Lecture 6

Postulate 4

IV. The state variables change with time according to Hamilton's equations:

$$\dot{x} = \frac{\partial \mathscr{H}}{\partial p}$$

$$\dot{p} = -\frac{\partial \mathscr{H}}{\partial x}$$

IV. The state vector $|\psi(t)\rangle$ obeys the *Schrödinger equation*

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

where $H(X, P) = \mathscr{H}(x \to X, p \to P)$ is the quantum Hamiltonian operator and $\mathscr{H}$ is the Hamiltonian for the corresponding classical problem.

$$\boxed{\text{TIME EVOLUTION POSTULATE 4}}$$

(1) EXPAND $|\psi(0)\rangle$ IN ENERGY EIGEN BASIS:

$$\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{2}{\sqrt{14}}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{3}{\sqrt{14}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\vec{e}$'s OF H OBEY TISE

$$H\,|E_i\rangle = E_i\,|E_i\rangle$$

$$H\,|E_i\rangle = i\hbar\frac{d}{dt}|E_i\rangle = E_i\,|E_i\rangle$$

$$|E_i(t)\rangle = e^{-iEt/\hbar}\,|E_i(0)\rangle$$

TIME EVOLUTION OF $\vec{e}$'S OF H IS VERY SIMPLE: ONLY THEIR PHASE CHANGES

(2) SIMPLY WRITE DOWN THE TIME DEPENDENCE:

$$|\psi(t)\rangle = \sum e^{-iE_it/\hbar}\,a_i\,|E_i\rangle$$

$$= \frac{1}{\sqrt{14}}e^{-i3t/\hbar}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{2}{\sqrt{14}}e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$+ \frac{3}{\sqrt{14}}e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

eF's

e$\vec{v}$'s OF H ARE CALLED STATIONARY STATES

eS's

ONLY THEIR PHASE VARIES

$$\xrightarrow{E=3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} \xrightarrow[\text{TWO}]{\text{MEASURE IMMEDIATELY} \quad E=3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} \xrightarrow[\text{THEN MEASURE}]{\text{WAIT } t'} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} \right) e^{-i3t'/\hbar}$$

ALWAYS GET $E=3$

ALWAYS IN STATE $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{i\theta_0}$

TIME EVOLUTION OF e$\vec{v}$ OF H:

$$e^{i\theta_0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{WAIT } t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{i\theta_0} \right] e^{-i3t/\hbar}$$

TIME EVOLUTION OF $\vec{e}\vec{v}$'S OF P

$$\text{E } p=2 \ \vec{e}\vec{v} \text{ OF } P_{op} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = E=3 \ \vec{e}\vec{v} \text{ OF } H_{op}$$

P=2 IS STATIONARY
BECAUSE $|P=2\rangle \equiv |E=3\rangle$
AND $|E=3\rangle$ IS STATIONARY

SO ITS TIME EVOLUTION IS EXACTLY THE SAME

AS THE $E=3$ $\vec{e}\vec{v}$ OF H

$$p=3 \ \vec{e}\vec{v} \text{ OF } P = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

NO EQUAL TO AN $\vec{e}\vec{v}$ OF H
SO $|p=3\rangle$ IS TIME
DEPENDENT

TO FIND
ITS TIME EVOLUTION

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

EXPAND IN $\vec{e}\vec{v}$'S OF H

BECAUSE WE KNOW THEIR
TIME EVOLUTION

PUT IN THEIR
TIME
DEPENDENCE

$$= \frac{1}{\sqrt{2}}\left[0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i3t/\hbar} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-i4t/\hbar} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-i5t/\hbar}\right]$$

IF $|\psi(0)\rangle = |p=3\rangle$ THEN $|\psi(t)\rangle \neq |p=3\rangle$

WHAT DOES THIS LOOK LIKE:

$$\frac{1}{\sqrt{2}}\left[e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

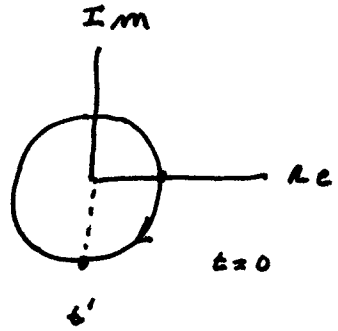
$$t=0 \quad \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

