of waves along any path around the center of the circle because the wavefunction must be single valued.

So the soultions to the Schrodinger equation are the 3D generalization of Bohr's atom.

field (for example, circular motion or oscillatory motion about an equilibrium point), the wave returns to its former path after a certain number of wave lengths.

Fig. 15 shows this behavior diagrammatically for a circular motion. The waves which have gone around 0, 1, 2, · · · times overlap and will, in general, destroy one another by interference (dotted waves in Fig. 15). Only in the

Because the wavefunction is single valued and its derivatives must be continuous, there must be an integer number of waves around any path

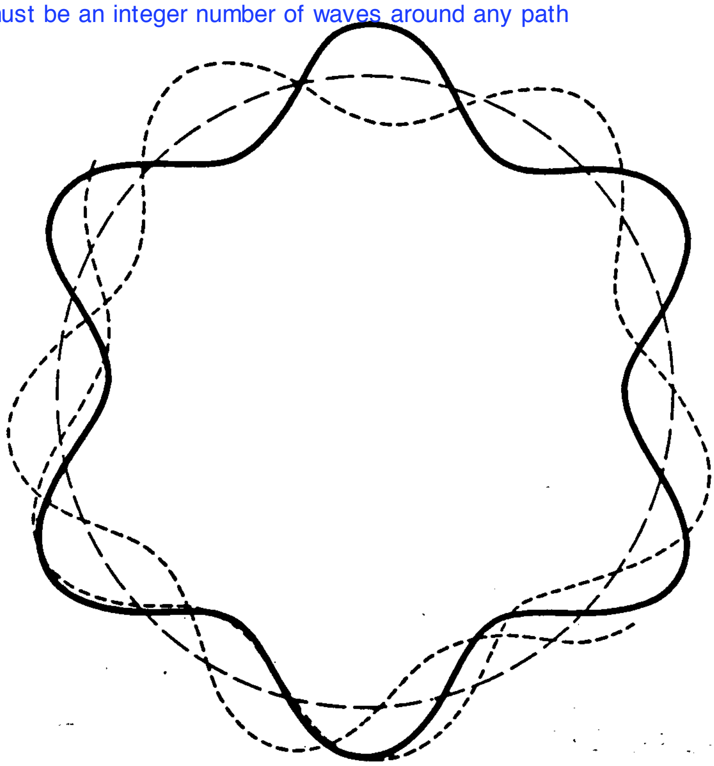


Fig. 15. De Broglie Waves for the Circular Orbits of an Electron about the Nucleus of an Atom (Qualitative). Solid line represents a stationary state (standing wave); dotted line, a quantum-theoretically impossible state (waves destroyed by interference).

special case where the frequency of the wave and, therefore, the energy of the corpuscle are such that an *integral* number of waves just circumscribe the circle (solid-line wave) do the waves which have gone around 0, 1, 2, · · · times reinforce one another so that a standing wave results. This standing wave has fixed *nodes*, and is analogous to the standing waves in a vibrating string which are possible only for certain definite frequencies, the fundamental frequency and its overtones (cf. Fig. 16). It follows, therefore, that *a stationary mode of vibration, together with a corresponding state of motion (orbit) of the corpuscle, is possible only for certain*

BOHR ATOM (CIRCULAR)

angular momentum is quantized with n=1, 2, 3, ...

$$L = m v r = n \hbar$$

the Coulomb attraction must provide the force
required to hold the electron in its circular orbit

$$\frac{e^2}{4\pi \epsilon_0 r^2} = m \frac{v^2}{r}$$

Bohr radius

solve for the radius of the n-th Bohr orbit

$$r_n = \left(\frac{4\pi \epsilon_0 \hbar^2}{m e^2} \right) n^2$$

the Bohr radius is
0.529 angstroms

$$r_n = n^2 a_0 \qquad a_0 = 0.529 \text{ Å}$$

$$E_n = \cancel{\frac{e^2}{8\pi \epsilon_0}}$$

the Rydberg

$$E_n = \boxed{-\frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi \epsilon_0} \right)^2} \frac{1}{n^2}$$

$$\boxed{E_n = \frac{-R_y}{n^2}} \qquad R_y = 13.6 \text{ eV}$$

Bohr's equation predicts the correct energy levels (one Rydberg over n squared)
and it also predicts the correct radius of the 1s state, which is the Bohr radius, but
it is not correct. It was however the beginning of quantum mechanics as we know it.

