February 15, 2012

Simple Harmonic Oscillator 2

(0) Differential Equation (last time)

(1) Qualitative Aspects

(2) Algebraic Solution

Use the Algebra of the Operators

The Operators are H, x, p, a+, a-

The Algebra is in the Commutators

Introduce the Ladder Operators

a+ and a- aka $a^\dagger$ and a

aka, raising and lowering operators

aka, creation and destruction operators

aka, creation and annihilation operators

Evaluate [H, a+], [H, a-], [a+, a-]

By reducing them to [x, p] = i hbar

Find the eigenenergies

Find the eigenkets

Translate into position space

LECTURE 19 7-24-07

SOLVED TISE IN POSITION SPACE

$$H|n\rangle = E_n|n\rangle$$

OPERATOR IN x-SPACE

EIGENENERGIES

WAVE FUNCTIONS $\varphi_n(x)$ IN x-SPACE

FOUND $E_n = (n + 1/2)\hbar\omega \qquad n = 0, 1, 2, 3, \ldots$

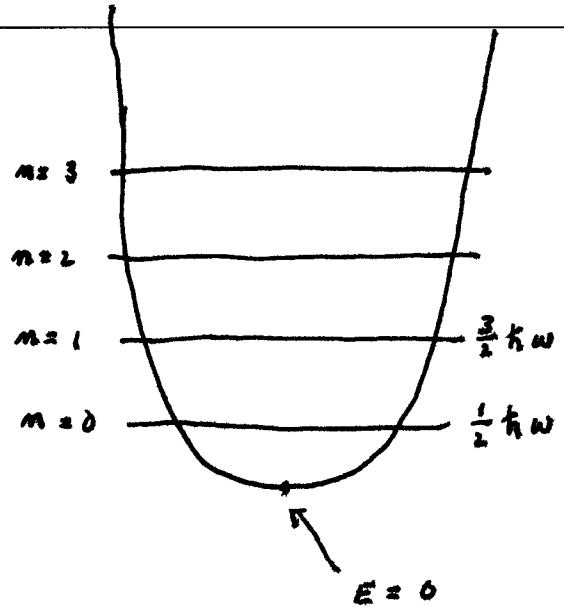
$$\varphi_n(y) = H_n(y)\, e^{-y^2/2}$$

STATIONARY STATES AKA ENERGY EIGENFUNCTIONS

HERMITE POLYNOMIALS

GAUSSIAN

Solution in Position Space obtained by solving the TISE differential equation



LADDER OF EQUALLY SPACED STATES

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

SPACING = $\hbar\omega$

GROUND STATE

$$E_0 = \frac{1}{2}\hbar\omega$$

ZERO POINT ENERGY

ZERO POINT MOTION

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

$\Rightarrow$ SYSTEM CAN NEVER STOP

Heisenberg Uncertainty

what the potential wants

what the KE wants

FIND TWO LINEAR OPERATORS

a^+ = RAISING OPERATOR = CREATION OPERATOR

a = LOWERING OPERATOR = DESTRUCTION OP

IN FIELD THEORY

PHONONS

PHOTONS

ELECTRONS

⋮

FEYNMAN STORY

The Qualitative Aspects of the SHO

The Dirac Delta Function

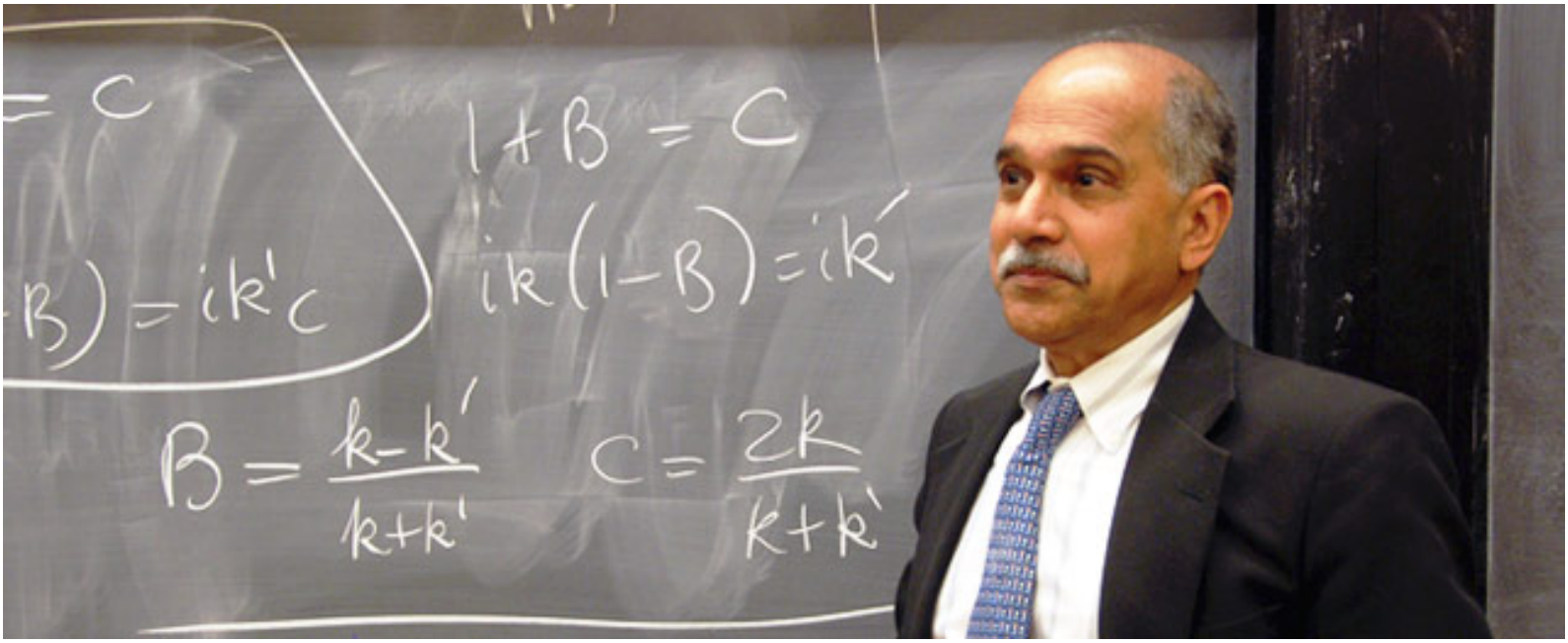
http://demonstrations.wolfram.com/RepresentationsOfTheDiracDeltafunction/