TIME-DEPENDENT PHASE FACTORS

$e^{-i4t/\hbar}$

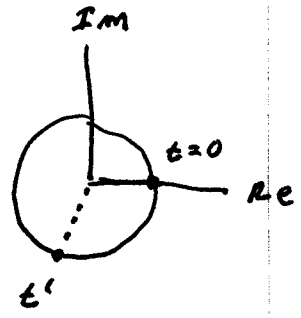
$e^{-i\omega t}$

$\omega = \frac{E}{\hbar}$



$e^{-i5t/\hbar}$

$e^{-i\omega' t}$



SO AT SOME TIME $e^{-i5t/\hbar}$ WILL BE 180°

AHEAD OF $e^{-i4t/\hbar}$

$$\frac{1}{\sqrt{2}}\left[e^{-i4t/\hbar}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i5t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$\frac{1}{\sqrt{2}}e^{-i4t/\hbar}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i\cancel{5}t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \nearrow \begin{pmatrix}0\\1\\1\end{pmatrix}e^{i\theta}$$

$$\searrow \begin{pmatrix}0\\1\\-1\end{pmatrix}e^{i\theta'}$$

START IN $p = 3$ $e\vec{v}$ OF P_{op}

← IN BETWEEN 50 : 50

LATER IN $p = 1$ $e\vec{v}$ OF P_{op}

← AGAIN IN BETWEEN

BACK IN $p = 3$ $e\vec{v}$ OF P_{op}

JUST LIKE COUPLED PENDULA

IN BETWEEN, YOU ARE

IN A LINEAR COMBINATION

OF $|E=4\rangle$ AND $|E=5\rangle$

Recipe for calculating time evolution

TIME EVOLUTION

(1) EXPAND $|\psi(0)\rangle$ IN ENERGY EIGENKETS

$$|\psi(0)\rangle = I\,|\psi(0)\rangle$$

$$= \left(\sum_i |E_i\rangle\langle E_i|\right)|\psi(0)\rangle$$

$$= \sum_i |E_i\rangle \underbrace{\langle E_i|\psi(0)\rangle}_{a_i}$$

$$= \sum_i a_i\,|E_i\rangle$$

(2) ADD THE TIME DEPENDENCE

$$|\psi(t)\rangle = \sum_i a_i\,|E_i\rangle\, \underbrace{e^{-iE_it/\hbar}}_{\text{TIME-DEP PHASE FACTORS}}$$

WORKS FOR ALL t

WHERE DOES THE SIMPLE TIME DEPENDENCE COME FROM?

THE $e\vec{v}$'s OF H

OBEY TDSE $\quad H|E_i\rangle = i\hbar \frac{d}{dt}|E_i\rangle$

OBEY TISE $\quad H|E_i\rangle = E_i|E_i\rangle$

$$|E_i(t)\rangle = e^{-iE_it/\hbar}|E_i(0)\rangle$$

PHASE EVOLUTION OF ENERGY EIGENKETS

ONLY EIGENKETS OF H ARE STATIONARY

LOOKED AT

$$|P=3\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$|P=3\rangle = \sum_i |E_i\rangle\langle E_i|P=3\rangle$$

$$= |E_1\rangle\langle E_1|P=3\rangle$$

$$+ |E_2\rangle\langle E_2|P=3\rangle$$

$$+ |E_3\rangle\langle E_3|P=3\rangle$$

$$= 0\,|E_1\rangle + \frac{1}{\sqrt{2}}|E_2\rangle + \frac{1}{\sqrt{2}}|E_3\rangle$$

$$|P=3,t\rangle = 0\,|E_1\rangle\, e^{-iE_1t/\hbar}$$

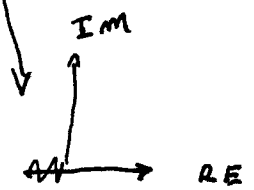
$$+ \frac{1}{\sqrt{2}}|E_2\rangle\, e^{-iE_2t/\hbar}$$