ORBITAL ANGULAR MOMENTUM

$\ell = 0, 1, 2, 3, \ldots$ **allowed values of l**

$m = -\ell, \ldots, 0, \ldots \ell$ **allowed values of m for each l**

Operator Equations

$$L^2 |\ell, m\rangle = \ell(\ell+1)\hbar^2 |\ell, m\rangle$$

$$L_z |\ell, m\rangle = m\hbar |\ell, m\rangle$$

$$L_\pm |\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\,\hbar\, |\ell, m\pm 1\rangle$$

SPECIAL CASES:

$\ell = 0 \Rightarrow m = 0$

$2\ell + 1 = 0$

SINGLET

$|0,0\rangle$ **L=0 Singlet**

$$L^2 |0,0\rangle = 0\hbar^2 |0,0\rangle$$

$$L_z |0,0\rangle = 0\hbar |0,0\rangle$$

$\ell = 1 \qquad m = -1, 0, +1$

$$\boxed{\begin{array}{c} 2\ell+1 = 3 \\ \text{TRIPLET} \end{array}}$$

$|1,1\rangle$

$|1,0\rangle$ **L=1 triplet**

$|1,-1\rangle$

$$L^2\,|1,m\rangle = 1(1+1)\hbar^2\,|1,m\rangle = 2\hbar^2\,|1,m\rangle$$

$$L_z\,|1,m\rangle = m\hbar\,|1,m\rangle$$

$\ell = 2 \qquad m = -2, -1, 0, +1, +2$

$$\boxed{\begin{array}{c} 2\ell+1 = 5 \\ \text{QUINTET} \end{array}}$$

$|2,2\rangle$

$|2,1\rangle$

$|2,0\rangle$ **L=5 quintet**

$|2,-1\rangle$

$|2,-2\rangle$

$$L^2\,|2,m\rangle = 6\hbar^2\,|2,m\rangle$$

$$L_z\,|2,m\rangle = m\hbar\,|2,m\rangle$$

L=1 Matrices

$\ell = 1$

L2

$$L^2 = \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

every vector is an $e\vec{v}$

Lz

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Lx

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \quad \frac{1}{\sqrt{2}}\,\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Ly

$$L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \quad \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2}\, i \\ -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2}\, i \\ -1 \end{pmatrix}$$

L+

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

L-

$$L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_0(x) \\ \psi_-(x) \end{pmatrix}$$

PROB AMP $S_z = 1$

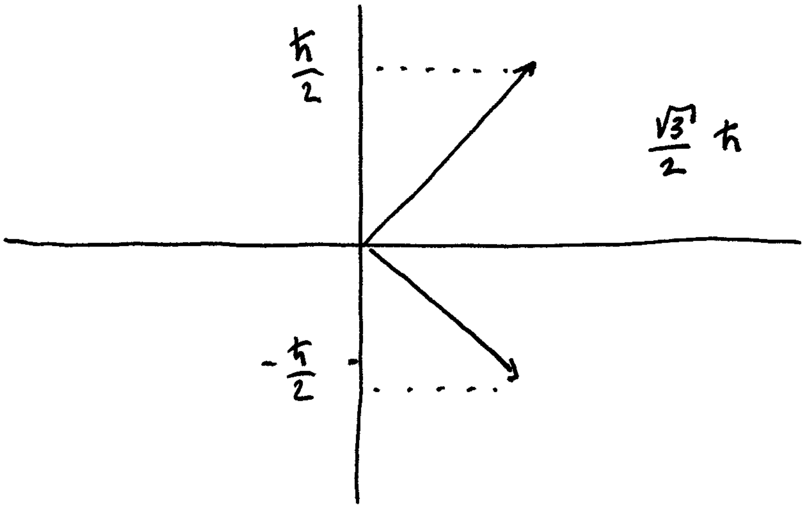
PROB AMP $S_z = 0$

PROB AMP $S_z = -1$

three component wavefunction

Spin 1/2

$2s+1 = 2$



$\langle s^2 \rangle = s(s+1)\hbar^2$ LENGTH VECTOR $= \frac{\sqrt{3}}{2}\hbar$

$\langle s_z \rangle = m\hbar$

$\langle s_x \rangle = 0$

$\langle s_y \rangle = 0$

For spin 1/2

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}$$

PROB AMP SPIN UP

PROB AMP SPIN DOWN

2 COMPONENT OBJECT IS CALLED A SPINOR

two component wavefunction

~~for spin~~

For spin 0

$$\Psi(x) = \left(\psi(x) \right)$$

one component wavefunction

Matrices for L=2

```
Jz[2] // MatrixForm
```

$$
\begin{array}{cc}
\text{Lz} & \begin{pmatrix} 2\hbar & 0 & 0 & 0 & 0 \\ 0 & \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar & 0 \\ 0 & 0 & 0 & 0 & -2\hbar \end{pmatrix}
\end{array}
$$

```
Jsq[2] // MatrixForm
```

$$
\begin{array}{cc}
\text{L2} & \begin{pmatrix} 6\hbar^2 & 0 & 0 & 0 & 0 \\ 0 & 6\hbar^2 & 0 & 0 & 0 \\ 0 & 0 & 6\hbar^2 & 0 & 0 \\ 0 & 0 & 0 & 6\hbar^2 & 0 \\ 0 & 0 & 0 & 0 & 6\hbar^2 \end{pmatrix}
\end{array}
$$

