February 22, 2012

Finish the harmonic oscillator

The beauty and power of the ladder operators

Start angular momentum

Qualitative introduction

4.4 The Harmonic Oscillator Revisited—Momentum Eigenfunctions

In Chapter 3 we solved the TISE for the bound state eigenfunctions and energy eigenvalues for the harmonic oscillator potential $U(x) = \frac{1}{2}m\omega^2 x^2$. In view of our discussion in this chapter, however, we should clarify the nature of wave functions that we obtained by noting that they are coordinate space wave functions. That is, in the context of this chapter, the wave functions that we obtained in Chapter 3 were $\psi(x)$'s rather than $\phi(p)$'s.

The harmonic oscillator is a very special problem in quantum physics. One reason for its uniqueness is that the coordinate x occurs in the TISE to exactly the same power as does the momentum p. Inserting the potential energy function into Equation 2.26 the TISE may be written

$$\left[\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2\right]\psi(x) = E\psi(x) \tag{4.45}$$

Because we are working in one-dimension we have again dropped the subscript x that denotes the component of the momentum. Examination of Equation 4.45 together with the (near) symmetric operator relations

$$\hat{p} \to \frac{\hbar}{i}\frac{d}{dx} \quad \text{and} \quad x \to -\frac{\hbar}{i}\frac{d}{dp} \tag{4.46}$$

makes it clear that the momentum and coordinate energy eigenfunctions must have the same general form. It is not necessary to actually solve the TISE in momentum space to obtain the set of $\phi_n(p)$ since we have already done the work. All we must do is replace x by its operator equivalent in p-space and put the resulting equation in the form of the differential equation that led us to the coordinate space eigenfunctions:

position space wave functions

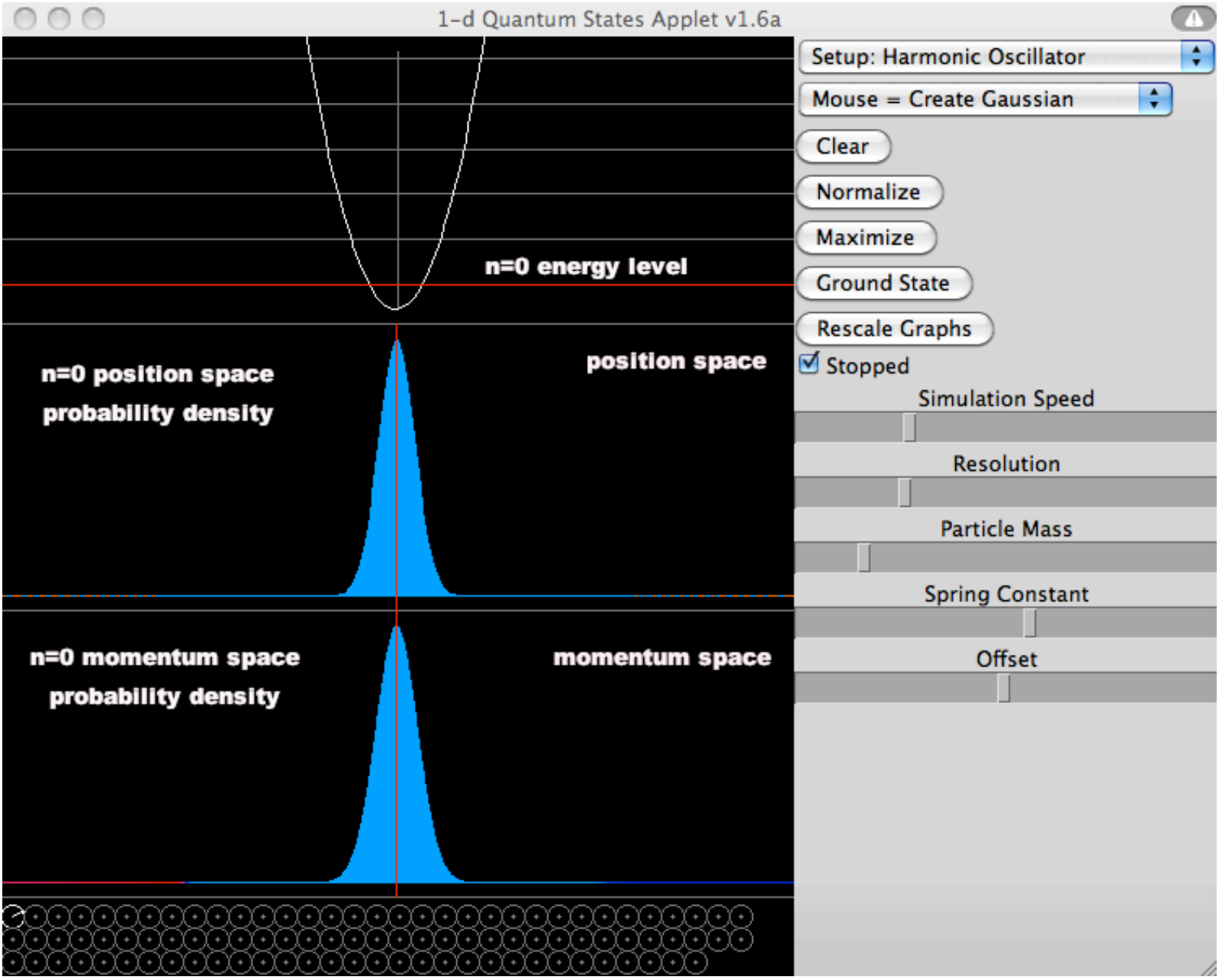
$$\boxed{\psi_n(x) = \sqrt{\frac{1}{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/(2\hbar)}} \tag{4.47}$$

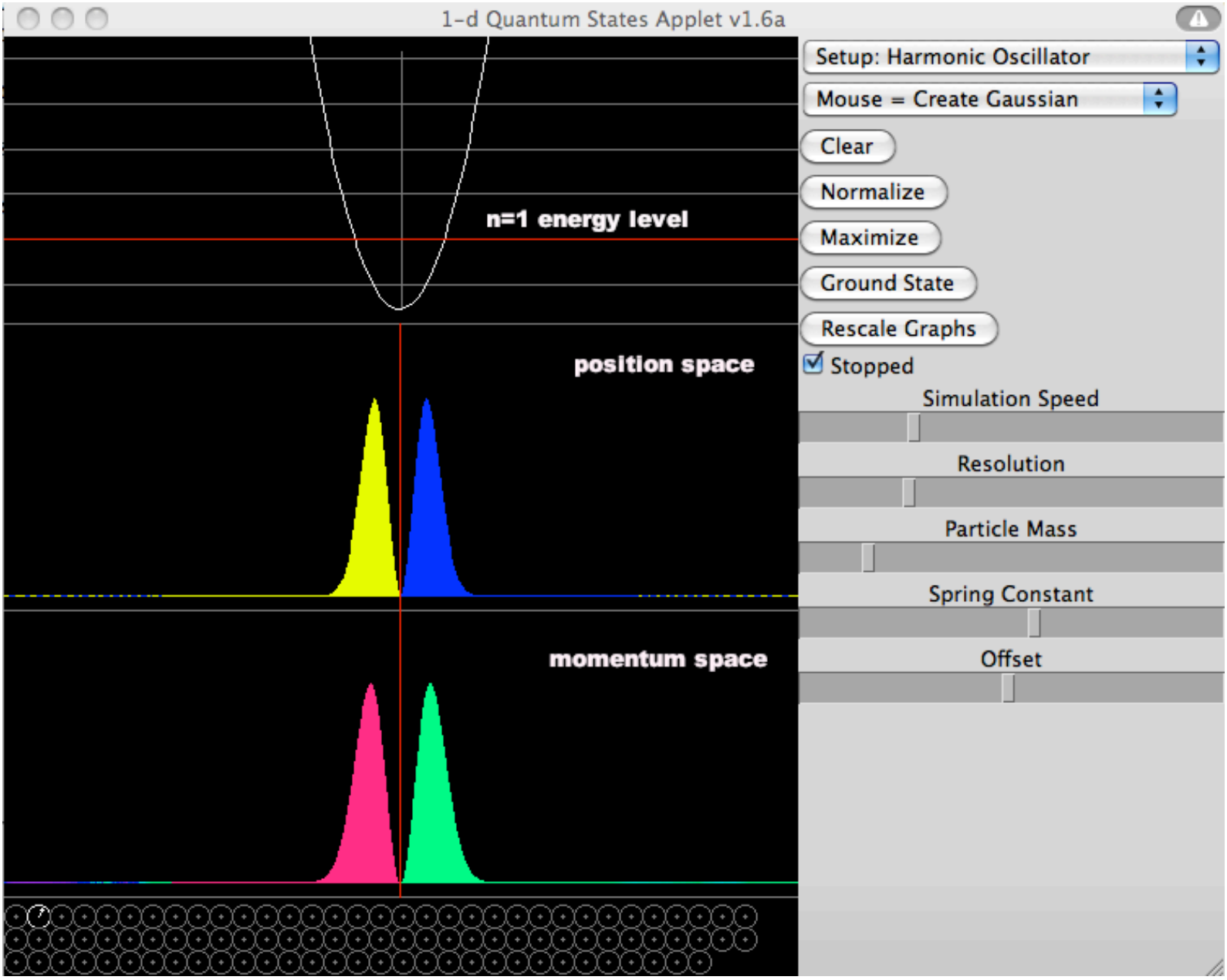
We then find the analogous constants and simply write the solutions (see Problem 11). The harmonic oscillator momentum eigenfunctions are found to be

