EXAMPLE 3

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & 2-\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = 0$$

$$(1-\omega)(2-\omega)(1-\omega) - 0 + (1)\left[0 - (2-\omega)(1)\right] = 0$$

$$(2-3\omega+\omega^2)(1-\omega) \qquad -2+\omega = 0$$

$$\cancel{2} - 3\omega + \omega^2 - \cancel{2}\omega + 3\omega^2 - \omega^3 \; - \cancel{2} + \cancel{\omega} = 0$$

$$-4\omega + 4\omega^2 - \omega^3 = 0 \qquad \text{CUBIC EQN}$$

$$\omega^3 - 4\omega^2 + 4\omega = 0$$

5

$$\omega(\omega^2 - 4\omega + 4) = 0 \qquad \text{"CE"}$$

$$\omega(\omega - 2)(\omega - 2) = 0$$

SO eV'S ARE $0, 2, 2$

FIND THE $e\vec{v}$'S

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a + c \\ 2b \\ a + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad b = 0$

$\Rightarrow \quad c = -a \qquad$ CHOOSE $a = +1$, $c = -1$

$$|\omega = 0\rangle = A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

NORMALIZATION CONSTANT

6

NORMALIZE

$$\langle w=0 \mid w=0 \rangle = 1$$

$$A^{*} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^{*} \quad A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1$$

$$AA^{*} \; (1+1) = 1$$

$$2\,|A|^{2} = 1$$

"PHASE CONVENTION"

conventional to choose real positive value

$$A = \frac{1}{\sqrt{2}}$$

$$|w=0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

7

TO FIND THE OTHER TWO $e\vec{v}$'s

ASSOCIATED WITH $\omega = 2$ degeneracy $\Rightarrow$ no unique pair of $e\vec{v}$'s

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a+c \\ 2b \\ a+c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$$\left.\begin{aligned} a+c &= 2a \\ a+c &= 2c \end{aligned}\right\} \Rightarrow \quad 2a = 2c$$

$a = c$

choose $a = c = 1$

$$2b = 2b \quad \} \Rightarrow \quad b = \text{anything}$$

INFINITELY MANY $e\vec{v}$'s

$$\begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}$$

~~SO WE CAN~~ LET'S CHOOSE ONE ARBITRARILY $\Rightarrow$ $b = 0$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

THEN CONSTRUCT SECOND $\perp$ TO FIRST

$$\begin{pmatrix} a & b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

any linear combination is an $e\vec{v}$ with $ev = 2$

$a = 0$ [redacted]

$b =$ anything

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

VERY IMPORTANT STEP!!!

CHECK YOUR ANSWER

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \checkmark$$

WHEW!!!

Eigenvalues and Eigenvectors

http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

http://web.mit.edu/18.06/www/Demos/eigen-applet-all/eigen_sound_all.html

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm

http://www.bluebit.gr/matrix-calculator/calculate.aspx

http://wims.unice.fr/wims/wims.cgi?session=6S051ABAFA.2&+lang=en&+module=tool%2Flinear%2Fmatrix.en

Fundamental theorem of algebra

http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

Every polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity

1702 Leibniz tried
1742 Bernoulli tried
1746 d'Alembert treid
1749 Euler tried
1772 Lagrange tried
1795 Laplace tried
1799 Gauss tried (had a topological gap)
1920 Gauss' topological gap filled by Ostrowski
1891 Weierstrass first constructive proof

The Abel-Ruffini Theorem (1824)
aka, Abel's Impossibility Theorem

http://en.wikipedia.org/wiki/Abel-Ruffini_theorem

There is no general algebraic solution—that is, solution in radicals—to polynomial equations of degree five or higher

1799 Ruffini tried (contained a gap)
1824 Abel
1885 proof using Galois theory

The quadratic, cubic, and quartic formulas are the general solutions to the quadratic, cubic, and quartic equations

Tutorials

http://en.wikipedia.org/wiki/Quartic_function

http://en.wikipedia.org/wiki/Cubic_function

http://en.wikipedia.org/wiki/Quadratic_function

Calculators

http://www.freewebs.com/brianjs/ultimateequationsolver.htm

http://www.akiti.ca/Quad3Deg.html

http://www.freewebs.com/brianjs/quinticequationcalculator.htm

Online Eigenvalue and Eigenvector Resources

Tutorials

http://web.mit.edu/18.06/www/Demos/eigen-applet-all/eigen_sound_all.html

http://ocw.mit.edu/ans7870/18/18.06/javademo/Eigen/

http://www.falstad.com/matrix/

http://abel.math.harvard.edu/archive/21b_fall_02/applets/index.html

http://www.math.ubc.ca/~cass/courses/m309-8a/java/m309gfx/eigen.html

Calculators

http://www.math.ubc.ca/~israel/applet/mcalc/matcalc.html

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm

http://www.bluebit.gr/matrix-calculator/

More Resources

http://archives.math.utk.edu/topics/linearAlgebra.html

http://www.math.harvard.edu/computing/java/links.html

http://www.educypedia.be/education/calculatorsmatrix.htm

Terrance Tao

http://www.math.ucla.edu/~tao/resource/general/115a.3.02f/linearMap.html

http://www.math.ucla.edu/~tao/resource/general/115a.3.02f/EigenMap.html

http://www.math.ucla.edu/~tao/java/index2.html

Please let me know if you find other comparably good or better online resources so I can add them to this page and forward the revised version to everyone. Thanks !!!