The Harmonic Oscillator

http://en.wikipedia.org/wiki/Quantum_harmonic_oscillator
http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/hosc6.html#c2
http://demonstrations.wolfram.com/QuantumClassicalCorrespondenceForTheHarmonicOscillator/
http://www-personal.umich.edu/~lorenzon/java_applets/spaceholder/applets/SHO-QM-example.html?D1=5
http://www.quantum-physics.polytechnique.fr/en/ Section 3
http://www.falstad.com/qm1d/
http://www.falstad.com/qm2dosc/
http://demonstrations.wolfram.com/EnergyLevelsOfAQuantumHarmonicOscillatorInSecondQuantization/
http://demonstrations.wolfram.com/CoherentStatesOfTheHarmonicOscillator/

The Hermite Polynomials

http://en.wikipedia.org/wiki/Hermite_polynomials
http://functions.wolfram.com/Polynomials/


= C
1 + B = C
ik(1 − B) = ik′
B = (k − k′)/(k + k′)
C = 2k/(k + k′)

Once we have bitten the quantum apple,
our loss of innocence is permanent.

In this section, I will put the harmonic oscillator
in its place—on a pedestal.

R. Shankar
Principles of Quantum Mechanics

http://boulder.research.yale.edu/Boulder-2008/Lectures/index.html#today
http://streaming.yale.edu:8080/ramgen/cmibroadcast/boulder/lectures/publecture.rm

Every competent physicist can"do"quantum mechanics,
but the stories we tell ourselves are as varied as
Scheherazade, and almost as implausible.

David Griffiths
Introduction to Quantum Mechanics

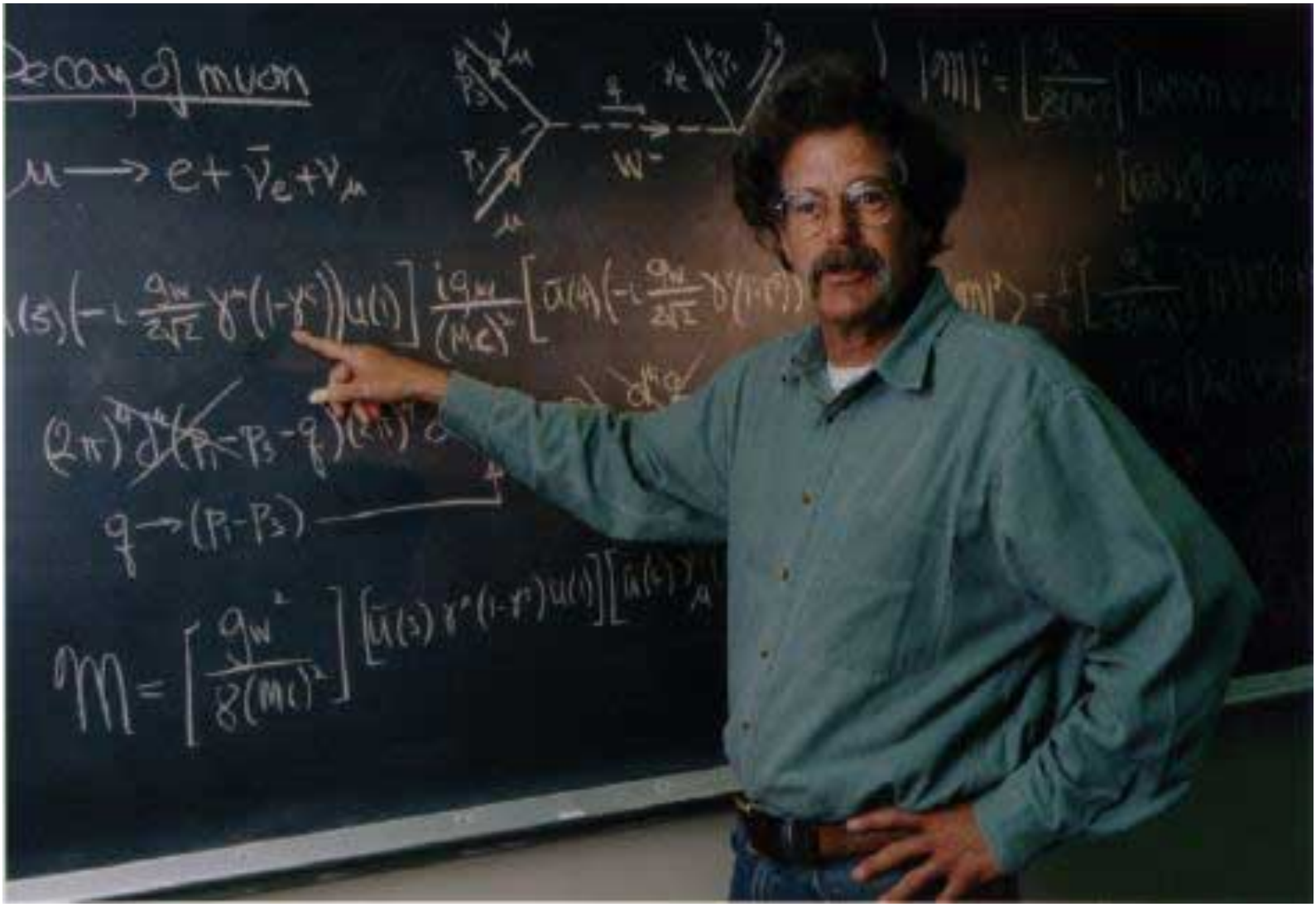
http://www.youtube.com/watch?v=xip-uGQx3gk