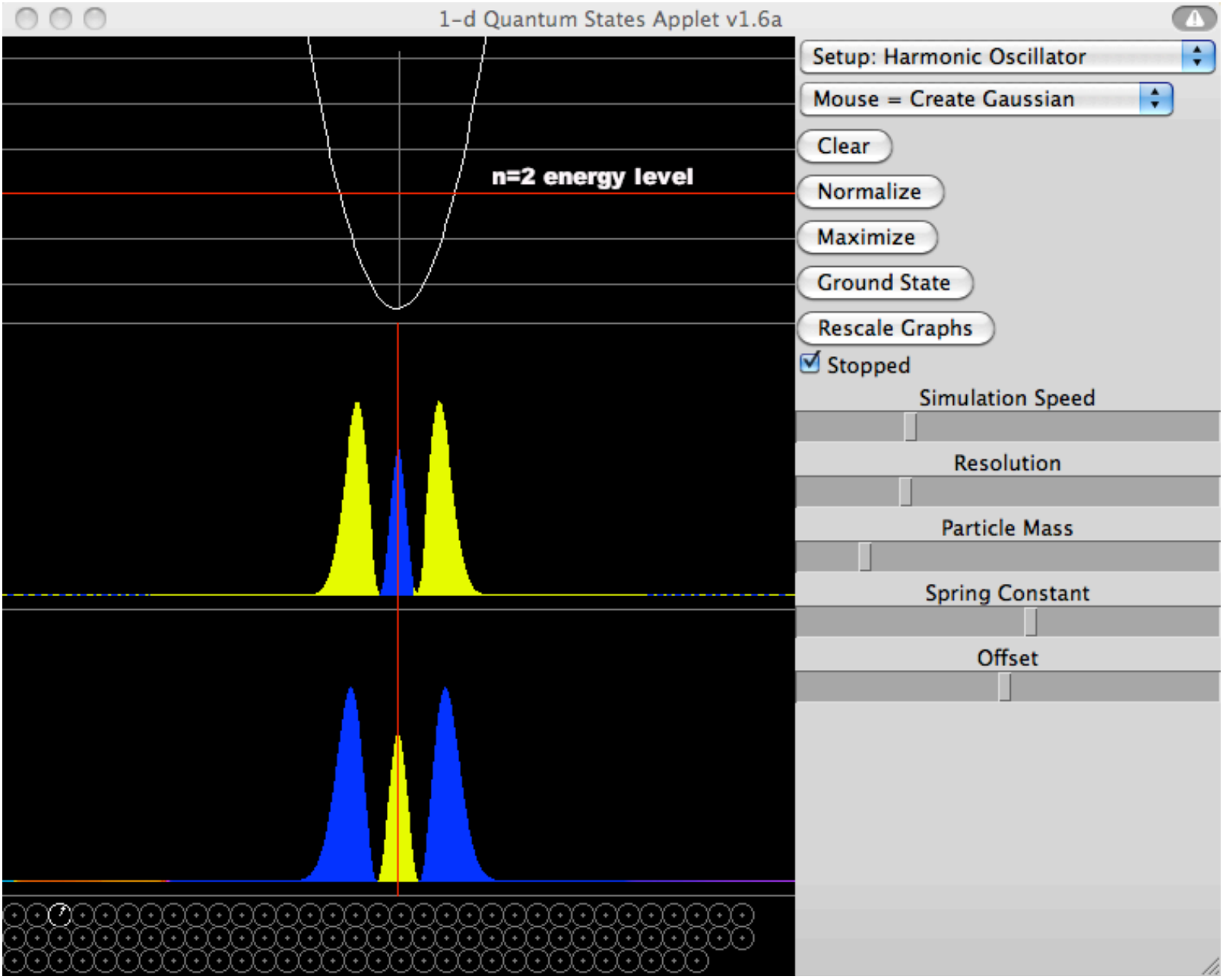
momentum space wave functions

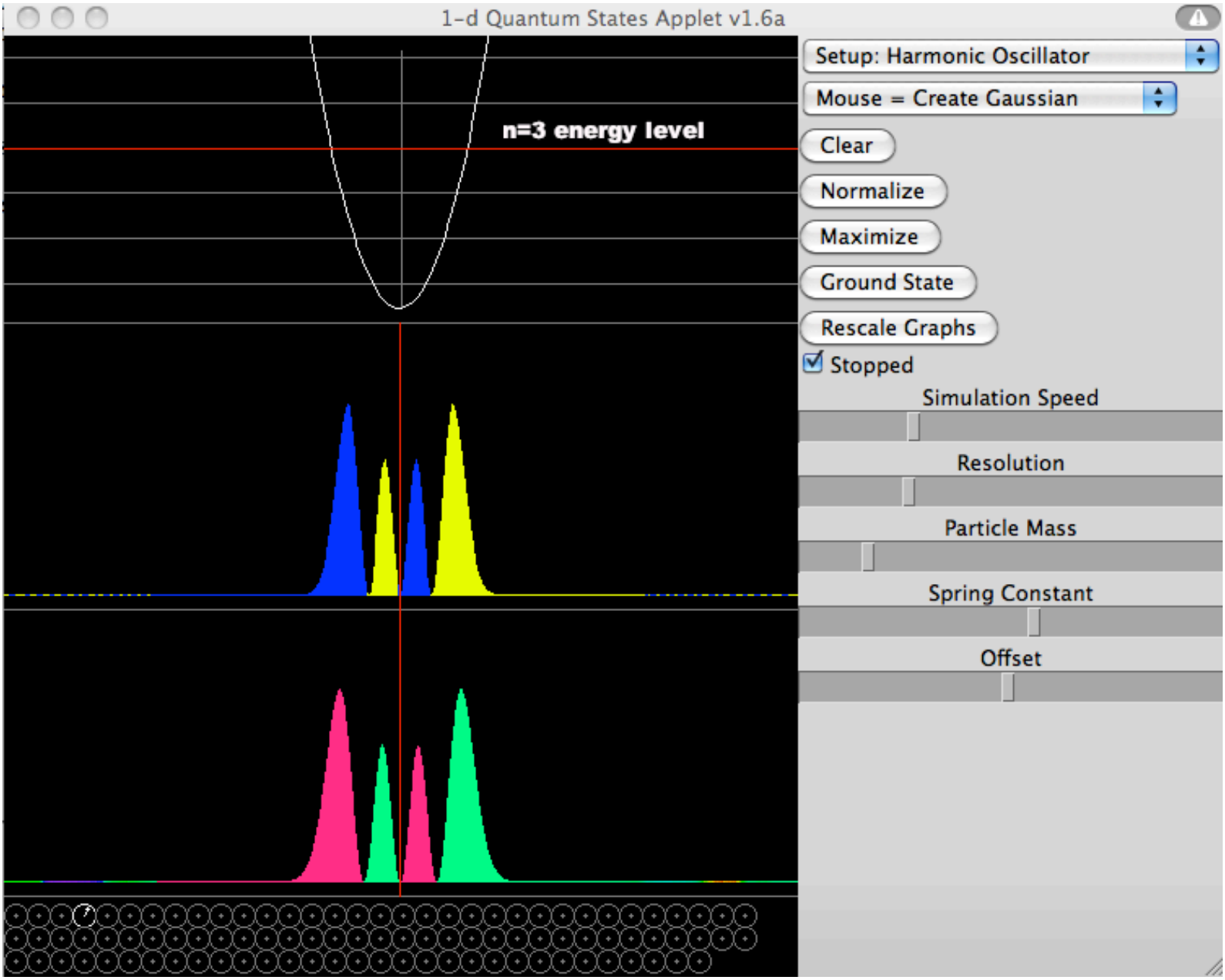
$$\boxed{\phi_n(p) = \sqrt{\frac{1}{2^n n!}}\left(\frac{1}{\pi m\omega\hbar}\right)^{1/4} H_n\left(\frac{1}{\sqrt{m\omega\hbar}}p\right)e^{-p^2/(2m\omega\hbar)}} \tag{4.48}$$

It may seem peculiar that in the midst of this chapter we have seemingly digressed to reexamine an already solved problem. If, however, the relationship between $\psi(x)$ and $\phi(p)$ that is contained in their Fourier transforms is valid, we should be able to check it with the easily derived oscillator eigenfunctions. We examine the ground state for which both the coordinate and momentum space


1-d Quantum States Applet v1.6a
Setup: Harmonic Oscillator
Mouse = Create Gaussian
Clear
Normalize
Maximize
Ground State
Rescale Graphs
Stopped
Simulation Speed
Resolution
Particle Mass
Spring Constant
Offset
n=0 energy level
n=0 position space
probability density
position space
n=0 momentum space
probability density
momentum space


1-d Quantum States Applet v1.6a
n=1 energy level
position space
momentum space
Setup: Harmonic Oscillator
Mouse = Create Gaussian
Clear
Normalize
Maximize
Ground State
Rescale Graphs
Stopped
Simulation Speed
Resolution
Particle Mass
Spring Constant
Offset


1-d Quantum States Applet v1.6a
n=2 energy level
Setup: Harmonic Oscillator
Mouse = Create Gaussian
Clear
Normalize
Maximize
Ground State
Rescale Graphs
Stopped
Simulation Speed
Resolution
Particle Mass
Spring Constant
Offset


1-d Quantum States Applet v1.6a
n=3 energy level
Setup: Harmonic Oscillator
Mouse = Create Gaussian
Clear
Normalize
Maximize
Ground State
Rescale Graphs
Stopped
Simulation Speed
Resolution
Particle Mass
Spring Constant
Offset

http://www.falstad.com/qm1d/

Expectation Values

http://www.st-andrews.ac.uk/~qmanim/embed_item_2.php?anim_id=5

http://www.st-andrews.ac.uk/~qmanim/embed_item_3.php?anim_id=4

But there are two potentials that can be handled in momentum space: first, for a *linear* potential $V(x) = -kx$, the momentum space analysis is actually easier—it's just a first-order equation. Second, for a particle in a *quadratic* potential—a simple harmonic oscillator—the two approaches yield the *same* differential equation. That means that the eigenfunctions in momentum space (scaled appropriately) must be *identical* to those in position space—the simple harmonic eigenfunctions are their own Fourier transforms!

Expectation Values

The expectation value of the position $< x > = < n \,|x\,|n >$ is zero in every stationary state because $x = \sqrt{\hbar/2m\omega}\ (a^\dagger + a)$ and because $a^\dagger$ and a produce states orthogonal to the original state.

For $n = 3$

$$< 3|(a^\dagger + a)|3 >= \sqrt{4} < 3|4 > +\sqrt{3} < 3|2 >= 0.$$

For arbitrary n

$$< n|(a^\dagger + a)|n >= \sqrt{n+1} < n|n+1 > +\sqrt{n} < n|n-1 >= 0.$$

The expectation value of the momentum in every stationary state is also zero because the momentum operator also contains $a^\dagger$ and a

$$p = i\sqrt{m\hbar\omega/2}(a^\dagger - a)$$

Step-by-step procedure to solve QM problems

(1) Solve the TISE

$$H\ |E_n>= E_n\ |E_n>$$

Diagonalize the Hamiltonian
Find the eigenvalues E_n
Find the eigenstates $|E_n>$

(2) Expand $|\psi(0)>$ in energy eigenstates

$$|\psi(0)>= \sum_n a_n\ |E_n>$$

The energy eigenstates are called stationary states

(3) Put in the time-dependent phase factors $e^{(-iE_n t/\hbar)}$

$$|\psi(t)>= \sum_n a_n\ e^{(-iE_n t/\hbar)}\ |E_n>$$

Expand $|\psi(0)>$ in the energy basis

$$|\psi(0)>=|E_1>+|E_2>+|E_3>$$

$$|\psi(0)>=I\ |\psi(0)>$$

$$I=\sum_n |E_n><E_n|$$

$$|\psi(0)>=\sum_n |E_n><E_n|\psi(0)>$$

$$a_n=<E_n|\psi(0)>$$

$$|\psi(0)>=\sum_n |E_n>a_n=\sum_n a_n\ |E_n>$$

Write down $|\psi(t)>$ by inspection in energy basis

$$|\psi(t)>=\sum_{all\ n} a_n\ |E_n>exp(-iE_nt/\hbar)$$

$$\psi(x,t)=\sum_{all\ n} a_n\ \psi_n(x)\ exp(-iE_nt/\hbar)$$

Just insert the time-dependent phase factors

Warning: This only works in the energy basis!!!