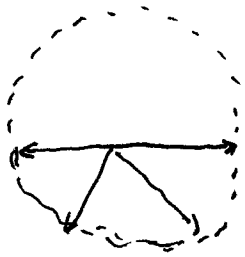
$$+ \frac{1}{\sqrt{2}}|E_3\rangle\, e^{-iE_3t/\hbar}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix} e^{-i4t/\hbar} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix} e^{-i5t/\hbar}$$

$$\frac{1}{\sqrt{2}} e^{-i4t/\hbar}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-i(1)t/\hbar}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

COEFF'S





$$\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\e^{-it/\hbar}\end{pmatrix}$$

DEGENERATE OPERATORS

$$H' = \begin{pmatrix} 3 & & \\ & 4 & \\ & & 4 \end{pmatrix}$$

$$\frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

POSSIBLE ENERGIES	3	4	4
RESULTING STATE	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	?	?

PROB $= \frac{1}{14}$ TOGETHER $\frac{13}{14}$

NEED TO (BOLDLY) GO BEYOND

$$\text{PROB}(E_i) = |\langle E_i | \psi \rangle|^2 \longrightarrow \text{STATE} = |E_i\rangle$$

GENERAL CASE

$$\text{PROB}(E_i) = \langle \psi | \mathbb{P}(E_i) | \psi \rangle$$

$$= \left| \mathbb{P}(E_i) | \psi \rangle \right|^2$$

WHAT IS $\mathbb{P}(E_i)$

$$\mathbb{P} = |2\rangle\langle 2| + |3\rangle\langle 3|$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{PROB}(E=4) = \left| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

$$= \frac{2^2 + 3^2}{14} = \frac{13}{14}$$

WHAT IS THE STATE?

$$|\psi(0^+)\rangle = \mathbb{P}\,|\psi(0^-)\rangle$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{14}} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$ NOT NORMALIZED

$$= \frac{1}{\sqrt{13}} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

MEASURE E AGAIN ALWAYS FIND $E = 4$

STAY IN SAME STATE

Statistical Description

Expectation Value aka Mean or Average

THE STATISTICAL DESCRIPTION

DICTIONARY

THE REST OF PHYSICS		QUANTUM MECHANICS	
$\bar{x}$	MEAN AVERAGE	$\langle x \rangle$	EXPECTATION VALUE
σ	STANDARD DEVIATION	Δx	UNCERTAINTY

EXPECTATION VALUE

$$\boxed{\langle \Omega \rangle_\psi = \sum_i P(\omega_i)\, \omega_i}$$

$$= \frac{\langle \psi | \Omega | \psi \rangle}{\langle \psi | \psi \rangle}$$

if $\langle \psi | \psi \rangle = 1$

$$\boxed{\langle \Omega \rangle_\psi = \langle \psi | \Omega | \psi \rangle}$$

WORKS IN ANY BASIS!

SHOW THE TWO DEF'S AGREE

$$\langle \Omega \rangle_\psi = \sum_i P(\omega_i)\, \omega_i$$

$$= \sum |\langle i | \psi \rangle|^2 \omega_i$$

$$= \sum_i \langle \psi | i \rangle \langle i | \psi \rangle \omega_i$$ just a number

$$= \underbrace{\sum_i}_{I} \langle \psi | \Omega \underbrace{| i \rangle \langle i |}_{I} \psi \rangle = \langle \psi | \Omega | \psi \rangle$$

BACK TO OUR TOY PROBLEM

$$H = \begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$P(E=3) = |\langle i | \psi \rangle|^2 = \left| (1\ 0\ 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