Feynman's Conversational Writing Style

Whenever I see a video of one of Feynman's lectures I am struck by the style of his presentation.

I am even more surprised when I read the associated text---it is almost word-for-word verbatim.

You will discover this when you read the following text for the video that you saw.

The lecture you saw is one of Feynman's seven Messenger Lectures. They are collected in written form in his book entitled **The Character of Physical Law** ($12.21 new; $3.13 used from Amazon).

Not only Feynman's published lectures, but also his textbooks and even his Nobel Prize Lecture share his unique conversational style. As you must know by now, this is in dramatic contrast to the writing style in all conventional physics textbooks.

As you will learn from Schrieffer's story on the following page from Gleick's book **Genius: The Life and Science of Richard Feynman** (Pantheon Books, 1992), Feynman clearly knew he was doing this---it was not accidental.

The following links lead to various assundry related material. Enjoy !!!

http://ezinearticles.com/?The-Art-of-Explaining-Things---Richard-Feynman-Style&id=2018884

http://thinking.bioinformatics.ucla.edu/2008/07/18/16/

http://headrush.typepad.com/creating_passionate_users/2005/09/conversational_.html

http://nerdwisdom.com/2007/08/31/richard-feynman/

Something has changed. Have books changed? Have lectures?

People have now a-days got a strange opinion that every thing should be taught by lectures. Now, I cannot see that lectures can do as much good as reading the books from which the lectures are taken.   — Samuel Johnson

A few more Dirac stories

Oppenheimer was working at Göttingen and the great mathematical physicist, Dirac, came to him one day and said: "Oppenheimer, they tell me you are writing poetry. I do not see how a man can work on the frontiers of physics and write a poetry at the same time. They are in opposition. In science you want to say something that nobody knew before, in words which everyone can understand. In poetry you are bound to say...something that everybody knows already in words that nobody can understand.

Once Peter Kapitza, the Russian physicist, gave Dirac an English translation of Dostoevski's Crime and Punishment. "Well, how do you like it?" asked Kapitza when Dirac returned the book. "It is nice," said Dirac, "but in one of the chapters the author made a mistake. He describes the Sun rising twice on the same day." This was his one and only comment on Dostoevski's novel.

Dirac politely refused Robert's [Robert Oppenheimer] two proffered books: reading books, the Cambridge theoretician announced gravely, 'interfered with thought'.   — Luis W. Alvarez used none of the technical apparatus for which he was now famous: no Feynman diagrams, no path integrals. Instead he began with mental pictures: this electron pushes that one; this ion rebounds like a ball on a spring. He reminded colleagues of an artist who can capture the image of a human face with three or four minimal and expressive lines. Yet he did not always succeed. As he worked on superfluidity, he also struggled with superconductivity, and here, for once, he failed. (Yet he came close. At one point, about to leave on a trip, he wrote a single page of notes, beginning, "Possibly I understand the main origin of superconductivity." He was focusing on a particular kind of phonon interaction and on one of the experimental signatures of superconductivity, a transition in a substance's specific heat. He could see, as he jotted to himself, that there was "something still a little haywire," but he thought he would be able to work out the difficulties. He signed the page: "*In case I don't return*. R. P. Feynman.") Three younger physicists, intensely aware of Feynman's competitive presence—John Bardeen, Leon Cooper, and Robert Schrieffer—invented a successful theory in 1957. The year before, Schrieffer had listened intently as Feynman delivered a pellucid talk on the two phenomena: the problem he had solved, and the problem that had defeated him. Schrieffer had never heard a scientist outline in such loving detail a sequence leading to failure. Feynman was uncompromisingly frank about each false step, each faulty approximation, each flawed visualization.

No tricks or fancy calculations would suffice, Feynman said. The only way to solve the problem would be to guess the outline, the shape, the quality of the answer.

> We have no excuse that there are not enough experiments, it has nothing to do with experiments. Our situation is unlike the field, say, of mesons, where we say, perhaps there aren't yet enough clues for even a human mind to figure out what is the pattern. We should not even have to look at the experiments. . . . It is like looking in the back of the book for the answer . . . The only reason that we cannot do this problem of superconductivity is that we haven't got enough imagination.

It fell to Schrieffer to transcribe Feynman's talk for journal publication. He did not quite know what to do with the incomplete sentences and the frank confessions. He had never read a journal article so obviously spoken aloud. So he edited it. But Feynman made him change it all back.