Eigenvector Example

In Dirac notation

$$|\psi(0)>= N\ (|E_1> + |E_2> + |E_3>)$$

In vector notation

$$\psi(0) => \frac{1}{\sqrt{3}}\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

In Dirac notation

$$|\psi(t)>= \frac{1}{\sqrt{3}}\ \left(e^{-iE_1t/\hbar}|E_1> + e^{-iE_2t/\hbar}|E_2> + e^{-iE_3t/\hbar}|E_3>\right)$$

In vector notation

$$\psi(t) = \frac{1}{\sqrt{3}}\ e^{-iE_1t/\hbar}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{3}}\ e^{-iE_2t/\hbar}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{3}}\ e^{-iE_3t/\hbar}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\psi(t) = \frac{1}{\sqrt{3}}\ \begin{pmatrix} e^{-iE_1t/\hbar} \\ e^{-iE_2t/\hbar} \\ e^{-iE_3t/\hbar} \end{pmatrix}$$

Eigenfunction Example

In Dirac notation

$$|\psi(0)>= N\left(|E_1> + |E_2> + |E_3>\right)$$

In eigenfunction notation

$$\psi(x,0) = N\Big(\psi_1(x) + \psi_2(x) + \psi_3(x)\Big)$$

In Dirac notation

$$|\psi(t)>= \frac{1}{\sqrt{3}}\left(e^{-iE_1t/\hbar}|E_1> + e^{-iE_2t/\hbar}|E_2> + e^{-iE_3t/\hbar}|E_3>\right)$$

In eigenfunction notation

$$\psi(x,t) = \frac{1}{\sqrt{3}}\left(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar} + \psi_3(x)e^{-iE_3t/\hbar}\right)$$

The $\psi_n(x)$ functions are called the eigenfunctions or the stationary states, and they are given by

$$\psi_1(x) = <x|E_1>$$
$$\psi_2(x) = <x|E_2>$$
$$\psi_3(x) = <x|E_3>$$

Possibilities and Probabilities

In Dirac notation

$$|\psi(0)>= N\ (|E_1> + |E_2> + |E_3>)$$

If you measure the energy, what are the possibilities and what are the corresponding probabilities?

The possibilities are the eigenvalues of the Hamiltonian.

So, the possibilities are E_1, E_2, E_3.

The probabilities are given by $|<E_i \mid \psi>|^2$.

So, the probabilities are $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.

What is the state of the system immediately after the measurement?

The state is in the eigenstate of H corresponding to the measured eigenvalue.

So, the state is $|E_1>$ or $|E_2>$ or $|E_3>$.

For the eigenvector example

$$|E_1 >=> \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|E_2 >=> \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|E_3 >=> \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For the eigenfunction example

$$\psi_1(x) = < x \mid E_1 >$$
$$\psi_2(x) = < x \mid E_2 >$$
$$\psi_3(x) = < x \mid E_3 >$$

To find the a_n expansion coefficients

$$|\psi(0)>=\sum_n a_n\,|E_n>$$

Compute the inner products

$$a_n=<E_n|\psi(0)>$$

In the vector example

$$a_1=<E_1|\psi(0)>=(1,0,0)\,\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}=\frac{1}{\sqrt{3}}$$

$$a_2=<E_2|\psi(0)>=(0,1,0)\,\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}=\frac{1}{\sqrt{3}}$$

$$a_3=<E_3|\psi(0)>=(0,0,1)\,\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}=\frac{1}{\sqrt{3}}$$

In the eigenfunction example

$$a_1=<E_1|\psi(0)>$$

$$a_2=<E_2|\psi(0)>$$

$$a_3=<E_3|\psi(0)>$$

Still compute the inner products, but .....

Still compute the inner products, but the inner products are now given by integrals

$$a_1 = < E_1|\psi(0) >= \int \psi_1^*(x)\ \psi(x,0)\ dx$$

$$a_2 = < E_2|\psi(0) >= \int \psi_2^*(x)\ \psi(x,0)\ dx$$

$$a_3 = < E_3|\psi(0) >= \int \psi_3^*(x)\ \psi(x,0)\ dx$$

Note that the probabilities are given by the magnitude squared of the expansion coefficients $Prob(E_n) = |a_n|^2$

$$Prob(E_1) = |a_1|^2 = |< E_1|\psi(0) >|^2 = \left| \int \psi_1^*(x)\ \psi(x,0)\ dx \right|^2$$

$$Prob(E_2) = |a_2|^2 = |< E_2|\psi(0) >|^2 = \left| \int \psi_2^*(x)\ \psi(x,0)\ dx \right|^2$$

$$Prob(E_3) = |a_3|^2 = |< E_3|\psi(0) >|^2 = \left| \int \psi_3^*(x)\ \psi(x,0)\ dx \right|^2$$

From Dyads to Dirac Notation

There is an old fashioned notation that is very similar to Dirac notation. It looks like this

$$I = \hat{i}\,\hat{i} + \hat{j}\,\hat{j} + \hat{k}\,\hat{k}$$

If you apply this identity operator I to an abstract vector $\vec{v}$ you find

$$\vec{v} = I\,\vec{v} = \left[\hat{i}\,\hat{i} + \hat{j}\,\hat{j} + \hat{k}\,\hat{k}\right]\vec{v} = \hat{i}\,\hat{i} \bullet \vec{v} + \hat{j}\,\hat{j} \bullet \vec{v} + \hat{k}\,\hat{k} \bullet \vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}.$$

This is called dyad notation. It takes an abstract vector $\vec{v}$, a vector not yet in any basis, and converts it into the representation of that vector in a specific basis. In this case, the basis is your friend, the good old, $\hat{i}, \hat{j}, \hat{k}$ basis.

You could, of course, write the same thing using the symbols $\hat{x}, \hat{y}, \hat{z}$ instead of $\hat{i}, \hat{j}, \hat{k}$.

$$\vec{v} = I\,\vec{v} = \left[\hat{x}\,\hat{x} + \hat{y}\,\hat{y} + \hat{z}\,\hat{z}\right]\vec{v} = \hat{x}\,\hat{x} \bullet \vec{v} + \hat{y}\,\hat{y} \bullet \vec{v} + \hat{z}\,\hat{z} \bullet \vec{v} = \hat{x}v_x + \hat{y}v_y + \hat{z}v_z = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}.$$

Now write the dyad notation as a sum over the three basis vectors

$$I = \sum_{l=1}^{3} \hat{l}\,\hat{l}$$

So the expansion in dyad notation is given by

$$\vec{v} = I\,\vec{v} = \left[\sum_{l=1}^{3} \hat{l}\,\hat{l}\right]\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Now look at the same thing, the exact analog, in Dirac notation: The identity operator I is the sum over the three basis ket vectors $|l>$

$$I = \sum_{l=1}^{3} |l><l|$$

So the expansion in the $|l>$ basis in Dirac notation is given by

$$|\psi> = I\,|\psi> = \left[\sum_{l=1}^{3} |l>\ <l|\right]|\psi> = a_1|1> + a_2|2> + a_3|3>.$$

Where the three expansion coefficients are given by

$$a_j =< j|\psi >$$

here $j = 1,\ 2,\ 3.$

From an old 441 exam: Using ladder operators to compute the time-dependent expectation values and uncertainties for x and p

1. Quantitative Aspects of the Harmonic Oscillator

Consider a particle moving in a simple harmonic oscillator well with the zero-time state vector

$$| \psi(t=0) > = N \left[| n=3 > + | n=4 > \right].$$

(a) Calculate the normalization constant N. Write down the equation for the normalized zero-time state vector $| \psi(0) >$ in terms of the energy eigenkets $| n >$. Using your equation for $| \psi(0) >$ in terms of the energy eigenkets $| n >$ write down the equation for the corresponding normalized time-dependent state vector $| \psi(t) >$ in terms of the energy eigenkets $| n >$. Convert your equation for the time-dependent state vector $| \psi(t) >$ in terms of the energy eigenkets $| n >$ into the corresponding equation for the time-dependent position-space wavefunction $\psi(x,t)$ in terms of the position-space stationary states $\psi_n(x)$.

(b) If you measure E at $t = 0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< E(t) >$. Calculate the time-dependent uncertainty $\Delta E(t)$. Explain the time-dependence, or lack thereof, of $< E(t) >$ and $\Delta E(t)$.

(c) If you measure x at $t = 0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< x(t) >$. Calculate the time-dependent uncertainty $\Delta x(t)$. Simulate the time evolution of this system using http://falstad.com. Do your expressions for $< x(t) >$ and $\Delta x(t)$ agree with your simulation?

(d) If you measure p at $t = 0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< p(t) >$. Calculate the time-dependent uncertainty $\Delta p(t)$. Simulate the time evolution of this system using http://falstad.com. Do your expressions for $< p(t) >$ and $\Delta p(t)$ agree with your simulation?

(e) Sketch the $t = 0$ probability density distributions $P(E,0)$, $P(x,0)$, and $P(p,0)$. Add your calculated expectation values and uncertainties to your sketches. Do they agree?

Three Ways to Solve:

(1) Do the integrals

(2) Use Matrix Method

(3) Use Dirac Notation

Do the integrals

the possibilities and the probabilities for position measurements

c.) The possible locations for a particle are unlimited, although ~~the~~ you are very unlikely to measure a particle far outside the potential well.

The probability is given by $\int_{x_1}^{x_2} \psi^*(x,0)\,\psi(x,0)\,dx$.

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

compute < x(t) >

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx \qquad d\xi = \sqrt{\frac{m\omega}{\hbar}}\,dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{x}{2^3\cdot 3!} H_3^2(\xi)\, e^{-\xi^2} + \frac{x}{2^4\, 4!} H_4^2(\xi)\, e^{-\xi^2} + \left[e^{\frac{i(E_3-E_4)t}{\hbar}} + e^{-\frac{i(E_3-E_4)t}{\hbar}}\right] x\, H_3(\xi) H_4(\xi) \frac{e^{-\xi^2}\, d\xi}{\sqrt{2^3 2^4 3!\, 4!}}$$

$$= \frac{1}{2\sqrt{\pi}} \left[\cancel{2\sqrt{\pi}} + \int_{-\infty}^{\infty} x \cos(\omega t)\, H_3(\xi) H_4(\xi) \frac{e^{-\xi^2}}{\sqrt{2^3 2^4 3!\, 4!}}\, d\xi \right] =$$

$$\langle x(t)\rangle = \frac{1}{\sqrt{\pi}\,2}\int_{-\infty}^{\infty} \frac{\xi}{2^3 3!} H_3^2(\xi)e^{-\xi^2} + \frac{\xi}{2^4\cdot 4!} H_4^2(\xi)e^{-\xi^2} + \frac{\xi\cos(\omega t)\,H_3(\xi)H_4(\xi)e^{-\xi^2}}{\sqrt{2^3 2^4\cdot 3!\cdot 4!}}\,d\xi$$

$$= \frac{1}{2\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\sqrt{\frac{\hbar}{m\omega}}\left[\frac{\xi}{2^3\cdot 3!}\left(8^2\xi^6 - \overset{\sim 0}{16\cdot 12\xi^4} + 12^2\xi^2\right)e^{-\xi^2} + \frac{\xi}{2^4 4!}\left(16^2\xi^8 - 2\cdot 16\cdot 48\xi^6 + \overset{\sim 0}{2\cdot 16\cdot 12\xi^4} + 48^2\xi^4 - 2\cdot 12\cdot 48\xi^2\right)e^{-\xi^2}\right]\right.$$

$$\sqrt{\frac{\hbar}{m\omega}}\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\cos(\omega t)\left[\frac{1}{\sqrt{2^7\cdot 3!\cdot 4!}}\left(8\cdot 16\xi^8 - 8\cdot 48\xi^6 - \cancel{12\cdot 16\xi^5} + 12\cdot 8\xi^4 + \cancel{12\cdot 48\xi^3} - \cancel{12\cdot 12\xi}\right)e^{-\xi^2}d\xi\right]$$

odd terms are 0

$$= \frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!\cdot\pi}}\cdot\frac{1}{2}\cdot\sqrt{\frac{\hbar}{m\omega}}\left[8\cdot 105\sqrt{\pi} + 15\cdot 48\sqrt{\pi} + 6\cdot 12\sqrt{\pi}\right]$$

UGLY

$\sqrt{\frac{2\hbar}{m\omega}}$ => why didn't you use a and a^+?