2L+1 = 6

- **J = 2**

```
Jx[2] // MatrixForm
```

Lx

$$\begin{pmatrix} 0 & \hbar & 0 & 0 & 0 \\ \hbar & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & \hbar \\ 0 & 0 & 0 & \hbar & 0 \end{pmatrix}$$

```
Jy[2] // MatrixForm
```

Ly

$$\begin{pmatrix} 0 & -i\,\hbar & 0 & 0 & 0 \\ i\,\hbar & 0 & -i\sqrt{\frac{3}{2}}\,\hbar & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}}\,\hbar & 0 & -i\sqrt{\frac{3}{2}}\,\hbar & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}}\,\hbar & 0 & -i\,\hbar \\ 0 & 0 & 0 & i\,\hbar & 0 \end{pmatrix}$$

WE ACTUALLY SOLVED A MORE GENERAL PROBLEM!

USED THE ALGEBRA

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

We solved every problem with this algebra !!!

SPIN ANGULAR MOMENTUM TOTAL ANGULAR MOMENTUM	} HAVE THE SAME ALGEBRA

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

$$L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$J^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$$

$$S^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$

L $0, 1, 2, 3$ integers **orbital must be integers**

J, S $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ **spin can be integer or half integer**

J = L + S so integer or half integer

CANNOT EXPRESS SPIN IN POSITION SPACE

IT LIVES IN SPIN SPACE

$$\vec{J} = \vec{L} + \vec{S}$$

CAN REPRESENT ORBITAL PART IN POSITION SPACE

$s = \frac{1}{2}$ $m_s = -\frac{1}{2}, \frac{1}{2}$

$\boxed{2s+1 = 2}$ DOUBLET

$|\frac{1}{2}, \frac{1}{2}\rangle$

s=1/2 doublet

$|\frac{1}{2}, -\frac{1}{2}\rangle$

$s = \frac{3}{2}$ $m_s = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$\boxed{2s+1 = 4}$ QUARTET

$|\frac{3}{2}, \frac{3}{2}\rangle$

$|\frac{3}{2}, \frac{1}{2}\rangle$

s=3/2 quartet

$|\frac{3}{2}, -\frac{1}{2}\rangle$

$|\frac{3}{2}, -\frac{3}{2}\rangle$

s=1/2 matrices

PAULI MATRICES

Sx

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Sy

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Sz

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

COMMON ev's $\quad + \frac{\hbar}{2}, - \frac{\hbar}{2}$

$$- m\hbar \qquad + m\hbar$$

S2

$$S^2 = s(s+1)\,\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

People did not like the idea of spin 1/2:

Sommerfeld turned the problem of the anomalous Zeeman effect over to young Heisenberg, a student in his seminar. Heisenberg, struggling with it, introduced half-integer quantum numbers into the old quantum theory for the first time. Sommerfeld was shocked, and urged Heisenberg not to publish. "If we know one thing about quantum numbers, we know they are integers!" Bohr, too, expressed displeasure. Pauli warned that today we allow half-integers, tomorrow it's quarter-integers, then eighths, sixteenths, and before you know it the quantum conditions have eroded away.

Paul Ehrenfest told Uhlenbeck and Goudsmit

"This is a good idea. Your idea may be wrong, but since both of you are so young without any reputation, you would not loose anything by making a stupid mistake."

Born told Stern

It took me quite a time before I took this idea seriously. I thought always that [space] quantization was a kind of symbolic expression for something which you don't understand. But to take this literally like Stern did, this was his own idea. . . . I tried to persuade Stern that there was no sense [in it], but then he told me that it was worth a try.

Poor Counsel from on High

In 1925 the young American Ralph Kronig derived a promising formula for "spin," hitherto undiscovered. Wolfgang Pauli, an imposing authority indeed, flatly told Kronig that his idea was impossible. Heisenberg was also sceptical. Kronig naturally didn't publish.[141]

Later that year in Leiden, two young Dutch physicists also thought of "spin." George Uhlenbeck and Samuel Goudsmit took their discovery to their professor, Paul Ehrenfest (one of Einstein's closest friends), who liked it. He suggested they show a copy of the paper to the grand old man of Dutch physics, H. A. Lorentz (Nobel 1902). Lorentz said the idea was impossible. Ehrenfest had apparently spotted the objections that Lorentz would raise, but had already sent the paper off anyway for publication. Thus Uhlenbeck and Goudsmit got full credit for discovering spin. "There is certainly no doubt," Uhlenbeck said, "that Kronig anticipated what was the main part of our ideas."[142] But Uhlenbeck and Goudsmit never won a Nobel Prize. Was it because the Nobel committee knew of Kronig's earlier idea? If so, why not honor all three? Goudsmit used to say that everyone always assumed he and Uhlenbeck had won a Nobel Prize.[143] Ironically, spin helped clinch confirmation of Pauli's exclusion principle of 1925, for which Pauli won the Nobel Prize in 1945.[144]