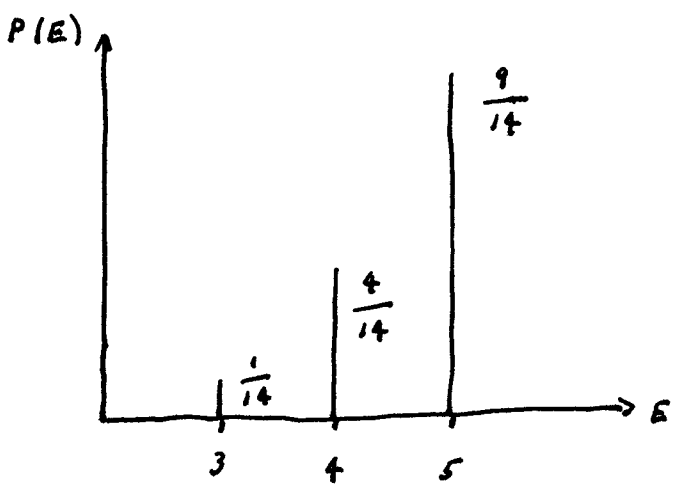
$$P(E=3) = \frac{1}{14}$$

$$P(E=4) = |\langle i | \psi \rangle|^2 = \left| (0\ 1\ 0) \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

$$P(E=4) = \frac{4}{14}$$

$$P(E=5) = |\langle i | \psi \rangle|^2 = \left| (0\ 0\ 1) \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right|^2$$

$$P(E=5) = \frac{9}{14}$$



FIRST WAY $\langle H \rangle$

$$\langle \Omega \rangle_\psi = \sum_i P(\omega_i)\, \omega_i$$

$$= \sum_i |\langle \omega_i | \psi \rangle|^2 \omega_i$$

$$\langle H \rangle = \langle E \rangle = \frac{1}{14}(3) + \frac{4}{14}(4) + \frac{9}{14}(5)$$

$$= \frac{64}{14} \approx 4.57$$

SECOND WAY

$$\langle \psi | H | \psi \rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 8 \\ 15 \end{pmatrix}$$

$$= \frac{64}{14}$$

$\Rightarrow$ SAME RESULT

BUT $\langle \psi | H | \psi \rangle$ DOES NOT DEPEND ON THE BASIS

CAN CALCULATE $\langle \psi | H | \psi \rangle$ WITHOUT EV'S AND EV'S!

CANNOT CALCULATE $\sum_i P(w_i)\, w_i$ W/O EV'S AND EV'S

EXPECTATION VALUE OF ENERGY VERSUS TIME

$$\langle E \rangle_{\psi(0)} = \langle \psi(0) | H | \psi(0) \rangle$$

$$\langle E(0) \rangle = \frac{64}{14}$$

$$\langle E \rangle_{\psi(t)} = \langle E(t) \rangle = \langle \psi(t) | H | \psi(t) \rangle$$

$$= \frac{1}{\sqrt{14}} \left(e^{+i3t/\hbar}, 2e^{+4it/\hbar}, 3e^{+i5t/\hbar} \right) \times$$

$$\begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix} \begin{pmatrix} e^{-i3t/\hbar} \\ 2e^{-i4t/\hbar} \\ 3e^{-i5t/\hbar} \end{pmatrix}$$

$$\langle E(t) \rangle = \langle E(0) \rangle$$

$\Rightarrow$ ENERGY IS CONSERVED!

GENERAL CASE: IF $[\Omega, H] = 0$

THEN $\langle \Omega(0) \rangle = \langle \Omega(t) \rangle$

$\Rightarrow$ Ω IS CONSERVED

$$[H, H] = 0$$

HAMILTONIAN GENERATES TIME TRANSLATION!

WE ALSO SAID

$$|\psi(t)\rangle = \sum_j |E_j\rangle\langle E_j|\psi(0)\rangle\, e^{-iE_jt/\hbar}$$

$$= \underbrace{\sum_j e^{-iE_jt/\hbar}|E_j\rangle\langle E_j|}_{U(t)} \quad |\psi(0)\rangle$$

$U(t)$ IS UNITARY $\Rightarrow$ STATES STAY NORMALIZED!