PRETTY!

$$= \frac{816}{\sqrt{18432}}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t) = \frac{17}{\cancel{2\sqrt{2}}}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t)$$

compute < x(t) x(t) >

$$\langle x^2(t)\rangle = \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty}\frac{\xi^2 e^{-\xi^2}}{2^3\cdot 3!}H_3^2(\xi) + \frac{\xi^2 e^{-\xi^2}}{2^4 4!}H_4^2(\xi) + \frac{\xi^2 e^{-\xi^2}}{\sqrt{2^7\cdot 3!\cdot 4!}}H_3(\xi)H_4(\xi)\cos(\omega t)\,d\xi$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\left[\int_{-\infty}^{\infty}\frac{e^{-\xi^2}}{2^3\cdot 3!}\left(8^2\xi^8 - 16\cdot 12\xi^6 + 12^2\xi^4\right) + \frac{e^{-\xi^2}}{2^4\cdot 4!}\left(16^2\xi^{10} - 2\cdot 16\cdot 48\xi^8 + 2\cdot 16\cdot 12\xi^6 + 48^2\xi^6 - 2\cdot 12\cdot 48\xi^4\right)d\xi\right.$$

$$\left. + \int_{-\infty}^{\infty}\frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!}}e^{-\xi^2}\left(\cancel{8\cdot 16\xi^7} - \cancel{8\cdot 48\xi^5} - 12\cdot 16\xi^6 + \cancel{12\cdot 8\xi^5} + 12\cdot 48\xi^4 - 12\cdot 12\xi^2\right)d\xi\right]$$

odd terms drop out

$$= \frac{1}{2\cancel{\sqrt{\pi}}}\sqrt{\frac{\hbar}{m\omega}}\left[\overset{\frac{35}{4}\sqrt{\pi}}{\frac{2\cdot 105\cdot 8^2\cdot\cancel{\sqrt{\pi}}}{2^3\cdot 3!\cdot 32}} - \overset{-15/2\sqrt{\pi}}{\frac{\cancel{16}\cdot\cancel{12}\cdot\cancel{2}\cdot 15}{\cancel{16}\cdot 2^3\cdot\cancel{3!}}\cancel{\sqrt{\pi}}} + \overset{9/4\sqrt{\pi}}{\frac{2\cdot 12^2\cdot 3}{8\cdot 2^3\cdot 3!}\cancel{\sqrt{\pi}}} + \frac{1}{2^4 4!}\left(\overset{\frac{315}{16}\sqrt{\pi}}{\frac{16^2\cdot 2\cdot 945}{64}\cancel{\sqrt{\pi}}} - \overset{-105/4\sqrt{\pi}}{\frac{4\cdot 16\cdot 48\cdot 105}{32}\cancel{\sqrt{\pi}}} + \overset{\frac{15}{8}\sqrt{\pi}}{\frac{4\cdot 16\cdot 12\cdot 15}{16}\cancel{\sqrt{\pi}}}\right.\right.$$

$$\left.\left. + \overset{45/4\sqrt{\pi}}{\frac{2\cdot 48^2\cdot 15}{16}\cancel{\sqrt{\pi}}} - \overset{9/4\sqrt{\pi}}{\frac{4\cdot 12\cdot 48\cdot 3}{8}\cancel{\sqrt{\pi}}}\right) + \frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot\cancel{4!}}}\left(-2\cdot 12\cdot 16\cdot\frac{15}{16}\cancel{\sqrt{\pi}} + 12\cdot 48\cdot 2\cdot\frac{3}{8}\cancel{\sqrt{\pi}} - \frac{12^2\cdot 2}{4}\cancel{\sqrt{\pi}}\right)\right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{197}{16} + \cos(\omega t)\,(0)\right) = \frac{197}{32}\sqrt{\frac{\hbar}{m\omega}}$$

compute the uncertainty in the position

$$\Delta x(t) = \sqrt{\frac{197}{32}\sqrt{\frac{\hbar}{m\omega}} - \frac{289}{8}\cdot\frac{\hbar}{m\omega}\cos^2(\omega t)}$$

$$\sqrt{\frac{2\hbar}{m\omega}}\left(1 + \sin^2\omega t\right)^{1/2}$$

Both seem to match falstad.com.

again ladder operators are more beautiful and easier!

the possibilities and the probabilities for momentum measurements

d) If you measure p at ~~any~~ time zero the value of p could be anywhere between $-\infty$ and ∞ with probability $\int_{p_1}^{p_2} \hat{\psi}^*(p,0)\,\hat{\psi}(p,0)\,dp$.

compute < p(t) >

$$\langle p(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, -i\hbar\frac{\partial}{\partial x}\, \psi(x,t)\, dx$$

$$= \frac{-i\hbar}{2}\int_{-\infty}^{\infty} \underbrace{\psi_3^*(x)\frac{\partial}{\partial x}\psi_3(x)}_{\text{part one}} + \underbrace{\psi_4^*(x)\frac{\partial}{\partial x}\psi_4(x)}_{\text{part two}} + \underbrace{e^{\frac{i(E_4-E_3)t}{\hbar}}\psi_4^*(x)\frac{\partial}{\partial x}\psi_3(x)}_{\text{part three}} + \underbrace{e^{\frac{-i(E_4-E_3)t}{\hbar}}\psi_3^*(x)\frac{\partial}{\partial x}\psi_4(x)}_{\text{part four}}\, dx$$

$A = \frac{m\omega}{\hbar}$

$$\text{part one} = \frac{-i\hbar}{2}\int_{-\infty}^{\infty} \frac{A}{\sqrt{\pi}}\cdot\frac{1}{2^3 3!}\left[(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2^4 3!}\frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\left[(24Ax^2-12)e^{-\frac{Ax^2}{2}} - Ax(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2^4 3!}\frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-Ax^2}\left[8\cdot 24A^2x^5 - 8\cdot 12Ax^3 - 8^2A^3x^7 + 8\cdot 12\cdot Ax^3 - 12\cdot 24Ax^3 + 12^2x + 12\cdot 8A^2x^5 - 12^2Ax^3\right]dx$$

odd terms drop out

$$= \frac{-i\hbar A^3}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^4 e^{-Ax^2}dx = \frac{-i\hbar A^2}{A^{5/2}}\cdot\frac{3}{4} = \frac{-i\hbar\cdot 3}{A^{1/2}\cdot 4} = 0$$

$$\text{part two} = \frac{-i\hbar}{2}\int_{-\infty}^{\infty}\frac{A^{1/2}}{\sqrt{\pi}}\frac{1}{2^4 4!}\left[(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}dx\right]$$

$$= \frac{-i\hbar A^{1/2}}{2^6 4!\sqrt{\pi}}\int_{-\infty}^{\infty}(16A^2x^4-48Ax^2+12)e^{-Ax^2}\left[(64A^2x^3-96Ax) - Ax(16A^2x^4-48Ax^2+12)\right]dx$$

$$= \frac{-i\hbar A^{1/2}}{2^8\cdot 3\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-Ax^2}\left[16\cdot 64A^4x^7 - 16\cdot 96A^3x^5 - 16^2A^4x^7 + 16\cdot 48A^3x^7 + 12\cdot 16A^3x^5 - \dots\right] = 0$$

$$\text{part three} = \frac{-i\hbar}{2}e^{\frac{i(E_4-E_3)t}{\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2^7 3!4!}}\frac{A^{3/2}}{\sqrt{\pi}}\left[(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2\cdot 3\sqrt{2^{11}}}\frac{A}{\sqrt{\pi}}e^{i\omega t}\int_{-\infty}^{\infty}(16A^2x^4-48Ax^2+12)e^{-Ax^2}\left[(3\cdot 8Ax^2-12) - Ax(8Ax^3-12x)\right]dx$$

$24Ax^2-12 \quad - 8A^2x^4+12Ax^2$

$$= \frac{-i\hbar A}{2\cdot 3\cdot\sqrt{\pi}}e^{i\omega t}\int_{-\infty}^{\infty}e^{-Ax^2}\left(16\cdot 24A^3x^6 - 12\cdot 16A^2x^4 - 8\cdot 16\cdot A^4x^8 + 12\cdot 16A^3x^6 - 48\cdot 24A^2x^4 + 12\cdot 48Ax^2 + 8\cdot 48A^3x^6 - 12\cdot 48A^2x^4 + 12\cdot 24Ax^2 - 12^2 - 8\cdot 12A^2x^4 + 12^2Ax^2\right)dx$$

$$= \frac{-i\hbar A}{2^{13/2}\cdot 3\cdot\sqrt{\pi}}e^{i\omega t}\sqrt{\pi}\left[\frac{-8\cdot 16\cdot A^4\cdot 2\cdot 105}{32\cdot A^{9/2}} + \frac{2\cdot(16\cdot 24\cdot A^3 + 12\cdot 16A^3 + 8\cdot 48A^3)\cdot 15}{16\,a^{5/2}} + \frac{2\cdot 3}{8\,a^{5/2}}\left(-12\cdot 16A^2 - 48\cdot 24A^2 - 12\cdot 48A^2 - 8\cdot 12A^2\right)\right.$$

$$+\frac{1}{2A^{3/2}}\left(12\cdot 48A + 12\cdot 24A + 12^2 A\right) + \frac{-12^2}{A^{1/2}}\Bigg]$$

$$= \frac{-i\hbar A^{1/2}}{2^{13/2}\cdot 3^{2}}\, e^{i\omega t}\Big[-840 + 1800 + \overset{-192}{-1512} + 504 - 144\Big] = i\hbar A^{1/2}\cdot\frac{\sqrt{2}}{2}\, e^{i\omega t} = \frac{i\hbar A^{1/2}}{\sqrt{2}}\, e^{i\omega t} = i\sqrt{\frac{m\omega\hbar}{2}}\, e^{i\omega t}$$

$$\text{part four} = \frac{-i\hbar}{2}\frac{e^{-i\omega t}A^{1/2}}{\sqrt{2^{11}\cdot 3^2}\sqrt{\pi}}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}\left(16A^2x^4 - 48Ax^2 + 12\right)e^{-\frac{Ax^2}{2}}\,dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\cdot 3\sqrt{\pi}}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-Ax^2}\Big[\left(16\cdot 4A^2x^3 - 48\cdot 2Ax\right) - Ax\left(16A^2x^4 - 48Ax^2 + 12\right)\Big]\,dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\cdot 3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\Big[8\cdot 16\cdot 4A^{7/2}x^6 - 8\cdot 48\cdot 2A^{5/2}x^4 - 8\cdot 16\cdot A^{7/2}x^8 + 8\cdot 48\cdot A^{7/2}x^6 - 12\cdot 8\cdot A^{5/2}x^4 - 12\cdot 16\cdot 4A^{5/2}x^4$$
$$+ 12\cdot 48\cdot 2A^{3/2}x^2 + 12\cdot 16A^{7/2}x^6 - 12\cdot 48A^{5/2}x^4 + 12^2A^{3/2}x^2\Big]\,dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\,3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\Big[-128A^{7/2}x^8 + 1088A^{7/2}x^6 - 2208A^{5/2}x^4 + 1296A^{3/2}x^2\Big]\,dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\cdot 3\cdot\cancel{\sqrt{\pi}}}\,2\cdot\cancel{\sqrt{\pi}}\Big[-420 + 1020 - 828 + 324\Big] = -i\sqrt{\frac{m\omega\hbar}{2}}\, e^{-i\omega t}$$