We have always had a great deal of difficulty understanding the
world view that quantum mechanics represents. At least I do,
because I'm an old enough man that I haven't got to the point
that this stuff is obvious to me. Okay, I still get nervous about it...
You know how it always is, every new idea, it takes a
generation or two until it is obvious that there's no real problem.
I cannot define the real problem, therefore I suspect there's no
real problem, but I'm not sure there's no real problem."

Richard Feynman (1982)

http://www.youtube.com/watch?v=J545tIw55bE


LEMMA 2
Proof:


Decay of muon
μ → e + ν̄e + νμ
W⁻
q → (p1 − p3)

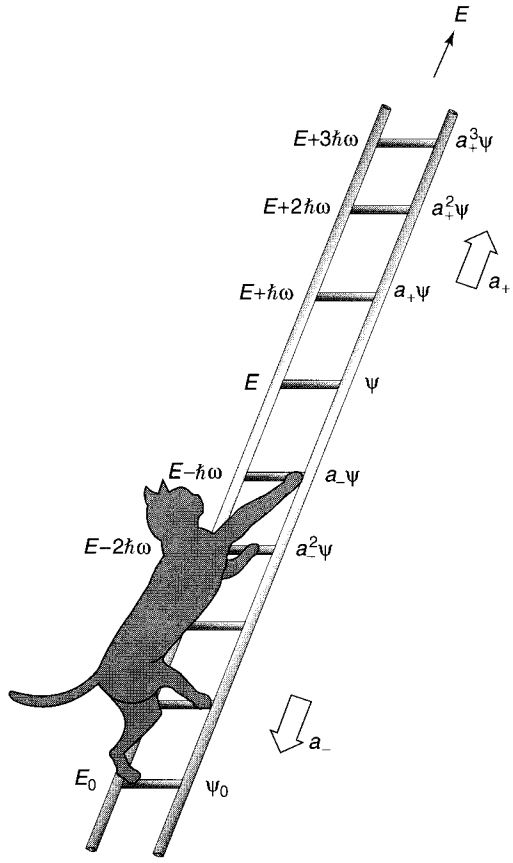


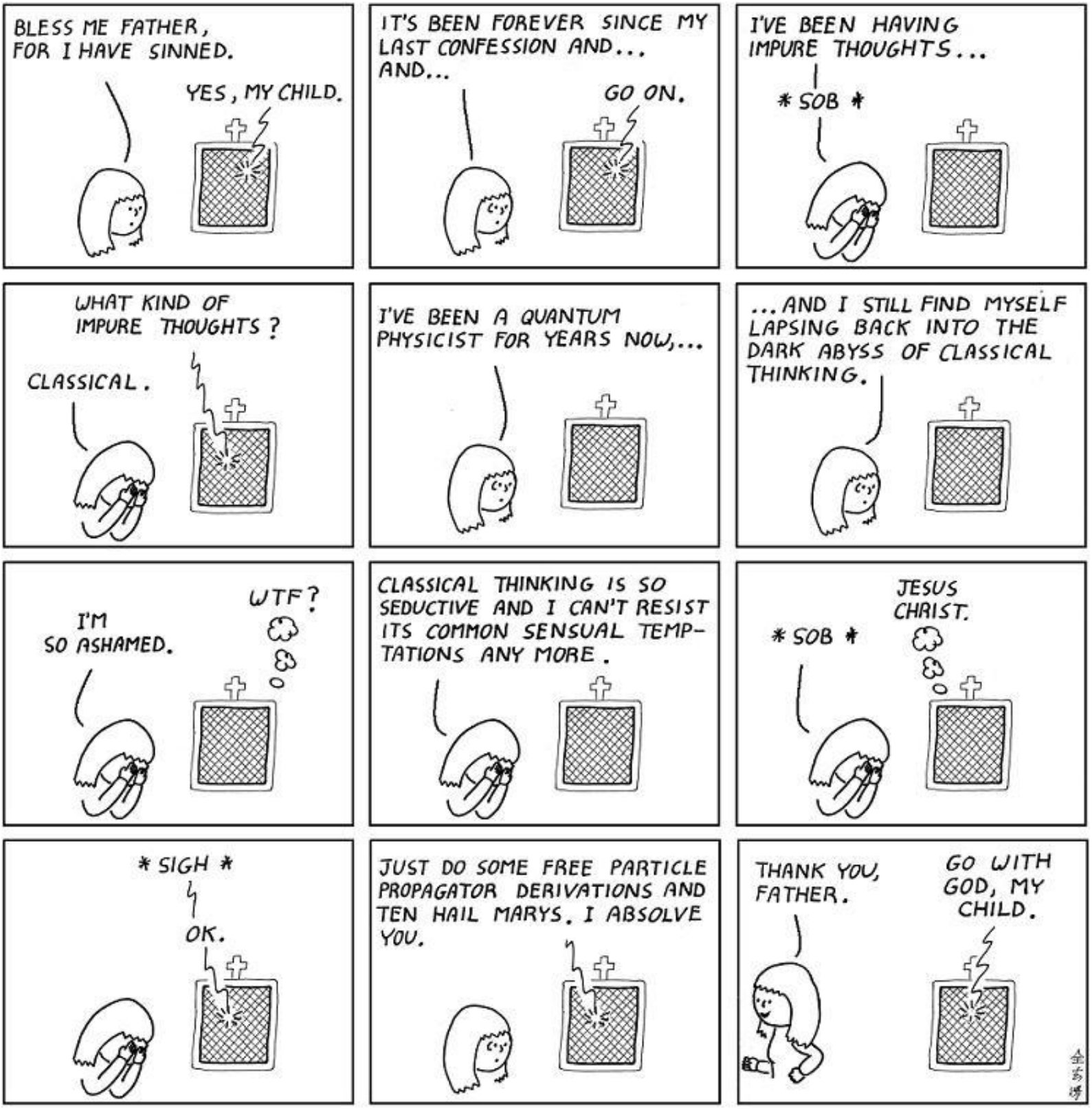
FIGURE 2.5: The "ladder" of states for the harmonic oscillator.

But wait! What if I apply the lowering operator repeatedly? Eventually I'm going to reach a state with energy less than zero, which (according to the general theorem in Problem 2.2) does not exist! At some point the machine must fail. How can that happen? We know that $a_-\psi$ is a new solution to the Schrödinger equation, but *there is no guarantee that it will be normalizable*—it might be zero, or its square-integral might be infinite. In practice it is the former: There occurs a "lowest rung" (call it ψ_0) such that

$$a_-\psi_0 = 0. \qquad [2.58]$$

We can use this to determine $\psi_0(x)$:

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0,$$


BLESS ME FATHER, FOR I HAVE SINNED.
YES, MY CHILD.
IT'S BEEN FOREVER SINCE MY LAST CONFESSION AND... AND...
GO ON.
I'VE BEEN HAVING IMPURE THOUGHTS...
* SOB *
WHAT KIND OF IMPURE THOUGHTS?
CLASSICAL.
I'VE BEEN A QUANTUM PHYSICIST FOR YEARS NOW,...