$$\langle\psi(0)|\psi(0)\rangle = 1$$

$$\uparrow$$

$$I$$

$$1 = \langle\psi(0)|\,U(t)\,U^\dagger(t)\,|\psi(0)\rangle$$

$$1 = \langle\psi(t)|\psi(t)\rangle$$

IN DIAGONAL REPRESENTATION

$$U(t) = \begin{pmatrix} e^{-iE_1 t/\hbar} & & & 0 \\ & e^{-iE_2 t/\hbar} & & \\ & & e^{-iE_3 t/\hbar} & \\ 0 & & & \end{pmatrix}$$

general unitary form $\quad e^{i\theta}$ on the diagonal

TIME-EVOLUTION OF THE STATIONARY STATES

$$U(t) = \begin{pmatrix} e^{-i3t/\hbar} & & \\ & e^{-i4t/\hbar} & \\ & & e^{-i5t/\hbar} \end{pmatrix}$$

$t=0$	$t=t$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} e^{-i3t/\hbar} \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ e^{-i4t/\hbar} \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ e^{-i5t/\hbar} \end{pmatrix}$

start in $e\vec{s}$ of H, stay in $e\vec{s}$ of H

STATIONARY STATES <=> ENERGY CONSERVATION

HOWEVER,

$e\vec{s}$ OF Ω WHERE $[H, \Omega] \neq 0$ ARE NOT STATIONARY

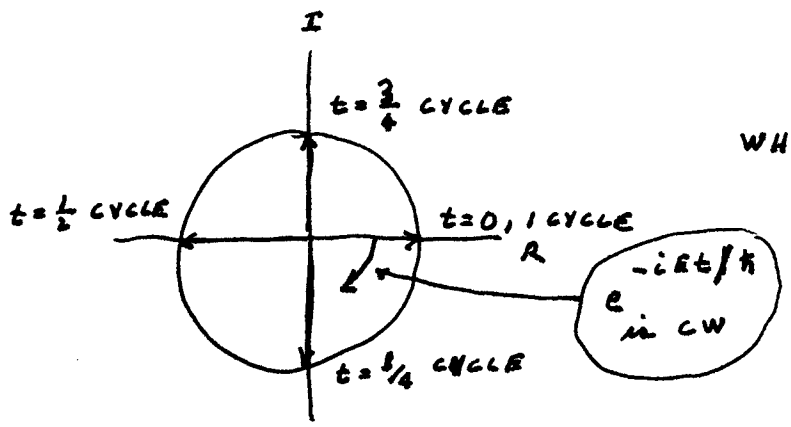
$t = 0$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$p = 3$

$t = t$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{-i4t/\hbar} \\ e^{-i5t/\hbar} \end{pmatrix}$$

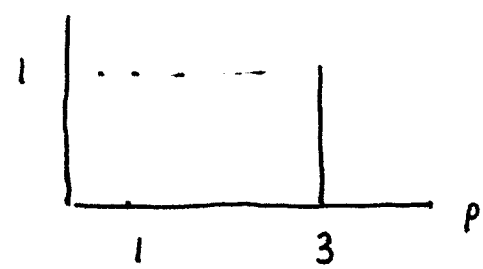


WHEN $\theta_5 - \theta_4 = \pi$

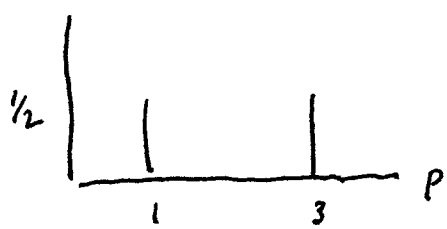
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

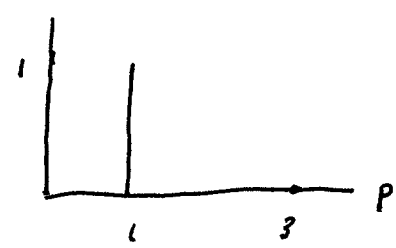
$p = 1$

PROB (P) AT $t = 0$



$\langle P \rangle$ DEPENDS ON TIME!