Again LADDER OP's!

$$\langle p(t)\rangle = \sqrt{\frac{m\omega\hbar}{2}}\Big[ie^{i\omega t} - ie^{-i\omega t}\Big] = \sqrt{2m\omega\hbar}\left[\frac{-e^{i\omega t} + e^{-i\omega t}}{2i}\right] = -\sqrt{2m\omega\hbar}\,\sin(\omega t)$$

compute < p(t) p(t) >

$$\langle p^2(t)\rangle = \int_{-\infty}^{\infty} -\hbar^2\,\psi^*(x,t)\frac{\partial^2}{\partial x^2}\psi(x,t)\,dx$$

$$= \frac{-\hbar^2}{2}\int_{-\infty}^{\infty}\left(\psi_3^*(x,t) + \psi_4^*(x,t)\right)\frac{\partial^2}{\partial x^2}\left(\psi_3(x,t) + \psi_4(x,t)\right)dx$$

$$= -\frac{\hbar^2}{2}\int_{-\infty}^{\infty}\psi_3^*(x,t)\frac{\partial^2}{\partial x^2}\psi_3(x,t) + \psi_4^*(x,t)\frac{\partial^2}{\partial x^2}\psi(x,t) + \psi_3^*(x,t)\frac{\partial^2}{\partial x^2}\psi_4(x,t) + \psi_4^*(x,t)\frac{\partial^2}{\partial x^2}\psi_3(x,t)\,dx$$

$$\int_{-\infty}^{\infty}\psi_3^*(x,t)\frac{\partial^2}{\partial x^2}\psi_3(x,t)\,dx = \int_{-\infty}^{\infty}\frac{A^{1/2}}{2^3\,3!\sqrt{\pi}}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\frac{\partial^2}{\partial x^2}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\,dx$$

$$= \frac{A^{1/2}}{2^3\,3!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\Big[\left(8A^{3/2}x^3 - 12A^{1/2}x\right)\Big[8A^{7/2}x^5 - 68A^{5/2}x^3 + 84A^{3/2}x\Big]\Big]\,dx$$

$$= \frac{A^{1/2}}{2^3\,3!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\Big[64A^8x^8 - 640A^4x^6 + 1488A^3x^4 - 1008A^2x^2\Big]\,dx$$

$$= \frac{A^{1/2}\, 2\cdot\sqrt{\pi}}{2^3\, 3!\,\sqrt{\pi}}\left[A^{1/2}\,210 - A^{1/2}\,600 + A^{1/2}\,558 - A^{1/2}\,252\right] = A\left[-\tfrac{7}{2}\right]$$

$$\int_{-\infty}^{\infty} \psi_4^*(x,t)\,\frac{\partial^2}{\partial x^2}\psi_4(x,t) = \frac{A^{1/2}}{2^4 4!}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\left(16A^2x^4 - 48Ax^2 + 12\right)\frac{\partial^2}{\partial x^2}\left(16A^2x^4 - 48Ax^2 + 12\right)dx$$

$$= \frac{A^{1/2}}{2^4 4!\,\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-Ax^2}\left[256A^6x^{10} - 3840A^5x^8 + 18624A^4x^6 - 31680A^3x^4 + 12096A^2x^2 - 1296A\right]dx$$

$$= \frac{A}{2^3 4!}\left[3780 - 12600 + 17460 - 11880 + 3024 - 648\right] = A\left(-\tfrac{9}{2}\right)$$

$$\int_{-\infty}^{\infty}\psi_3^*(x,t)\,\frac{\partial^2}{\partial x^2}\psi_4(x,t)\,dx = \frac{A^{1/2}}{\sqrt{2^7\, 3!\, 4!\,\pi}}\,e^{-i\omega t}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\frac{\partial^2}{\partial x^2}\left(16A^2x^4 - 48Ax^2 + 12\right)e^{-\frac{Ax^2}{2}}dx$$

$$= \frac{A^{1/2}e^{-i\omega t}}{\sqrt{2^7\, 3!\, 4!\,\pi}}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)\left(16A^4x^6 + -192A^3x^4 + 576A^2x^2 - 108A\right)dx = 0$$

odd terms drop out

Likewise $\displaystyle\int_{-\infty}^{\infty}\psi_4^*(x,t)\,\frac{\partial^3}{\partial x^2}\left(\psi_3(x,t)\right)dx = 0$

compute the uncertainty in the momentum

AGAIN LADDER OPS!

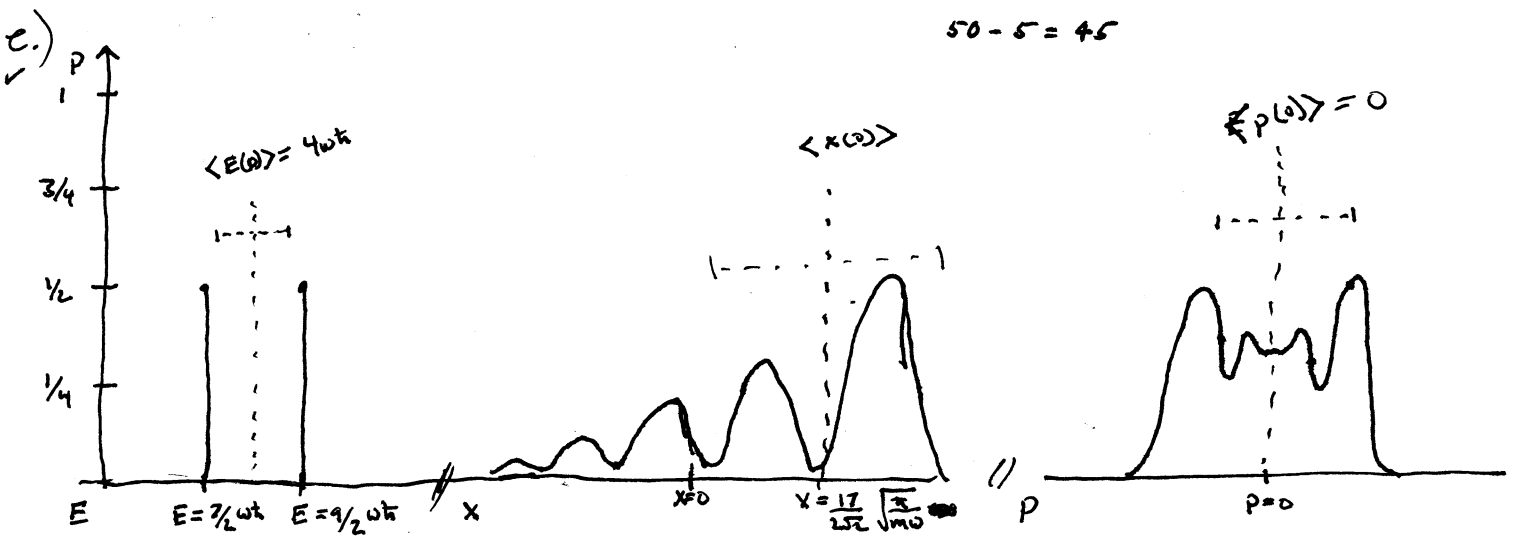
$$\langle p^2(t)\rangle = -\frac{\hbar^2}{2}\left[-8A\right] = 4m\omega\hbar$$

$$\Delta p(t) = \sqrt{\langle p^2(t)\rangle - \langle p(t)\rangle^2} = \sqrt{4m\omega\hbar - 2m\omega\hbar\sin^2(\omega t)} = \sqrt{2m\omega\hbar\left(2 - \sin^2(\omega t)\right)}$$

The expectation of p is 90° out of phase with $\langle x(t)\rangle$ exactly as the simulation on falstad.com shows. The uncertainty seems to match as well.

50 - 5 = 45

e.)



Matrix Method

1) Simple harmonic Oscillator:

$$|\psi(t=0)\rangle = N\left[|N=3\rangle + |N=4\rangle\right]$$

✓ a) $$\langle\psi(0)|\psi(0)\rangle = (0\ 0\ 0\ 1\ 1\ 0\ 0\ ,\ ,\ ,\ )\, N \cdot N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = 1$$

$$= |N|^2 (1 + 1) = 2|N|^2$$

$$\Rightarrow \boxed{N = \frac{1}{\sqrt{2}}}$$

compute the normalization factor and then write down I psi(0) >, I psi(t) >, and psi(x,t)

- $$\boxed{|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|N=3\rangle + |N=4\rangle\right]}$$

$E_N = (N + 1/2)\hbar\omega$

$\Rightarrow E_N = \frac{7}{2}\hbar\omega$ or $\frac{9}{2}\hbar\omega$

- $$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-7i\omega t/2}|3\rangle + e^{-9i\omega t/2}|4\rangle\right]}$$

- $$\langle x|\psi(t)\rangle = \psi(x,t)$$

$$= \frac{1}{\sqrt{2}}\left(\langle x|3\rangle e^{-7i\omega t/2} + \langle x|4\rangle e^{-9i\omega t/2}\right)$$

$$= \boxed{\frac{1}{\sqrt{2}}\left(\psi_3(x)\, e^{-7i\omega t/2} + \psi_4(x)\, e^{-9i\omega t/2}\right)}$$

1b) ✓ E at $t=0$, possibilities & probabilities ?

the possibilities and the probabilities for energy measurements

$\boxed{\text{possibilities:}}$ $E_N = \frac{7}{2}\hbar\omega, \frac{9}{2}\hbar\omega$ → only possible eigenvalues @ $t=0$.