...AND I STILL FIND MYSELF LAPSING BACK INTO THE DARK ABYSS OF CLASSICAL THINKING.
I'M SO ASHAMED.
WTF?
CLASSICAL THINKING IS SO SEDUCTIVE AND I CAN'T RESIST ITS COMMON SENSUAL TEMPTATIONS ANY MORE.
* SOB *
JESUS CHRIST.
* SIGH *
OK.
JUST DO SOME FREE PARTICLE PROPAGATOR DERIVATIONS AND TEN HAIL MARYS. I ABSOLVE YOU.
THANK YOU, FATHER.
GO WITH GOD, MY CHILD.

The Ladder Operator Method

a = [a,H] = aH-Ha—oh, my, gotcha!!!

The ladder operator solution to the simple harmonic oscillator problem is subtle, exquisite, and rather slippery—so I thought you might appreciate a recapitulation of what I said in class . . . . You might want to go through the argument line-by-line until it clicks!

There were three steps in the argument:

1

1. The first step was to show that the eigenvalues of the Hamiltonian H are equal to $\hbar\omega$ times $\frac{1}{2}$ plus the eigenvalues of the number operator $N = a^\dagger a$ (which will turn out to be n, so we will end up with $(n + \frac{1}{2})\hbar\omega$). We did this by showing that the Hamiltonian H is $\hbar\omega$ times the sum of the number operator plus one half the identity operator,

$$H \;=\; \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega.$$

find the eigenvalue of the ground state

We showed this by defining the $a^\dagger$ and a operators, and then calculating $a^\dagger a$. Note that once we found that $H = (a^\dagger a + \frac{1}{2})\,\hbar\omega$, we immediately knew that the eigenvectors of H would be the same as the eigenvalues of $a^\dagger a$—because every vector is an eigenvector of the identity operator! We also immediately knew that the eigenvalues of H would be equal to $\hbar\omega$ times the eigenvalues of the $a^\dagger a$ operator plus $\frac{1}{2}\hbar\omega$.