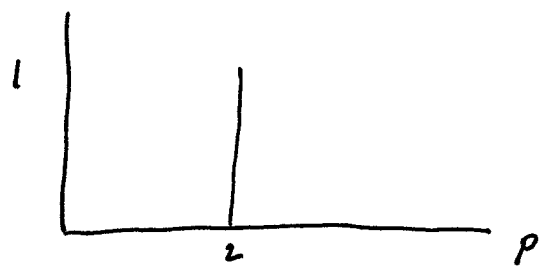




WHAT ABOUT $p=2$ STATE

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i3t/\hbar} \\ 0 \\ 0 \end{pmatrix}$$

PROB (P)



MEASURE $p=2$ INDEPENDENT OF TIME

Statistical Description

Uncertainty aka Standard Deviation

$\bar{x}$ $\qquad$ $\langle x \rangle$

σ_x $\qquad$ Δx

EXPECTATION VALUE

$$\langle x \rangle_\psi = \langle \psi | x_{op} | \psi \rangle$$

$$= \sum_i P(x_i)\, x_i \qquad \text{DISCRETE}$$

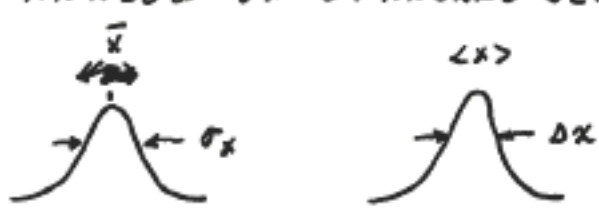
$$= \int_{x_i - \epsilon}^{x_i + \epsilon} dx \; P(x)\, x \qquad \text{CONTINUOUS}$$

MORE LATER ABOUT CONTINUOUS

TODAY: UNCERTAINTY

QM ANALOG OF STANDARD DEVIATION

CONTINUOUS



PROB DENSITIES

DISCRETE

UNCERTAINTY

$$\Delta \Omega = \langle (\Omega - \langle \Omega \rangle)^2 \rangle^{1/2}$$

S M R

R = ROOT

M = MEAN

S = SQUARE

AGAIN, TWO FORMS

$$(\Delta \Omega)^2 = \sum_i P(\omega_i) \left[\omega_i - \langle \Omega \rangle \right]^2$$

$$(\Delta \Omega)_\psi = \sqrt{\langle \psi | \Omega - \langle \Omega \rangle | \psi \rangle}$$

FIRST WAY $\Delta \Omega$

$$(\Delta H)^2 = \sum_i p(E_i)\left[E_i - \langle E \rangle\right]^2$$

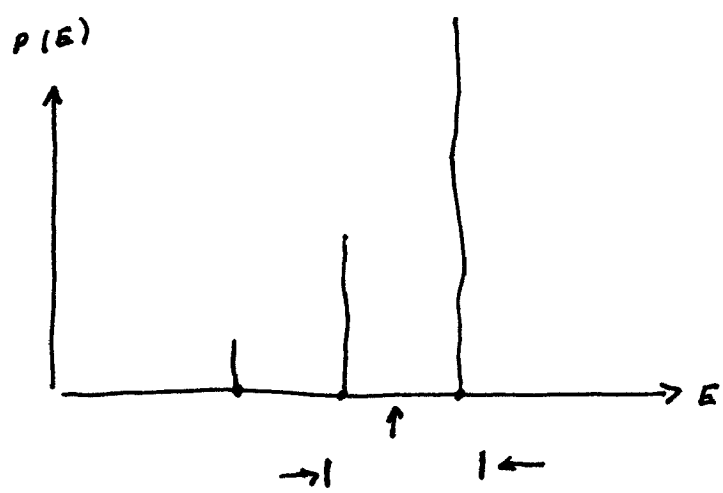
$$= \frac{1}{14}\left[3 - 4.57\right]^2$$

$$+ \frac{4}{14}\left[4 - 4.57\right]^2$$

$$+ \frac{9}{14}\left[5 - 4.57\right]^2$$

$$(\Delta H)^2 = \frac{5.43}{14} = 0.39$$

$$\Delta H = 0.62$$



$E = 4.57 \pm 0.62$

[redacted]

SECOND WAY $\Delta\Omega$

$$\Delta\Omega^2 = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \left[\begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix} - \begin{pmatrix} 4.57 & & \\ & 4.57 & \\ & & 4.57 \end{pmatrix} \right]^2 \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1.57 & & \\ & -0.57 & \\ & & +0.43 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{14} \left[(1)(-1.57)^2 + 4(-0.57)^2 + 9(0.43)^2 \right]$$

$$\Delta\Omega^2 = \frac{5.43}{14} = 0.39$$

$$\Delta\Omega = 0.62$$

± 0.62

[redacted]

18

THIRD ~~SECOND~~ WAY $\Delta\Omega$ (SECOND WAY)' ?