Reactions to the Stern–Gerlach Experiment

The following quotes from James Franck, Niels Bohr, and Wolfgang Pauli are among the messages that Walther Gerlach received in immediate response to postcards (like the one shown in figure 4) he had sent;[10] the quote from Arnold Sommerfeld appeared in the 1922 edition of his classic book;[17] that from Albert Einstein is in a March 1922 letter to Born;[18] that from I. I. Rabi is from reference 8, page 119. (See also Rabi's obituary for Otto Stern in PHYSICS TODAY, October 1969, page 103.)

Through their clever experimental arrangement Stern and Gerlach not only demonstrated *ad oculos* [for the eyes] the space quantization of atoms in a magnetic field, but they also proved the quantum origin of electricity and its connection with atomic structure.

—***Arnold Sommerfeld*** **(1868–1951)**

The most interesting achievement at this point is the experiment of Stern and Gerlach. The alignment of the atoms without collisions via radiative [exchange] is not comprehensible based on the current [theoretical] methods; it should take more than 100 years for the atoms to align. I have done a little calculation about this with [Paul] Ehrenfest. [Heinrich] Rubens considers the experimental result to be absolutely certain.

—***Albert Einstein*** **(1879–1955)**

More important is whether this proves the existence of space quantization. Please add a few words of explanation to your puzzle, such as what's really going on.

—***James Franck*** **(1882–1951)**

I would be very grateful if you or Stern could let me know, in a few lines, whether you interpret your experimental results in this way that the atoms are oriented only parallel or opposed, but not normal to the field, as one could provide theoretical reasons for the latter assertion.

—***Niels Bohr*** **(1885–1962)**

This should convert even the nonbeliever Stern.

—***Wolfgang Pauli*** **(1900–58)**

As a beginning graduate student back in 1923, I . . . hoped with ingenuity and inventiveness I could find ways to fit the atomic phenomena into some kind of mechanical system. . . . My hope to [do that] died when I read about the Stern–Gerlach experiment. . . . The results were astounding, although they were hinted at by quantum theory. . . . This convinced me once and for all that an ingenious classical mechanism was out and that we had to face the fact that the quantum phenomena required a completely new orientation.

—***Isidor I. Rabi*** **(1898–1988)**

spin 3/2 matrices

```
Jz[3 / 2] // MatrixForm
```

$$S_z \quad \begin{pmatrix} \frac{3\hbar}{2} & 0 & 0 & 0 \\ 0 & \frac{\hbar}{2} & 0 & 0 \\ 0 & 0 & -\frac{\hbar}{2} & 0 \\ 0 & 0 & 0 & -\frac{3\hbar}{2} \end{pmatrix}$$

```
Jsq[3 / 2] // MatrixForm
```

$$S2 \quad \begin{pmatrix} \frac{15\hbar^2}{4} & 0 & 0 & 0 \\ 0 & \frac{15\hbar^2}{4} & 0 & 0 \\ 0 & 0 & \frac{15\hbar^2}{4} & 0 \\ 0 & 0 & 0 & \frac{15\hbar^2}{4} \end{pmatrix}$$

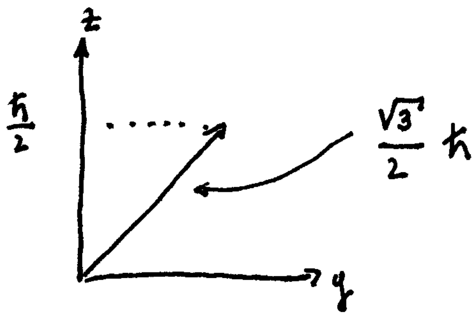
```
Jx[3 / 2] // MatrixForm
```

$$
\begin{pmatrix}
0 & \frac{\sqrt{3}\ \hbar}{2} & 0 & 0 \\
\frac{\sqrt{3}\ \hbar}{2} & 0 & \hbar & 0 \\
0 & \hbar & 0 & \frac{\sqrt{3}\ \hbar}{2} \\
0 & 0 & \frac{\sqrt{3}\ \hbar}{2} & 0
\end{pmatrix}
\quad \mathbf{Sx}
$$

```
Jy[3 / 2] // MatrixForm
```

$$
\mathbf{Sy} \quad
\begin{pmatrix}
0 & -\frac{1}{2}\, i \sqrt{3}\ \hbar & 0 & 0 \\
\frac{1}{2}\, i \sqrt{3}\ \hbar & 0 & -i\,\hbar & 0 \\
0 & i\,\hbar & 0 & -\frac{1}{2}\, i \sqrt{3}\ \hbar \\
0 & 0 & \frac{1}{2}\, i \sqrt{3}\ \hbar & 0
\end{pmatrix}
$$