2

2. The second step was to show that when the $a^\dagger$ and a operators act on any eigenvector of H, we get back another eigenvector of H one step up or down the ladder of states. We showed this by calculating the three commutators:

$$\begin{aligned}[a, a^\dagger] &= +1\\ [a, H] &= +a\\ [a^\dagger, H] &= -a^\dagger\end{aligned}$$

find all of the other eigenvalues

and considering the action of the last two commutators on any eigenvector of the Hamiltonian

$$\begin{aligned}[a, H]\;|\text{eigenvector of H}> &= +a\;\;|\text{eigenvector of H}>\\ [a^\dagger, H]\;|\text{eigenvector of H}> &= -a^\dagger\;|\text{eigenvector of H}>\,.\end{aligned}$$

By expanding the commutators, we found

$$\begin{aligned}\left(aH - Ha\right)\;|\text{eigenvector of H}> &= +a\;\;|\text{eigenvector of H}>\\ \left(a^\dagger H - Ha^\dagger\right)\;|\text{eigenvector of H}> &= -a^\dagger\;|\text{eigenvector of H}>\,.\end{aligned}$$

which allowed us to conclude that

$$\begin{aligned}a\;|\epsilon> &= (\epsilon - 1)\;|\epsilon - 1>\\ a^\dagger\;|\epsilon> &= (\epsilon + 1)\;|\epsilon + 1>\,.\end{aligned}$$

This showed us that the eigenvalues of H are separated by $\pm\hbar\omega$. Combining this with the $\frac{1}{2}\hbar\omega$ from step one, we then concluded that the eigenvalues of the Hamiltonian are given by

$$E_n \;=\; \left(n + \frac{1}{2}\right)\hbar\omega$$

where n is any integer (*i.e.,* positive, negative, or zero!!!). However, in step three, we found the smallest eigenvalue of the number operator is equal to zero.

So the only other thing we did not know yet was whether the raising and lowering operators return normalized eigenvectors of the Hamiltonian, *i.e.,* are the vectors $a|\epsilon>$ and $a^\dagger|\epsilon>$ normalized eigenvectors of H? We did know that they are eigenvectors of H with eigenvalues of $(\epsilon-1)\hbar\omega$ and $(\epsilon+1)\hbar\omega$, respectively, but we did not know whether they are normalized—and, in fact, they are not!

3

3. The third step was to calculate the normalization coefficients. To do this we started with two adjacent normalized states, $|n>\equiv|E=(n+\frac{1}{2})\hbar\omega>$ and $|n-1>\equiv|E=((n-1)+\frac{1}{2})\hbar\omega>$ and then we calculated the expectation value of the number operator in two different ways:

(i) First, we started with the lowering operator equation

calculate the normalization factors for a+ and a-

$$a\;|n> = \;c_n\;|n-1>$$

and then we calculated the adjoint of this equation

$$<n|\;a^\dagger \;=<n-1|\;c_n^*.$$

We combined these to evaluate the expectation value of the number operator

$$\begin{aligned} <n|\;a^\dagger a\;|n> \;&=<n-1|\;c_n^* c_n\;|n-1> = \;|c_n|^2\;\;<n-1|n-1> \\ &= \;|c_n|^2. \end{aligned}$$

(ii) Second, we replaced $a^\dagger a$ by $\widehat{H}-\frac{1}{2}$ and recalculated the expectation value of N

$$\begin{aligned} <n|\;\widehat{H}-\frac{1}{2}\;|n> \;&=<n-1|\;\left[\;(n+\frac{1}{2})+\frac{1}{2}\;\right]\;|n> = \;n\;\;<n|n> \\ &= \;n. \end{aligned}$$

By combining these two calculations, we found

$$\begin{aligned} |c_n|^2 = n \;&\Rightarrow\; c_n = \sqrt{n} \\ \Rightarrow\; a\;|n> = \sqrt{n}\;|n-1>\;. \end{aligned}$$

Finally, to see that the lowest eigenvalue of the number operator is zero, we considered

$$a\;|0> = \;\sqrt{0}\;|0-1> = \;0\;|-1> = \;\vec{0}.$$

So $|0>$ is the bottom rung on the ladder (lowering it we obtain the zero vector), and consequently the lowest eigenvalue of H is $\frac{1}{2}\hbar\omega$, which is the zero point energy of the oscillator.