$\boxed{\text{Probabilities:}}$

$$P(7/2\,\hbar\omega) = \left|\langle N=3|\psi(t=0)\rangle\right|^2 = \left|(0\,0\,0\,1\,0\ldots)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\\1\\0\\\vdots\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(1\cdot 1 + 0\cdot 1)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \boxed{\frac{1}{2}}$$

$$P(9/2\,\hbar\omega) = \left|\langle N=4|\psi(t=0)\rangle\right|^2 = \left|(0\,0\,0\,0\,1\,0\ldots)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\1\\1\\0\\\vdots\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(0\cdot 1 + 1\cdot 1)\right|^2 = \left|\frac{1}{\sqrt{2}}\right| = \boxed{\frac{1}{2}}$$

compute $\langle E(t)\rangle$

$$\langle E(t)\rangle = \langle\psi(t)|\mathcal{H}|\psi(t)\rangle$$

$$= \langle\psi(t)|\begin{array}{c}0\\1\\2\\3\\4\\\vdots\end{array}\begin{pmatrix}1/2 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 3/2 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 5/2 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 7/2 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 9/2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\hbar\omega\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\e^{-i\cdot 7\omega t/2}\\e^{-i\cdot 9\omega t/2}\\\vdots\end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\left(0\quad 0\quad 0\quad \cancel{e^{7i\omega t/2}}\quad \cancel{e^{9i\omega t/2}}\quad \cdots\right)\begin{pmatrix}0\\0\\0\\7/2\,\cancel{e^{-i7\omega t/2}}\\9/2\,\cancel{e^{-i9\omega t/2}}\\\vdots\end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\left[7/2 + 9/2\right] = \boxed{4\hbar\omega}$$

1b) cont

$$\Delta E(t) = \left[\langle \psi(t) | \mathcal{H}^2 | \psi(t) \rangle - \langle \psi(t) | \mathcal{H} | \psi(t) \rangle^2 \right]^{1/2}$$

– 1st calculate $\mathcal{H}^2$: $\langle \mathcal{H} \rangle = \left(a^+a + \frac{1}{2}\right)\hbar\omega \left(a^+a + \frac{1}{2}\right)\hbar\omega$

$$= \left(a^+a\,a^+a + \frac{1}{2}a^+a + a^+a\,\frac{1}{2} + \frac{1}{4}\right)\hbar^2\omega^2 \qquad a^+a = N$$

$$= \boxed{\left(n^2 + n + 1/4\right)\hbar^2\omega^2}$$

compute < H(t) H(t) >

– $$\langle \psi(t) | \mathcal{H}^2 | \psi(t) \rangle = \frac{\hbar^2\omega^2}{2}\left[\langle 3| e^{7i\omega t/2} + \langle 4| e^{9i\omega t/2}\right] \cdot$$

$$\left(n^2 + N + 1/4\right) \cdot \left[|3\rangle e^{-7i\omega t/2} + |4\rangle e^{-9i\omega t/2}\right]$$

$$= \frac{\hbar^2\omega^2}{2}\left[\langle 3| e^{7i\omega t/2} + \langle 4| e^{9i\omega t/2}\right] \cdot \left[\left(3^2 + 3 + 1/4\right)|3\rangle e^{-7i\omega t/2} +\right.$$

$$\left.\left(4^2 + 4 + 1/4\right)|4\rangle e^{-9i\omega t/2}\right] = \frac{\hbar^2\omega^2}{2}\left[\langle 3| \cancel{e^{7i\omega t/2}} + \langle 4| \cancel{e^{9i\omega t/2}}\right]$$

$$\cdot \left[\frac{49}{4}|3\rangle \cancel{e^{-7i\omega t/2}} + \frac{81}{4}|4\rangle \cancel{e^{-9i\omega t/2}}\right]$$

$$= \frac{\hbar^2\omega^2}{2}\left[\cancelto{1}{\langle 3|3\rangle}\,\cancelto{1}{e^0} \cdot \frac{49}{4} + \frac{81}{4}\cancelto{1}{\langle 4|4\rangle}\,\cancelto{1}{e^0}\right] = \boxed{\frac{65}{4}\hbar^2\omega^2}$$

compute the uncertainty in the energy

$$\Delta E(t) = \left[\frac{65}{4}\hbar^2\omega^2 - (4\hbar\omega)^2\right]^{1/2} = \left[\frac{65}{4}\hbar^2\omega^2 - \frac{64}{4}\hbar^2\omega^2\right]^{1/2}$$

$$= \boxed{\frac{\hbar\omega}{2}}$$

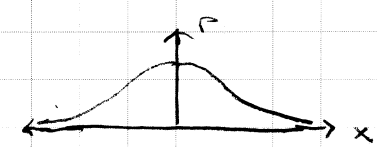
– Both $\langle E(t) \rangle$ + $\Delta E(t)$ should be free of time dependence, because of the symmetrical Bra/Ket relation for the Hamiltonian + the orthonormality of all energy eigenvectors.

1c) ✓ **the possibilities and the probabilities for position measurements**

The possible values for x @ $t=0$ are any and all values of x between $-\infty$ + ∞. This is due to the continuous nature of position in the universe. The probabilities of the position values will be governed by the gaussian distribution, given by the square of the wave fxn. However, to obtain a single precise value of x will cause the probability to goto zero, as shown by the equation below, because $\Delta x \to 0$.

$$N \int_a^a |\psi(x,t)|^2 \, dx = 0 \Rightarrow \text{No possible probability except zero for a single position measurement.}$$

possibilities and probabilities



compute < x(t) >

$$\langle x(t) \rangle = \langle \psi(t) | X | \psi(t) \rangle$$

$$= \langle \psi(t) | \left(\frac{\hbar}{2m\omega}\right)^{1/2} \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & \sqrt{1} & 0 & 0 & 0 \cdots \\ 1 & \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \cdots \\ 2 & 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \cdots \\ 3 & 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \cdots \\ 4 & 0 & 0 & 0 & \sqrt{4} & 0 \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i7\omega t/2} \\ e^{-i9\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \langle \psi(t) | \frac{1}{\sqrt{2}} \left(\frac{\hbar}{2m\omega}\right)^{1/2} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{4}\, e^{-i7\omega t/2} \\ \sqrt{4}\, e^{-i9\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \frac{\cancel{2}}{\cancel{2}} \left(\frac{\hbar}{2m\omega}\right)^{1/2} \cdot \begin{pmatrix} 0 & 0 & 0 & e^{+i7\omega t/2} & e^{+i9\omega t/2} & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i9\omega t/2} \\ e^{-i9\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \left[\frac{\hbar}{2m\omega}\right]^{1/2} \left(e^{i7\omega t/2} \cdot e^{-i9\omega t/2} + e^{i9\omega t/2} \cdot e^{-i7\omega t/2} \right) = \left[\frac{\hbar}{2m\omega}\right]^{1/2} \frac{\left(e^{i\omega t} + e^{-i\omega t}\right)}{2} \cdot 2$$

$$\langle x(t) \rangle = \boxed{\left[\frac{2\hbar}{m\omega}\right]^{1/2} \cos(\omega t)}$$

1c) cont $\Delta x(t) = \left[\langle \psi(t) | X^2 | \psi(t) \rangle - \left(\langle \psi(t) | X | \psi(t) \rangle \right)^2 \right]^{1/2}$

$$X^2 = X \cdot X = \left(\frac{\hbar}{2m\omega} \right) \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 1 & 0 & 3 & 0 & \sqrt{6} & 0 \\ 2 & \sqrt{2} & 0 & 5 & 0 & \sqrt{12} \\ 3 & 0 & \sqrt{6} & 0 & 7 & 0 \\ 4 & 0 & 0 & \sqrt{12} & 0 & 9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \cdots$$

– from HW 5, ch. 8 soln's

compute < x(t) x(t) >

$$\langle \psi(t) | X^2 | \psi(t) \rangle = \frac{1}{2} \cdot \frac{\hbar}{2m\omega} \langle \psi(t) | \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} \\ 0 & \sqrt{6} & 0 & 7 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i7\omega t/2} \\ e^{-i9\omega t/2} \end{pmatrix}$$

$$= \frac{\hbar}{4m\omega} \left(0 \quad 0 \quad 0 \quad \cancel{e^{7i\omega t/2}} \quad \cancel{e^{9i\omega t/2}} \quad \cdots \right) \begin{pmatrix} 0 \\ \sqrt{6}\, e^{-7i\omega t/2} \\ \sqrt{12}\, e^{-9i\omega t/2} \\ 7\, \cancel{e^{-i\omega 7 t/2}} \\ 9\, \cancel{e^{-9i\omega t/2}} \end{pmatrix} = \boxed{\frac{4\hbar}{m\omega}}$$

compute the uncertainty in the position

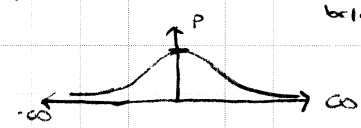
$$\Delta x(t) = \left[\frac{4\hbar}{m\omega} - \left(\left[\frac{2\hbar}{m\omega} \right]^{1/2} \cos(\omega t) \right)^2 \right] = \boxed{\left(\frac{2\hbar}{m\omega} \left[2 - \cos^2(\omega t) \right] \right)^{1/2}}$$

consistent with simulation

– according to http://Falstad.com, my values of $\langle x(t) \rangle$ + $\Delta x(t)$ are consistent w/ the simulation.

the possibilities and the probabilities for momentum measurements

1d) 8 Similiar to the position possibilities + probabilities, measurement of p @ time t=0 could have an infinite number of possibilities, in the interval $-\infty$ to ∞. (so, p) If we were able to pinpoint one value for p then the corresponding probability will goto zero for the reasons listed in part 1c. This all boils down to p being directly proportional to $\dot{x}$ (ie velocity), and x's continous nature in the real spectrum, p will still have a gaussian probability centered at zero + falling off exponential as shown below



compute < p(t) >

$$- \langle p(t) \rangle = \langle \Psi(t) | P | \Psi(t) \rangle$$

$$= \frac{1}{2} \langle \Psi(t) | \, i \left(\frac{m\omega\hbar}{2} \right)^{1/2} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & i & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & i\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & i\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & i\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-7i\omega t/2} \\ e^{-9i\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \frac{i}{2} \left(\frac{m\omega\hbar}{2} \right)^{1/2} \left(\cancel{0} \;\; 0 \;\; \cancel{0} \;\; \cancel{e^{7i\omega t/2}} \;\; e^{9i\omega t/2} \right) \begin{pmatrix} \cancel{0} \\ \cancel{0} \\ \cancel{i\sqrt{3}\, e^{-9i\omega t/2}} \\ 2i\, e^{-9i\omega t/2} \\ 2e^{-7i\omega t/2} \end{pmatrix}$$

$$= \frac{i}{\cancel{2}} \left(\frac{m\omega\hbar}{2} \right)^{1/2} \left(e^{-i\omega t} + e^{i\omega t} \right) \cdot \frac{2}{2}$$

$$= \boxed{\left(\frac{+m\omega\hbar}{2} \right)^{1/2} \cos(\omega t)}$$

$2\sin(\omega t)$

to compute the uncertainty use

$$- \Delta p(t) = \left[\langle \Psi(t) | P^2 | \Psi(t) \rangle - \left(\langle \Psi(t) | P | \Psi(t) \rangle \right)^2 \right]^{1/2}$$