Expectation Values



$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **in the Sz=+1/2 state**

$$\langle S_z \rangle = (1 \quad 0)^* \; \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle S_z \rangle = \frac{\hbar}{2} = m\hbar$$

$$\langle S_x \rangle = (1 \quad 0)^* \; \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle S_x \rangle = 0$$

$$\langle S_y \rangle = (1 \quad 0)^* \; \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle S_y \rangle = 0$$

MEASURE S_x	1/2	TIME	$+\frac{\hbar}{2}$
	1/2	TIME	$-\frac{\hbar}{2}$

MEASURE S_y	1/2	TIME	$+\frac{\hbar}{2}$
	1/2	TIME	$-\frac{\hbar}{2}$

Old Exam Problem

2. Consider a system initially in the state $\mid \psi(0) >$ with the Hamiltonian H, where

$\mid \psi(0) > = N \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the L_z basis, and where $H = (2\omega/\hbar)L^2 + (3\omega)L_z$.

(a) What angular momentum is described by the 3-component vector $\mid \psi(0) >$? What is the length of this vector? What are the allowed z-projections?

(b) Calculate the normalization constant N and the Hamiltonian matrix.

Hint: $L^2 \mid l, m > = l(l+1)\hbar^2 \mid l, m >$ and $L_z \mid l, m > = m\hbar \mid l, m >$.

(c) Calculate the eigenvalues and the eigenvectors of the Hamiltonian.

Hint: H, L^2 and L_z all commute.

(d) Calculate the time evolution of the state vector $\mid \psi(t) >$ by expanding $\mid \psi(0) >$ as a sum of energy eigenvectors and using the time evolution of the energy eigenvectors.

(e) If the energy is measured at time t, what results can be found and with what probabilities will these results be found?

(f) Calculate $< E >$ and ΔE. Plot $P(E)$ *vs.* E and indicate $< E >$ and ΔE on your plot.

(g) If L^2 and L_z are measured at time t, what results can be found and with what probabilities will these results be found?

FOR $\ell = 1$

$$H = A L^2 + B L_z$$

$$L^2 \longrightarrow \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$L_z \longrightarrow \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$H \longrightarrow \begin{pmatrix} 2A\hbar^2 + B\hbar & 0 & 0 \\ 0 & 2A\hbar^2 & 0 \\ 0 & 0 & 2A\hbar^2 - B\hbar \end{pmatrix}$$

GENERALIZED UNCERTAINTY RELATION

FOR ANY OBSERVABLES A and B

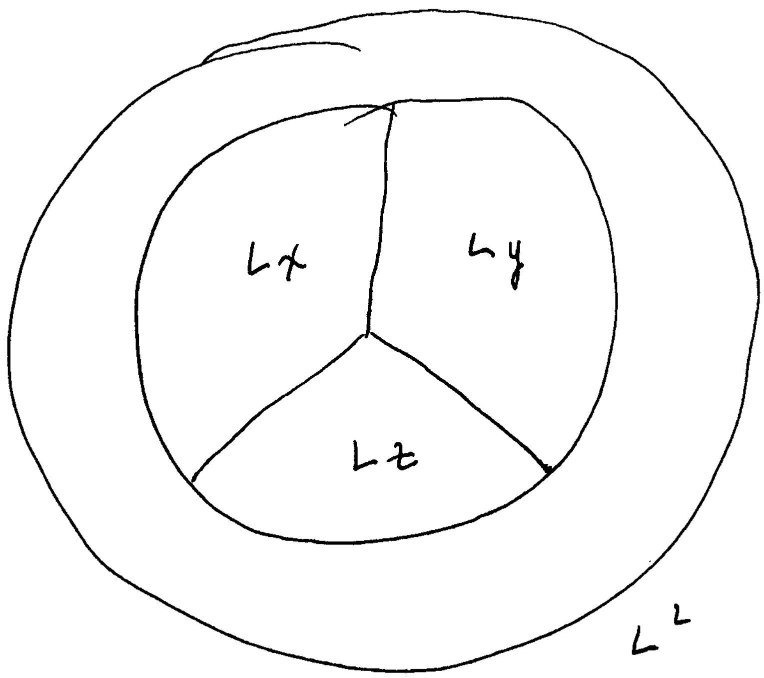
$$(\Delta A)^2 (\Delta B)^2 \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|^2$$