(2) OPERATOR METHOD

$$H\,|E_n\rangle = E_n\,|E_n\rangle$$

$$\left[\frac{p_{op}^2}{2m} + \frac{1}{2} m\omega^2 x_{op}^2\right]|E_n\rangle = E_n\,|E_n\rangle$$

USE THE ALGEBRA OF THE OPERATORS

BASICALLY, REDUCE EVERYTHING BACK TO THE FUNDAMENTAL COMMUTATION RELATIONSHIP

canonical commutation relation

$$[x_{op}, p_{op}] = i\hbar I = i\hbar$$

THIS IS BASIS INDEPENDENT!

x BASIS

$$x_{op}\,p_{op} - p_{op}\,x_{op} \rightarrow x\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)x$$

$$= i\hbar\left[\cancel{-x\frac{\partial}{\partial x}} + \frac{\partial x}{\partial x} + \cancel{x\frac{\partial}{\partial x}}\right]$$

$$= i\hbar$$

P BASIS

$$x_{op}\, p_{op} - p_{op}\, x_{op} \rightarrow \left(i\hbar \frac{\partial}{\partial p}\, p\right) - \left(p\, i\hbar \frac{\partial}{\partial p}\right)$$

$$= i\hbar \left[\frac{\partial p}{\partial p} + \cancel{p \frac{\partial}{\partial p}} - \cancel{p \frac{\partial}{\partial p}} \right]$$

$$= i\hbar$$

Factor the Hamiltonian Operator

IDEA : FACTOR H

FACTORIZATION METHOD : OLD
LADDER OPERATOR METHOD : NEW

$$H = p^2 + x^2 = (x + ip)(x - ip)$$

INTRODUCE LADDER OPERATORS

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x_{op} + i \left(\frac{1}{2m\omega\hbar}\right)^{1/2} p_{op}$$

$$a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\, x_{op} - i \left(\frac{1}{2m\omega\hbar}\right)^{1/2} p_{op}$$

The Ladder Operators

NOW x_{op} AND p_{op} ARE HERMITEAN,

BUT a AND a^+ ARE NOT! $\Rightarrow$ a and a^+ are not observable

$$x \longleftrightarrow p$$

$$m\omega \qquad\qquad \frac{1}{m\omega}$$

EXPRESS H IN TERMS OF a AND a^+

$$a^+ a = [c\, x_{op} - i d\, p_{op}][c\, x_{op} + i d\, p_{op}]$$

$$= c^2 x_{op}^2 - i^2 d^2 p_{op}^2 + i c d\,(x_{op} p_{op} - p_{op} x_{op})$$

$$= \frac{m\omega}{2\hbar} x_{op}^2 + \frac{1}{2m\omega\hbar} p_{op}^2$$

$$+ i\sqrt{\frac{m\omega}{2\hbar}\,\frac{1}{2m\omega\hbar}}\,\underbrace{[x_{op}, p_{op}]}_{i\hbar}$$

$$\boxed{a^+ a = \frac{H_{op}}{\hbar\omega} - \frac{1}{2}}$$

Hamiltonian in terms of the Ladder Operators

Step 1: Find the eigenvalues of the Hamiltonian

$$\boxed{H = \left(a^{+} a + \frac{1}{2}\right) \hbar \omega}$$

define $\hat{H} = \dfrac{H}{\hbar \omega}$ dimensionless Hamiltonian

dimensionless Hamiltonian

$$\boxed{\hat{H} = a^{+} a + \frac{1}{2}}$$

$E \rightarrow \epsilon = \dfrac{E}{\hbar \omega}$

SOLVE ~~[illegible]~~ eV, e$\vec{V}$ PROBLEM for $\hat{H}$, then translate back to H

$$\boxed{\hat{H} \, | \epsilon_n \rangle = \epsilon_n | \epsilon_n \rangle}$$

Solve TISE in Hilbert space

Do everything in Dirac notation

NOW WE CAN (and WILL) SOLVE THIS USING ONLY THE ALGEBRA OF THE COMMUTATORS

need 3 commutators

$$[a, a^{+}] = ?$$

$$[a, \hat{H}] = ?$$

$$[a^{+}, H] = ?$$

Step 2: Calculate the three commutators

reduce each one to ~~[illegible]~~ $[x_{op}, p_{op}]$

FIRST ONE

$$[a, a^+] = a a^+ - a^+ a$$ **commutator 1**

$$= (cx + idp)(cx - idp) - (cx - idp)(cx + idp)$$

$$= [c^2 x^2 + d^2 p^2 - icd\,(xp - px)]$$

$$- [c^2 x^2 + d^2 p^2 + icd\,(xp - px)]$$

$$[a, a^+] = -2i\,cd\,[x, p]$$

$$= -2i \sqrt{\frac{m\omega}{2\hbar}\,\frac{1}{2m\omega\hbar}}\; i\hbar$$

$$\boxed{[a, a^+] = I = 1}$$ **commutator 1**

SECOND ONE

$$[a, \hat{H}] = [a, a^+ a + 1/2]$$ **commutator 2**

$$= [a, a^+ a]$$

$$= a\,a^+ a - a^+ a\,a = [a, a^+]\,a$$

$$\boxed{[a, \hat{H}] = a}$$ **commutator 2**

THIRD ONE

$$[a^+, \hat{H}] = [a^+, a^+a + \tfrac{1}{2}]$$ **commutator 3**

$$= [a^+, a^+a]$$

$$= a^+ \underbrace{a^+a} - a^+ \underbrace{a a^+}$$

$$= a^+ \underbrace{[a^+, a]}_{-1}$$

$$\boxed{[a^+, \hat{H}] = -a^+}$$ **commutator 3**

WHAT GOOD ARE a AND a^+?