1d) cont

$$p^2 = p \cdot p = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & i & 0 & 0 & 0 & \cdots \\ 1 & 0 & \sqrt{2}i & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & i\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 2i & \cdots \\ 0 & 0 & 0 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array} \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2}i & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3}i & 0 \\ 0 & 0 & \sqrt{3} & 0 & 2i \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 3i & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 5i & 0 & -2\sqrt{3} \\ 0 & -\sqrt{3}\sqrt{2} & 0 & 7i & 0 \\ 0 & 0 & 2\sqrt{3} & 0 & 4i \end{pmatrix}$$

$$i^2 \left(\frac{m\omega\hbar}{2}\right)$$

compute < p(t) p(t) >

$$- \quad \langle \psi(t) | p^2 | \psi(t) \rangle = \frac{1}{2} \cdot \frac{-m\omega\hbar}{2} \langle \psi(t) | \begin{pmatrix} i & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 3i & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 5i & 0 & -2\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 7i & 0 \\ 0 & 0 & 2\sqrt{3} & 0 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-7i\omega t/2} \\ e^{-9i\omega t/2} \end{pmatrix}$$

$$= \frac{-m\omega\hbar}{4} \langle \cancel{0} \; \cancel{0} \; \cancel{0} \; \cancel{e^{7i\omega t/2}} \; \cancel{e^{9i\omega t/2}} | \begin{pmatrix} \cancel{0} \\ \cancel{-\sqrt{3}\sqrt{2}\, e^{-7i\omega t/2}} \\ \cancel{-2\sqrt{3}\, e^{-9i\omega t/2}} \\ 7i\, \cancel{e^{-7i\omega t/2}} \\ 4i\, \cancel{e^{-9i\omega t/2}} \end{pmatrix}$$

$$= -\frac{m\omega\hbar}{4}(7i + 4i)$$

compute the uncertainty in the momentum

$$\Rightarrow \Delta p(t) = \left[-\frac{11}{4} i m\omega\hbar - \left(i \left(\frac{m\omega\hbar}{2}\right)^{1/2} \cos(\omega t) \right)^2 \right]^{1/2}$$

$$= \left[-\frac{11}{4} i m\omega\hbar + \frac{m\omega\hbar}{2} \cos^2(\omega t) \right]^{1/2} = \boxed{\left(\frac{m\omega\hbar}{2}\right)^{1/2} \left(\cos(\omega t) - \frac{11}{2} i\right)^{1/2}}$$

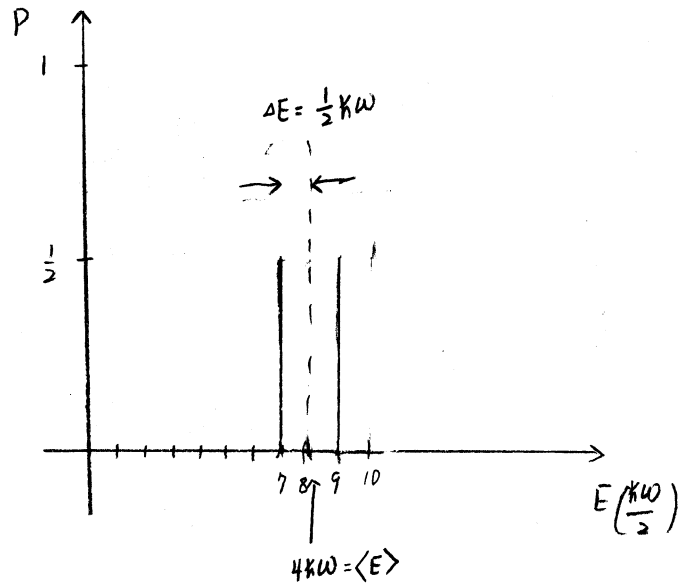
$$\sqrt{2m\omega\hbar}\,(1 + \cos^2 \omega t)^{1/2}$$

consistent with simulation

- After checking http://falstad.com, my expressions for $\Delta p(t)$ & $\langle p(t) \rangle$ appear to agree w/ the simulations.

✓ (e)

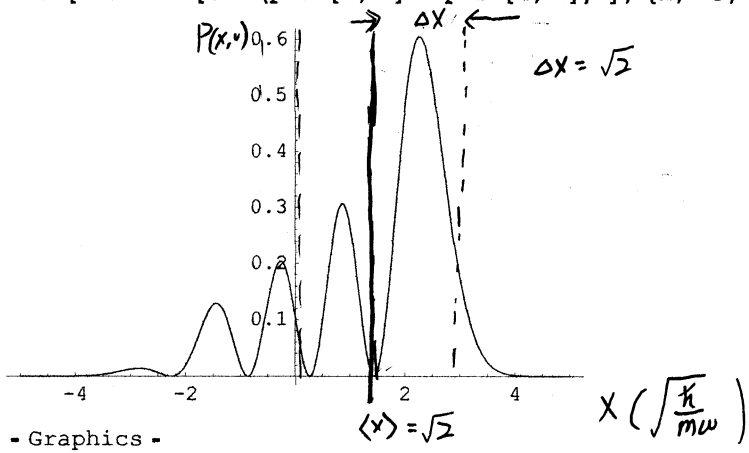
(1) $P(E,0)$


P
1
$\frac{1}{2}$
$\Delta E = \frac{1}{2}\hbar\omega$
7
8
9
10
$E\left(\frac{\hbar\omega}{2}\right)$
$4\hbar\omega = \langle E \rangle$

■ **(2.) P(x,0): (assume $\frac{m\omega}{\text{hBar}}$ =1)**

```
In[1]:= phi1[x_, n_] = (1/(π (2^(2 n)) ((n!)^2)))^0.25 E^(-x^2/2) HermiteH[n, x];

In[2]:= Plot[Evaluate[0.5 (phi1[x, 3] + phi1[x, 4])^2], {x, -5, 5}]
```

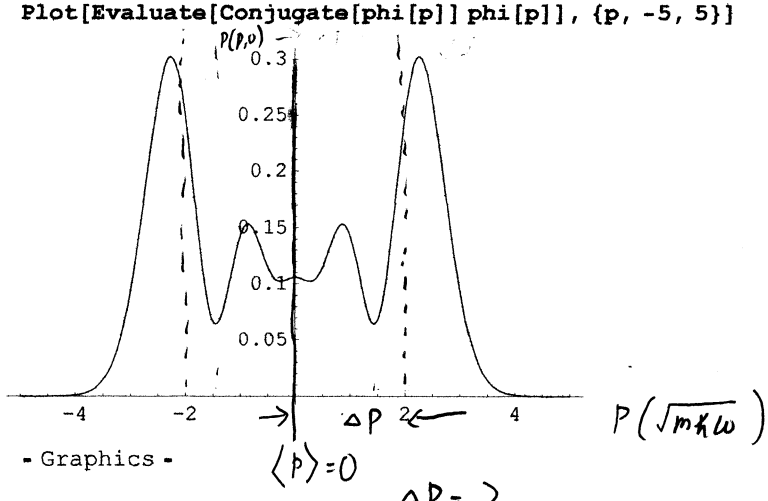



```
Out[2]= - Graphics -
```

■ **(2.) P(p,0): (assume (m ω hBar) =1)**

```
In[10]:= phi3[p_] = FourierTransform[phi1[x, 3], x, p];
         phi4[p_] = FourierTransform[phi1[x, 4], x, p];
         phi[p_] = 1/(2^0.5) (phi3[p] + phi4[p]);

In[15]:= Plot[Evaluate[Conjugate[phi[p]] phi[p]], {p, -5, 5}]
```



50 - 0 = 50

```
Out[15]= - Graphics -
```

Yes, they agree

Dirac Notation

2. Quantitative aspects of SHO

✓ (a) **calculate the normalization constant**

(1) we have $\langle\psi(0)|\psi(0)\rangle = N^*N(\langle 4| + \langle 3|)(|3\rangle + |4\rangle) = 2N^2 = 1$

$\Rightarrow N$ can be $\frac{1}{\sqrt{2}}$

write down the t=0 normalized state vector in Dirac notation

(2) the normalized wave function at $t=0$ is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle) \quad \text{——— ①}$$

write down the corresponding time-dependent state vector in Dirac notation

(3) Assuming the ground state is $|0\rangle$, we have the time-dependent state vector

$|\psi(t)\rangle$ be : $$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|3\rangle e^{-\frac{iE_3t}{\hbar}} + |4\rangle e^{-\frac{iE_4t}{\hbar}}\right), \text{ where} \quad \text{——— ②}$$

insert the time-dependent phase factors

$$E_3 = (3+\tfrac{1}{2})\hbar\omega = \tfrac{7}{2}\hbar\omega$$
$$E_4 = (4+\tfrac{1}{2})\hbar\omega = \tfrac{9}{2}\hbar\omega$$

write down the corresponding position-space wavefunction

(4) Using $\langle x|\psi(t)\rangle = \psi(x,t)$ we have

$$\psi(x,t) = \langle x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\langle x|3\rangle e^{-\frac{iE_3t}{\hbar}} + \langle x|4\rangle e^{-\frac{iE_4t}{\hbar}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\psi_3(x) e^{-\frac{iE_3t}{\hbar}} + \psi_4(x) e^{-\frac{iE_4t}{\hbar}}\right)$$

✓ (b) (1) Since $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$ contains only eigenstates corresponding to E_3 and E_4,

the possibilities and the probabilities of energy measurements

the possible energy measurements are only :

$$E_3 = \frac{7}{2}\hbar\omega \quad , \text{ with probability } |\langle 3|\psi(0)\rangle|^2 = \frac{1}{2}$$

$$E_4 = \frac{9}{2}\hbar\omega \quad , \quad \text{- - - -} \quad |\langle 4|\psi(0)\rangle|^2 = \frac{1}{2}$$

the time-dependent statistics for energy measurements

(2) $\langle E(t)\rangle = \langle\psi|H|\psi\rangle = \langle\psi(t)|\left(\frac{1}{\sqrt{2}}E_3|3\rangle e^{-\frac{iE_3t}{\hbar}} + \frac{1}{\sqrt{2}}E_4|4\rangle e^{-\frac{iE_4t}{\hbar}}\right)$

$=$ (using $e^{i\gamma}\cdot e^{-i\gamma}=1$ and $\langle n|m\rangle=\delta_{nm}$ $\quad \frac{1}{2}E_3+\frac{1}{2}E_4 = \frac{1}{2}\left(\frac{7}{2}+\frac{9}{2}\right)\hbar\omega$

compute < E(t) > $= 4\hbar\omega$ ※

(3) $\Delta E(t)^2 = \langle H^2\rangle - \langle H\rangle^2 = \langle\psi(t)|\left(\frac{1}{\sqrt{2}}E_3^2|3\rangle e^{-\frac{iE_3t}{\hbar}} + \frac{1}{\sqrt{2}}E_4^2 e^{-\frac{iE_4t}{\hbar}}|4\rangle\right) - 16\hbar^2\omega^2$

$= \frac{1}{2}E_3^2 + \frac{1}{2}E_4^2 - 16(\hbar\omega)^2 = \frac{1}{4}\hbar^2\omega^2$

$\Delta E(t) = \frac{1}{2}\hbar\omega$ ※

compute the uncertainty in the energy

[H, H] = 0

(4) Since the energy E is the eigenvalue of the stationary states of the system, we expect that $\langle E\rangle$ and ΔE will not change with t. Our calculation in (2), (3) verify this.

superposition of stationary states

the possibilities and the probabilities for position measurements

✓ (c)

(1) Since the potential energy is not infinite (unless at $x \to \pm\infty$), the possible measurement of position

possibilities can be $X \in [-\infty, \infty]$, with probability density be

probability density

$$|\langle x|\psi(0)\rangle|^2 = \frac{1}{2}\left(\psi_3(x) + \psi_4(x)\right)^2, \text{ where}$$