FOR L_x AND L_y

$$(\Delta L_x)^2 (\Delta L_y)^2 \geq \left| \frac{1}{2i} \langle i\hbar L_z \rangle \right|^2$$

$$\geq \frac{\hbar^2}{4} |\langle L_z \rangle|^2$$

$$\Delta L_x \, \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle|$$


Lx
Ly
Lz
L^2

EIGENVALUE PROBLEM

$$L^2 | \ell, m \rangle = \ell(\ell+1)\hbar^2 | \ell, m \rangle$$

$$L_z | \ell, m \rangle = m\hbar | \ell, m \rangle$$

IF H IS ROTATIONALLY INVARIANT

$$[H, L_i] = 0$$

$$[H, L^2] = 0$$

$$[H, L_z] = 0$$

(H eigenfcns) = (radial eigenfcn)(spherical harmonic)

LADDER OPERATOR SOLUTION

$$L^2 = \vec{L} \cdot \vec{L} = L_x^2 + L_y^2 + L_z^2$$

$$\boxed{[L_i, L_j] = i\hbar \, \epsilon_{ijk} L_k}$$

commutation relation

sum over repeated indices

EINSTEIN SUMMATION OVER K.

ϵ_{ijk} CALLED TOTALLY ANTISYMMETRIC TENSOR

http://en.wikipedia.org/wiki/Levi-Civita_symbol

1 2 3	2 1 3	any two equal
2 3 1	1 3 2	
3 1 2	3 2 1	
$\epsilon_{ijk} = +1$	$\epsilon_{ijk} = -1$	$\epsilon_{ijk} = 0$
cyclic permutations	**non-cyclic permutations**	**any 2 indices equal**

1 2 3

x y z

ANOTHER VIEW : $\epsilon_{123} = +1$

changes sign when interchange two indices

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

$$\boxed{\vec{L} \times \vec{L} = i\hbar \vec{L}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix}$$

CROSS PRODUCTS

$$\vec{c} = \vec{a} \times \vec{b}$$

$$c_i = \epsilon_{ijk} a_j b_k$$

EINSTEIN SUM OVER j, k.

SINCE L_x, L_y, L_z DO NOT COMMUTE...

CHOOSE L^2, L_z

RAISING AND LOWERING OPERATORS

$$L_+ = L_x + i L_y$$

$$L_- = L_x - i L_y$$

WORK OUT COMMUTATORS

$$[L^2, L_i] = 0$$

$$[L^2, L_\pm] = 0$$

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

GOAL

$$\boxed{\begin{aligned} L^2 |\ell, m\rangle &= \ell(\ell+1)\hbar^2 |\ell, m\rangle \\ L_z |\ell, m\rangle &= m\hbar |\ell, m\rangle \\ L_\pm |\ell, m\rangle &= \sqrt{\ell(\ell+1) - m(m\pm 1)}\,\hbar\, |\ell, m\pm 1\rangle \end{aligned}}$$

COMMUTATORS

$$[L^2, L_z] = 0 \qquad L^2 L_z - L_z L^2 = 0$$

$$[L^2, L_\pm] = 0 \qquad L^2 L_+ - L_+ L^2 = 0$$

$$[L_z, L_\pm] = \pm\hbar L_\pm \qquad L_z L_+ - L_+ L_z = \hbar L_+$$

don't know

know

don't know

ASSUME

$$\boxed{\begin{aligned} L^2 |\alpha\,\beta\rangle &= \alpha |\alpha\,\beta\rangle \\ L_z |\alpha\,\beta\rangle &= \beta |\alpha\,\beta\rangle \end{aligned}}$$

Like the SHO

[a , H] = a

aH - Ha = a

know

don't know

$$[L_z, L_+] = \hbar L_+$$

$$L_z L_+ - L_+ L_z = \hbar L_+$$

$$L_z L_+ |\alpha \beta\rangle = L_+ L_z |\alpha \beta\rangle + \hbar L_+ |\alpha \beta\rangle$$

$$= \beta L_+ |\alpha \beta\rangle + \hbar L_+ |\alpha \beta\rangle$$

$$= (\beta + \hbar) L_+ |\alpha \beta\rangle$$

$$L_z \left(L_+ |\alpha \beta\rangle \right) = (\beta + \hbar) \left(L_+ |\alpha \beta\rangle \right)$$

$L_+ |\alpha \beta\rangle$ IS AN $e\vec{v}$ OF L_z WITH $ev = \beta + \hbar$

L_- " " " " " " " $ev = \beta - \hbar$

STEP SIZE = β

L+ goes up the ladder

L- goes down the ladder

$$[L^2, L_+] = 0$$

$$L^2 L_+ - L_+ L^2 = 0$$

$$L^2 L_+ |\alpha, \beta\rangle = L_+ L^2 |\alpha, \beta\rangle$$

$$= L_+ (\alpha \; |\alpha, \beta\rangle)$$

$$L^2 (L_+ |\alpha, \beta\rangle) = \alpha (L_+ |\alpha, \beta\rangle)$$

$$\boxed{\begin{array}{l} (L_+ \,|\alpha, \beta\rangle) \text{ is an } e\vec{v} \text{ of } L^2 \text{ with } ev = \alpha \\ (L_- \,|\alpha, \beta\rangle) \;\text{" " " " " "}\; ev = \alpha \end{array}}$$