GIVEN ANY EIGENSTATE OF $\hat{H}$,

a AND a^+ GENERATE THE ADJACENT

STATES

Show that

a^+	goes up the ladder	**Raising Operator**
a	goes down the ladder	**Lowering Operator**

TO SEE THIS, CONSIDER AN EV OF $\hat{H}$

$$\hat{H}\,|\epsilon\rangle = \epsilon\,|\epsilon\rangle$$

CALCULATE

$$a\,|\epsilon\rangle = \;?$$

$$a^{+}\,|\epsilon\rangle = \;?$$

Using a = aH-Ha

we know what H does to its eigenkets

we know how H is related to a

use this to determine what a does to the eigenkets of H

$$\hat{H}\left[a\,|\epsilon\rangle\right] = \left(a^{+}a + \tfrac{1}{2}\right)\left[a\,|\epsilon\rangle\right]$$

$$\boxed{[a, H] = aH - Ha = a \;\Rightarrow\; Ha = aH - a}$$

$$H\,a\,|\epsilon\rangle = \left[aH - a\right]|\epsilon\rangle$$

we want to know what Ha does to an eigenket of H

we know what H does to one of its eigenkets

$$= \left[a\,\epsilon - a\right]|\epsilon\rangle$$

$$= (\epsilon - 1)\left[a\,|\epsilon\rangle\right]$$

$\Rightarrow$ $a\,|\epsilon\rangle$ is an ev of H with ev $\epsilon - 1$!

$$a\,|\epsilon\rangle \propto |\epsilon - 1\rangle$$

$$a\,|\epsilon\rangle = C(\epsilon)\,|\epsilon - 1\rangle$$

norm factor

NORM COEFF

$$C(E) = \sqrt{n}$$

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$$

normalized lowering operator

IN JUST THE SAME WAY

$$[a^+, H] = a^+H - Ha^+ = -a^+$$

$$Ha^+ = a^+H + a^+$$

$$a^+|\epsilon\rangle = D(\epsilon)\,|\epsilon+1\rangle$$

WE WILL SHOW

$$D(\epsilon) = \sqrt{n+1}$$

$$\boxed{a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

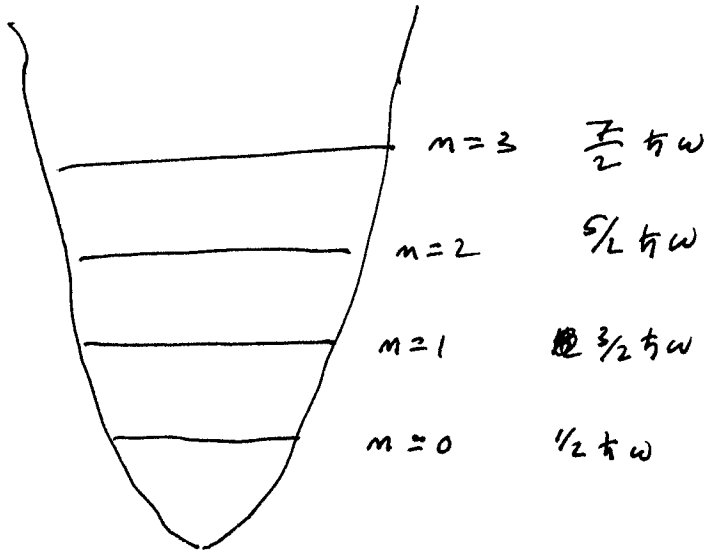
normalized raising operator

NOTATION:

$|n\rangle$ HAS EIGENVALUE $(n + 1/2)$

$$\hat{H}\,|n\rangle = (n + 1/2)\,|n\rangle$$

$$H\,|n\rangle = (n + 1/2)\hbar\omega\,|n\rangle$$



∞ number of states

$\frac{1}{2}\hbar\omega$

g.s. energy $\neq 0$!

ZERO POINT ~~MOTION~~ ENERGY

$\Rightarrow$ ZERO POINT MOTION

Heisenberg Uncertainty

what the potential wants

what the KE wants

TO COMPLETE THE SOLUTION, WE NEED TO FIND THE GROUND STATE

$$a\,|gs\rangle = a\,|\epsilon_0\rangle = \vec{0}$$

$$a^+\,\vec{0} = \vec{0}$$

$$a^+ a\,|\epsilon_0\rangle = \vec{0}$$

$$\left[\hat{H} - \tfrac{1}{2}\right]|\epsilon_0\rangle = \vec{0}$$

$$\boxed{\hat{H}\,|\epsilon_0\rangle = \tfrac{1}{2}\,|\epsilon_0\rangle}$$

ϵ_0 HAS EIGEN VALUE $+\frac{1}{2}$!