$\psi_n(x)$ is the stationary energy eigenstate in position space

(2) using $X = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a + a^+)$, where a, a^+ are the lowering/raising operators,

we have

compute < x(t) >

$$\langle X(t)\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)\left(1e^{\frac{it}{\hbar}(E_3 - E_4)} + 1e^{\frac{it}{\hbar}(E_4 - E_3)}\right)$$

$$= \left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\cos\left(\frac{t}{\hbar}(E_4 - E_3)\right) = \left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\cos(\omega t)$$

(3) We have $\langle X^2\rangle = \frac{\hbar}{2m\omega}\langle\psi(t)|(a+a^+)^2|\psi(t)\rangle = \frac{\hbar}{2m\omega}\langle(a+a^+)\psi(t)|(a+a^+)\psi(t)\rangle$

compute < x(t) x(t) >

$$= \frac{\hbar}{2m\omega}\left|\frac{1}{\sqrt{2}}e^{-\frac{iE_3t}{\hbar}}\left(\sqrt{3}|2\rangle+2|4\rangle\right)+\frac{1}{\sqrt{2}}e^{-\frac{iE_4t}{\hbar}}\left(2|3\rangle+\sqrt{5}|5\rangle\right)\right|^2$$

$$= \frac{\hbar}{4m\omega}(7+9) = \frac{4\hbar}{m\omega}$$

Thus, $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{4\hbar}{m\omega} - \frac{2\hbar}{m\omega}\cos^2(\omega t)$

compute the uncertainty in the position

$$= \frac{2\hbar}{m\omega}(1+\sin^2\omega t)$$

$$\Rightarrow \Delta x = \sqrt{\frac{2\hbar}{m\omega}}(1+\sin^2\omega t)^{\frac{1}{2}}$$

do the simulations agree?

(4) The simulations do agree with my calculation (both $\langle x\rangle$, Δx are periodic, with the frequency of Δx is twice as frequency of $\langle x\rangle$.

✓ (d) **the possibilities and the probabilities for momentum measurements**

(1) From the symmetric of x, p in the Hamiltonian of H we know that $\psi_n(p) = \langle p | n \rangle$ is only a translation of $\psi_n(x)$ to $\psi_n\left(\frac{p}{m\omega}\right)$. Thus, from (c) we know that

possibilities the possible measurement of momentum can be $p \in [-\infty, \infty]$, with the probability

probability density density be $|\langle p | \psi(0) \rangle|^2 = \frac{1}{2}\left(\hat{\psi}_3\left(\frac{p}{m\omega}\right) + \hat{\psi}_4\left(\frac{p}{m\omega}\right)\right)^2$, where

$\psi_n(x)$ is the nth eigenstates in position space

(2) using $p = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}(a^+ - a)$, we have

compute < p(t) > $\langle p(t) \rangle = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} \langle \psi(0) | (a^+ - a) | \psi(0) \rangle$

$$= i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}\left(e^{i(E_4 - E_3)\frac{t}{\hbar}} - e^{-i(E_4 - E_3)\frac{t}{\hbar}}\right)$$

$$= 2i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} i\sin(\omega t) = -2\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}\sin(\omega t) \ \ast$$

(3) for $\langle p^2 \rangle$, Using $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ we have

$$\langle H \rangle = 4\hbar\omega = \frac{1}{2m}\langle p^2 \rangle + \frac{1}{2} m\omega^2 \langle x^2 \rangle$$

compute < p(t) p(t) > (he should have used ladder operators)

$$= \frac{1}{2m}\langle p^2 \rangle + \frac{1}{2} m\omega^2 \frac{4\hbar}{m\omega} = \frac{1}{2m}\langle p^2 \rangle + 2\hbar\omega$$

$$\Rightarrow \langle p^2 \rangle = 2m\left(4\hbar\omega - 2\hbar\omega\right) = 4m\hbar\omega$$

Thus, we have $\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 = 4m\hbar\omega - 2m\hbar\omega \sin^2\omega t$

$$= 2m\hbar\omega\left(1 + \cos 2\omega t\right)$$

compute the uncertainty in the momentum

$$\Rightarrow \Delta p = \sqrt{2m\hbar\omega}\left(1 + \cos 2\omega t\right)^{\frac{1}{2}}$$

do the simulations agree?

(4) The simulation agrees with my expressions of $\langle p(t) \rangle$ and $\Delta p(t)$

(both $\langle p \rangle$, Δp are periodic, with frequency $\omega_{(\Delta p)} = 2\,\omega_{(\langle p \rangle)}$.
There is a $\frac{\pi}{2}$ phase difference between $\langle p \rangle$ and $\langle x \rangle$, which is consistent to the $\sin\omega t$ and $\cos\omega t$ expressions

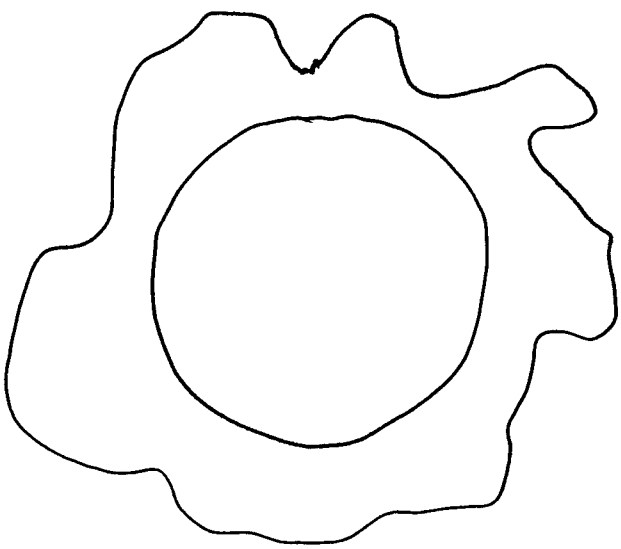
Lecture 12 Angular Momentum

Math: **Spherical harmonics are a complete set of orthonormal functions on the sphere**

Physics: **Spherical harmonics describe the orbital angular momentum**

MATH : SPHERICAL HARMONICS ARE A COMPLETE ORTHONORMAL SET OF BASIS FUNCTIONS ON THE SPHERE.

PHYSICS : SPHERICAL HARMONICS DESCRIBE THE ORBITAL ANGULAR MOMENTUM



$$\text{ANY FCN}(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=+\ell} a_{\ell m} Y_{\ell m}(\theta,\varphi)$$

$$a_{\ell m} = \int \text{ANY FCN}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi)\, d\Omega$$

$$\langle\theta,\varphi|\,|A\rangle = \overset{\langle\theta,\varphi|}{\sum_{\ell}} \sum_{m} |\ell,m\rangle\langle\ell,m|A\rangle$$

Spherical Harmonics

The Meaning of the Spherical Harmonics
http://infovis.uni-konstanz.de/research/projects/SimSearch3D/images/harmonics_img.jpg

The Spherical Harmonics
http://oak.ucc.nau.edu/jws8/dpgraph/Yellm.html
http://www.bpreid.com/applets/poasDemo.html
http://www.du.edu/~jcalvert/math/harmonic/harmonic.htm

Encyclopedia
http://en.wikipedia.org/wiki/Spherical_harmonics
http://en.wikipedia.org/wiki/Table_of_spherical_harmonics
http://mathworld.wolfram.com/SphericalHarmonic.html

Applications of Spherical Harmonics
http://www.falstad.com/qmrotator/
http://www.falstad.com/qmatom/
http://www.falstad.com/qmatomrad/
http://www.falstad.com/qm2dosc/
http://www.falstad.com/qm3dosc/

Legendre Polynomials

The Meaning of the Legendre Polynomials
http://physics.unl.edu/~tgay/content/multipoles.html

Encyclopedia
http://en.wikipedia.org/wiki/Legendre_polynomials
http://mathworld.wolfram.com/LegendrePolynomial.html

Wolfram Demonstrations
http://demonstrations.wolfram.com/SphericalHarmonics/
http://demonstrations.wolfram.com/VisualizingAtomicOrbitals/
http://demonstrations.wolfram.com/HydrogenOrbitals/
http://demonstrations.wolfram.com/PlotsOfLegendrePolynomials/
http://demonstrations.wolfram.com/PolarPlotsOfLegendrePolynomials/
http://demonstrations.wolfram.com/DipoleAntennaRadiationPattern/

Other Examples

The Earth's Magnetic Field

http://cgc.rncan.gc.ca/geomag/nmp/early_nmp_e.php?p=1
http://en.wikipedia.org/wiki/Earth%27s_magnetic_field
http://www.ngdc.noaa.gov/geomag/WMM/DoDWMM.shtml
http://www.geomag.us/info/Declination/magnetic_lines_2010.gif

The Earth's Gravitional Field

http://en.wikipedia.org/wiki/Geoid
http://www.esri.com/news/arcuser/0703/graphics/geoid1_lg.gif
http://www.geomag.us/models/pomme5.html
http://earth-info.nga.mil/GandG/images/ww15mgh2.gif
http://op.gfz-potsdam.de/champ/
http://www.gfy.ku.dk/~pditlev/annual_report/matematiker.jpg

The Universe

http://abyss.uoregon.edu/~js/21st_century_science/lectures/lec27.html
http://wmap.gsfc.nasa.gov/media/080997/080997_5yrFullSky_WMAP_4096B.tif

Computer Lighting and Games

https://buffy.eecs.berkeley.edu/PHP/resabs/images/2006//101194-5.jpg
http://www.cg.tuwien.ac.at/research/publications/2008/Habel_08_SSH/
http://www.planetlara.com/underworld/renders/lara/full.jpg
http://casuallyhardcore.com/blog/index.php?s=shader

Art

http://www.math.hawaii.edu/~dale/bleecker/bleecker.html
http://cricketdiane.files.wordpress.com/2009/04/cricketdiane-castle-in-the-sky-2006-1.jpg

The Meaning of the Legendre Polynomials
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http://www.geomag.us/info/Declination/magnetic_lines_2010.gif

The Earth's Gravitional Field
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http://www.esri.com/news/arcuser/0703/graphics/geoid1_lg.gif
http://www.geomag.us/models/pomme5.html
http://earth-info.nga.mil/GandG/images/ww15mgh2.gif
http://op.gfz-potsdam.de/champ/
http://www.gfy.ku.dk/~pditlev/annual_report/matematiker.jpg

The Universe
http://abyss.uoregon.edu/~js/21st_century_science/lectures/lec27.html
http://www.asiaa.sinica.edu.tw/~lychiang/index/node10.html
http://wmap.gsfc.nasa.gov/media/080997/080997_5yrFullSky_WMAP_4096B.tif

Computer Lighting and Games
https://buffy.eecs.berkeley.edu/PHP/resabs/images/2006//101194-5.jpg
http://www.cg.tuwien.ac.at/research/publications/2008/Habel_08_SSH/
http://www.planetlara.com/underworld/renders/lara/full.jpg
http://casuallyhardcore.com/blog/index.php?s=shader

Art
http://www.math.hawaii.edu/~dale/bleecker/bleecker.html
http://cricketdiane.files.wordpress.com/2009/04/cricketdiane-castle-in-the-sky-2006-1.jpg