THIS LADDER HAS A TOP AND A BOTTOM

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 - L_z^2 = L_x^2 + L_y^2$$

$$\langle \alpha, \beta | (L^2 - L_z^2) | \alpha, \beta \rangle = \langle \alpha, \beta | (L_x^2 + L_y^2) | \alpha, \beta \rangle$$

$$\langle \alpha, \beta | (\alpha - \beta^2) | \alpha, \beta \rangle$$

$$(\alpha - \beta^2) \langle \alpha, \beta | \alpha, \beta \rangle$$

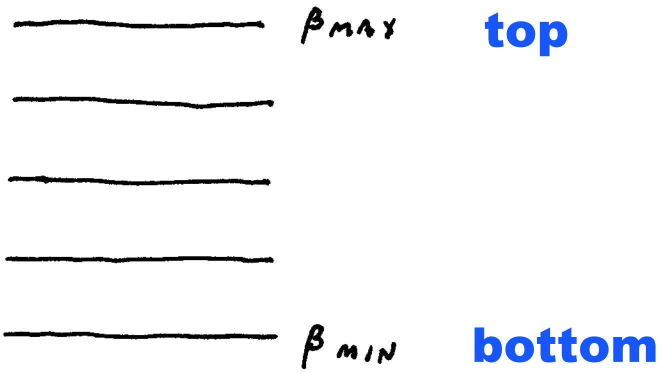
$$(\alpha - \beta^2) = \langle \alpha, \beta | (L_x^2 + L_y^2) | \alpha, \beta \rangle \geq 0$$

$$\beta^2 \leq \alpha$$

PHYSICALLY: GIVEN TOTAL ANGULAR MOMENTUM SQUARED α, ITS z PROJECTION β CANNOT BE GREATER THAN α

$\boxed{\text{TOP AND BOTTOM TO LADDER} \quad L_+ |\alpha, \beta_{MAX}\rangle = 0 \quad L_- |\alpha, \beta_{MIN}\rangle = 0}$

NEXT STEP: SHOW $\beta_{MAX} = -\beta_{MIN}$



STEP SIZE $= \hbar$

$$\beta_{MAX} - \beta_{MIN} = 2\beta_{MAX} = (\text{integer})\,\hbar$$

$$\beta_{MAX} = \frac{1}{2}\hbar\,(\text{integer}) \qquad \text{integer} = 0, 1, 2, \ldots$$

$$\alpha = (\beta_{MAX})(\beta_{MAX} + \hbar) = \hbar^2 \left(\frac{k}{2}\right)\left(\frac{k}{2} + 1\right)$$

$$\boxed{\alpha = \ell(\ell+1)\hbar^2}$$

eigenvalue of the L squared operator

WE ACTUALLY SOLVED A MORE GENERAL PROBLEM!

USED THE ALGEBRA

$$[L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k$$

We solved every problem with this algebra !!!

SPIN ANGULAR MOMENTUM
TOTAL ANGULAR MOMENTUM } HAVE THE SAME ALGEBRA

$$[S_i, S_j] = i\hbar\,\epsilon_{ijk} S_k$$

$$[J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k$$

$$L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$J^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$$

$$S^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$

L $0, 1, 2, 3$ integers orbital must be integers

J, S $0, 1/2, 1, 3/2, 2, \ldots$ spin can be integer or half integer

J = L + S so integer or half integer

CANNOT EXPRESS SPIN IN POSITION SPACE

IT LIVES IN SPIN SPACE

$$\vec{J} = \vec{L} + \vec{S}$$

CAN REPRESENT ORBITAL PART IN POSITION SPACE

NEXT STEP ... go into real space

$$|\ell, m\rangle$$

$$\langle \theta, \varphi | \ell, m\rangle = Y_{\ell m}(\theta, \varphi)$$

$$\langle \ell, m | \theta, \varphi\rangle = Y^*_{\ell m}(\theta, \varphi)$$

orthonormal basis

$$\boxed{\langle \ell', m' | \ell, m\rangle = \delta_{\ell \ell'}\, \delta_{m m'}}$$

$$\int Y^*_{\ell' m'}(\theta, \varphi)\, Y_{\ell m}(\theta, \varphi)\, d(\cos\theta)\, d\varphi = \delta_{\ell\ell'}\delta_{mm'}$$

Q: HOW TO FIND ALL THE $Y_{\ell m}$'s ?