find the ground state energy

NOW WE CAN FIND ALL THE REST OF THE STATES BY APPLYING a^+

$$a^+\,|\epsilon_0\rangle \sim |\epsilon_0+1\rangle$$

$$a^+ a^+\,|\epsilon_0\rangle \sim |\epsilon_0+2\rangle$$

$$(a^+)^n\,|\epsilon_0\rangle \sim |\epsilon_0+n\rangle$$

can find the rest of the eigenstates by raising

WANT A NORMALIZED SET OF EIGENKETS

$$\left.\begin{array}{l} |0\rangle \\ |1\rangle \\ |2\rangle \end{array}\right\} \Rightarrow \langle n|m\rangle = \delta_{nm} \quad \text{ORTHO NORMAL}$$

want orthonormal basis

CONSIDER

compute the normalization coefficents

$$a|n\rangle = c_n|n-1\rangle$$

ADJOINT

$$\langle n|a^\dagger = \langle n-1|c_n^*$$

MAKE SANDWICH

$$\langle n|a^\dagger a|n\rangle = \langle n-1|c_n^* c_n|n-1\rangle$$

$$= |c_n|^2 \underbrace{\langle n-1|n-1\rangle}_{1}$$

$$\langle n|\hat{H} - \tfrac{1}{2}|n\rangle = \langle n|(n+\tfrac{1}{2}) - \tfrac{1}{2}|n\rangle$$

$$\boxed{|c_n|^2 = n \qquad \text{CHOOSE } c_n = \sqrt{n}}$$

In Matrix Language

LADDER OPERATORS

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{DOWN}$$

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \text{UP}$$

lowering

raising

TRANSLATE ALL THIS INTO MATRIX LANGUAGE

Lowering Operator

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a = \begin{bmatrix} 0 & \sqrt{1} & & & 0 \\ 0 & 0 & \sqrt{2} & & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ & & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{matrix} n=0 \\ n=1 \\ n=2 \end{matrix}$$

MATRIX ELEMENTS

Kronecker delta function

$$\langle n'|a|n\rangle = \langle n'|\sqrt{n}\,|n-1\rangle$$

$$= \sqrt{n}\;\delta_{n',n-1}$$

Raising Operator

$$a^{+} = \begin{bmatrix} 0 & & & & \\ \sqrt{1} & 0 & & & 0 \\ 0 & \sqrt{2} & 0 & & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \end{bmatrix}$$

matrix elements

$$\langle n' | a^{+} | n \rangle = \sqrt{n+1}\ \delta_{n', n+1}$$

Position Operator

$$x_{op} = \sqrt{\frac{\hbar}{2 m \omega}}\ (a + a^{+})$$

$$x_{op} = \begin{bmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ & & & & \end{bmatrix} \sqrt{\frac{\hbar}{2 m \omega}}$$

Momentum Operator

$$p_{op} = i\sqrt{\frac{m\hbar\omega}{2}}\left[a^{+} - a\right]$$

$$p_{op} = i\sqrt{\frac{m\hbar\omega}{2}}\begin{bmatrix} 0 & -\sqrt{1} & & \\ \sqrt{1} & 0 & -\sqrt{2} & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ & & & \end{bmatrix}$$

Hamiltonian Operator

$$H = \begin{bmatrix} 1/2 & & & & \\ & 3/2 & & & \\ & & 5/2 & & \\ & & & 7/2 & \\ & & & & \ddots \end{bmatrix} \hbar\omega$$

SUMMARY

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^{+}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$H = (a^{+}a + 1/2)\,\hbar\omega$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^{+})^{n}\,|0\rangle$$

$\longrightarrow$ generating fcn

NUMBER OPERATOR

$$a^{+}a = N$$

$$N = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$

Number Operator

To find the ground state wavefcn in position space

TO FIND $\psi_0(x)$

Apply Lowering Operator to the ground state

$$a|0\rangle = 0$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x_{op} + i\sqrt{\frac{1}{2m\omega\hbar}}\, p_{op}$$

Translate Lowering Operator into position space

$$= \sqrt{\frac{m\omega}{2\hbar}}\, x + i\sqrt{\frac{1}{2m\omega\hbar}}\left(-i\hbar\frac{\partial}{\partial x}\right)$$

CHANGE VARIABLES

$$y = \sqrt{\frac{m\omega}{2\hbar}}\, x$$

$$dy = \sqrt{\frac{m\omega}{2\hbar}}\, dx$$

Change variables to get rid of the ugly constants, at least for a while

$$a \rightarrow \frac{1}{\sqrt{2}}\left[y + \frac{d}{dy}\right]$$

$$a|0\rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}\left[y + \frac{d}{dy}\right]\psi_0(y) = 0$$

FIRST ORDER DIFF EQ!

Solve first order diff eq by integrating

$$\frac{d\,\psi_0(y)}{\psi_0(y)} = -y\,dy$$

$$\ln \psi_0 = -\frac{y^2}{2} + c$$

$$\psi_0(y) = e^{c}\, e^{-y^2/2} = A_0\, e^{-y^2/2}$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}}$$

ground state wavefcn in position space

TO CALCULATE HIGHER STATES

calculate the higher states with the Raising Operator

$$a^+ = \frac{1}{\sqrt{2}}\left[y - \frac{d}{dy}\right]$$

$$\boxed{\psi_n(y) = \frac{(a^+)^n}{\sqrt{n!}}\,\psi_0(y)}$$

apply a+ to the dimensionless equation

$$\psi_n(y) = H_n(y)\, e^{-y^2/2}$$

Gaussian

Hermite polynomials

http://demonstrations.wolfram.com/FundamentalCommutationRelationsInQuantumMechanics/

http://demonstrations.wolfram.com/SchroedingersCatOnCatnip/

http://www.fen.bilkent.edu.tr/~yalabik/applets/collapse.html