USEFUL ALGEBRA

$$\Delta\Omega^2 = \langle (\Omega - \langle\Omega\rangle I)^2 \rangle = \langle (\Omega^2 - 2\langle\Omega\rangle \Omega^I + \langle\Omega\rangle^2 I) \rangle$$

$$= \langle \Omega^2 \rangle - 2\langle\Omega\rangle\langle\Omega\rangle + \langle\Omega\rangle^2$$

$$= \langle \Omega^2 \rangle - \langle\Omega\rangle^2$$

$$\left[\frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 9 & & \\ & 16 & \\ & & 25 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right]$$

$$- \left[\frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right]^2 = \frac{5.43}{14} = 0.39$$

$$\Delta\Omega = \sqrt{(\Delta\Omega)^2} = 0.69$$

NOTE THAT

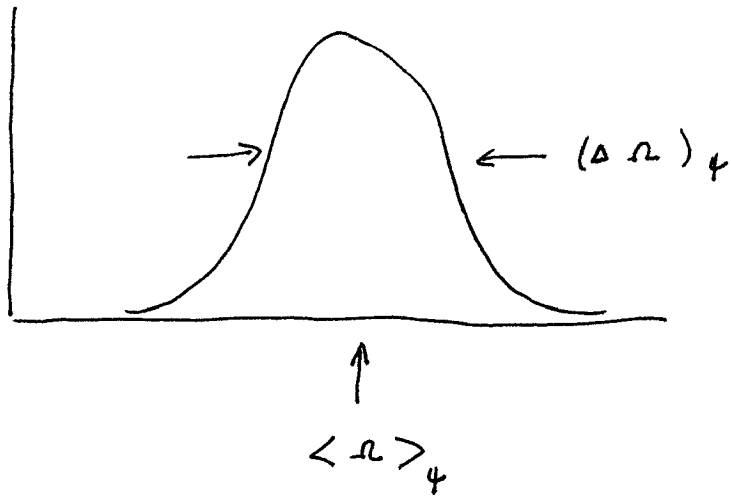
$$\langle \Omega \rangle_\psi = \frac{\langle \psi | \Omega | \psi \rangle}{\langle \psi | \psi \rangle}$$

AND

$$(\Delta \Omega)_\psi = \sqrt{\frac{\langle \psi | (\Omega - \langle \Omega \rangle)^2 | \psi \rangle}{\langle \psi | \psi \rangle}}$$

DO NOT REQUIRE US TO KNOW THE EV'S OR $\vec{EV}$'S OF Ω !

DISCRETE $\longrightarrow$ CONTINUOUS



CONTINUOUS

$$\langle \psi | \Omega | \psi \rangle \longrightarrow \int \psi^*(x)\, \Omega\, \psi(x)\, dx$$

$$(\Delta \Omega)^2_\psi = \int p(\omega) \left[\omega - \langle \Omega \rangle\right]^2 d\omega$$

USEFUL ALGEBRA

$$\langle (\Omega - \langle \Omega \rangle I)^2 \rangle = \langle \Omega^2 - 2 \langle \Omega \rangle \Omega + \langle \Omega \rangle^2 I \rangle$$

$$= \langle \Omega^2 \rangle - \langle \Omega \rangle^2$$

CONTINUOUS CASE

$$\langle \Omega^2 \rangle - \langle \Omega \rangle^2$$

$$\int \psi^*(x)\, \Omega^2\, \psi(x)\, dx - \left(\int \psi^*(x)\, \Omega\, \psi(x)\, dx \right)^2$$