A: Solve the differential equation ... OR

Two ways to find the spherical harmonics

JUST LIKE SHO, WE HAVE TWO CHOICES

$$L^2 \, |l, m\rangle = l(l+1)\hbar^2 \, |l, m\rangle$$

$$L_z \, |l, m\rangle = m\hbar \, |l, m\rangle$$

(1) solve differential equation

$$-\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{lm}(\theta, \varphi) =$$

$$l(l+1)\hbar^2 \, Y_{lm}(\theta, \varphi)$$

$$-i\hbar \frac{\partial}{\partial \varphi} Y_{lm}(\theta, \varphi) = m\hbar \, Y_{lm}(\theta, \varphi)$$

OR

(2) use ladder operators as generating function

$$L_+ Y_{ll}(\theta, \varphi) = 0 \quad \Rightarrow \quad \begin{array}{l} Y_{ll} \\ \downarrow L_- \\ Y_{l,l-1} \\ \downarrow L_- \end{array}$$

$$\text{or} \quad L_- Y_{l-l}(\theta, \varphi) = 0 \quad \Rightarrow \quad \begin{array}{l} L_{l-l} \\ \downarrow L_+ \end{array}$$

separate the phi dependence

IMPORTANT TECHNIQUE # 137

SEPARATION OF VARIABLES

$$Y_{\ell m}(\theta,\varphi) = f_{\ell m}(\theta)\, g_{\ell m}(\varphi)$$

$$L_z |\ell, m\rangle = m\hbar\, |\ell, m\rangle$$

$$-i\hbar \frac{\partial}{\partial\varphi}\Big(f_{\ell m}(\theta)\, g_{\ell m}(\varphi)\Big) = m\hbar \Big(f_{\ell m}(\theta)\, g_{\ell m}(\varphi)\Big)$$

$$-i\hbar\, f_{\cancel{\ell m}}(\theta) \frac{\partial g_{\ell m}(\varphi)}{\partial\varphi} = m\hbar\, f_{\cancel{\ell m}}(\theta)\, g_{\ell m}(\varphi)$$

$$\boxed{\frac{\partial g}{\partial\varphi} = i m g \quad \Rightarrow \quad g = e^{i m \varphi}}$$

does not depend on L

So it separates ~~and~~

and it does not depend on ℓ

$\boxed{\text{next step find } F_{\ell m}(\theta)\text{'s}}$

$$L_+ \, Y_{\ell\ell}(\theta, \varphi) = 0$$

will yield a simple differential equation

$$\cancel{\hbar e^{i\varphi}} \left[\frac{\partial}{\partial \theta} + i \cot\theta \frac{d}{d\varphi} \right] \left[F_{\ell\ell}(\theta) \, e^{i\ell\varphi} \right] = 0$$

$$\frac{d F_{\ell\ell}}{d\theta} \cancel{e^{i\ell\varphi}} + i \cot\theta \, F_{\ell\ell} \, i\ell \, \cancel{e^{i\ell\varphi}} = 0$$

The simple differential equation

$$\left[\frac{d}{d\theta} - \ell \cot\theta \right] F_{\ell\ell}(\theta) = 0$$

Use the Freshman Physics Method

TRY $\quad F_{\ell\ell} = A(\sin\theta)^\ell$

$$\frac{d}{d\theta} A(\sin\theta)^\ell = A(\sin\theta)^{\ell-1} \cos\theta$$

$$A(\sin\theta)^{\ell-1}\cos\theta - \ell \frac{\cos\theta}{\sin\theta} A(\sin\theta)^\ell \stackrel{?}{=} 0$$

yep!

$$\Rightarrow \boxed{Y_{\ell\ell}(\theta, \varphi) = A_\ell (\sin\theta)^\ell \, e^{i\ell\varphi}}$$

TO GET THE REST

Apply L- to get the rest

$$L_- \, Y_{\ell\ell} = a \, Y_{\ell\,\ell-1}$$

then normalize using

$$\int Y_{\ell m}^* \, Y_{\ell m} \, d\Omega = 1$$

$$L_- = \hbar \, e^{-i\varphi} \left(\frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right)$$

SPHERICAL HARMONICS

$\ell = 0$

$$\boxed{Y_{00}(\theta, \varphi) = \langle \theta, \varphi | 0, 0 \rangle = \frac{1}{\sqrt{4\pi}}}$$

$$\int \left(\frac{1}{\sqrt{4\pi}} \right)^* \frac{1}{\sqrt{4\pi}} \, d\Omega = 1 \quad \checkmark$$

$\ell = 1$

$$Y_{11}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{i\varphi}$$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

note symmetry

$$Y_{1\,-1}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{-i\varphi}$$

$\ell = 2$

$$Y_{2\,\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} (\sin\theta)^2 \; e^{\pm 2i\varphi}$$

$$Y_{1\,\pm 1}(\theta, \varphi) = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \; e^{\pm i\varphi}$$

$$Y_{1\,0}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$