LECTURE 4 JUNE 26, 2007

YESTERDAY eV's AND $e\vec{v}$'s

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$e\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad eV = 0$$

$$e\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad eV = 2$$

$$e\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad eV = 2$$

$\{ e\vec{v} \}$ ORTHONORMAL BASIS

MUTUALLY PERP

NORMALIZED

CAN MAKE ANY 3d VECTOR

TODAY: MACHINERY TO CHANGE BASIS

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$$\Omega \longrightarrow U^{\dagger}\,\Omega\, U$$

$U =$ UNITARY OP BUILT OUT OF ORTHONORMAL BASIS OF IT'S $e\vec{v}$'s

$$\underset{\omega = 2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \qquad \underset{\omega = 2}{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \qquad \underset{\omega = 0}{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

$$U = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

HIGHEST ev . . . . . LOWEST ev

CONVENTION

WHY DOES THIS WORK?

THM: THE eV'S OF ANY HERMITEAN OPERATOR ARE REAL

$\Rightarrow$ ANY HERMITEAN OPERATOR HAS (AT LEAST ONE) SET OF ORTHONORMAL $e\vec{v}$'S THAT ARE A BASIS

IN THIS BASIS, Ω IS DIAGONAL WITH ITS ω_i'S ON THE DIAGONAL

BUT HOW DOES IT WORK?

$$\langle w | v \rangle \qquad \text{inner product}$$

number $= |\langle w|| \, |v\rangle| \cos\theta$

CHANGE BASIS

$$\big\downarrow$$

$$\langle w' | v' \rangle = \text{SAME NUMBER}$$

$$\langle w | v \rangle = \langle w | I | v \rangle$$

$$= \langle w | U U^+ | v \rangle$$

$$= \langle w' | v' \rangle$$

$U U^+ = I$ identity operator

CHANGE BASIS FOR OPERATORS

$$\langle w | \Omega | v \rangle = \langle w | I \, \Omega \, I | v \rangle$$

$$\langle w | U U^{\dagger} \Omega \; U U^{\dagger} | v \rangle$$

$$\left(\langle w | U \right) \left(U^{\dagger} \Omega U \right) \left(U^{\dagger} | v \rangle \right)$$

$$\langle w' | \quad \Omega' \quad | v' \rangle$$

$$U = \begin{pmatrix} \begin{pmatrix} \\ e_{\vec{v}_1} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ e_{\vec{v}_2} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ e_{\vec{v}_3} \\ \\ \end{pmatrix} \end{pmatrix}$$

$$U^{\dagger} = \left(U^{T} \right)^{*}$$

$$U^{\dagger} = \begin{bmatrix} \left(\quad e_{\vec{v}_1} \quad \right)^{*} \\ \left(\quad e_{\vec{v}_2} \quad \right)^{*} \\ \left(\quad e_{\vec{v}_3} \quad \right)^{*} \end{bmatrix}$$

APPLY U^+ TO THE $e\vec{v}$'s OF Ω

$$\begin{bmatrix} \left(e\vec{v}_1 \right)^* \\ \left(e\vec{v}_2 \right)^* \\ \left(e\vec{v}_3 \right)^* \end{bmatrix} \begin{pmatrix} e\vec{v}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e\vec{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e\vec{v}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

SO U^+ CHANGES THE KET BASIS

$\Rightarrow$ U " " BRA BASIS

$$U^+ \Omega U = \Omega$$

DOES IT WORK? CALCULATE $U^+ \Omega U$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 2\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \checkmark$$

it works

SAY:

FIND ev's AND $e\vec{v}$'s OF H

DIAGONALIZE THE H

THM 10: FOR EVERY HERMITEAN OPERATOR, THERE EXISTS A BASIS OF ITS ORTHONORMAL EIGENVECTORS

x, p, H are Hermitian

$\Rightarrow$ unique expansion in x
p
E

ANY COMMON EIGENVECTORS?

THM 13: IF Ω and Λ are commuting Hermitian operators then there exists a common basis of eigenvectors that diagonalizes them both.

COMMUTATOR

$$[\Omega, \Lambda] = 0 = \Omega\Lambda - \Lambda\Omega$$

THREE CASES:

(1) COMPATIBLE: SHARE ALL $e\vec{v}$

(2) INCOMPATIBLE: SHARE NO $e\vec{v}$

(3) MIXED: SHARE SOME $e\vec{v}$ BUT NOT ALL

$$\Omega \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$ev = 1 \quad e\vec{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad ev = 2 \quad e\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$ev = 3 \quad e\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\Lambda \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$ev = 1 \quad e\vec{v} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

many different

$ev = 2 \quad e\vec{v} = \cancel{\begin{pmatrix}0\\1\\0\end{pmatrix}}$

$$\Omega\Lambda = \begin{pmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 3\cdot 2 \end{pmatrix}$$

$$\Lambda\Omega = \begin{pmatrix} 1\cdot 1 & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 2\cdot 3 \end{pmatrix}$$

$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = 0 \quad \Rightarrow$ THERE EXISTS A UNIQUE SET OF $e\vec{v}$'s THAT DIAGONALIZE BOTH

CSCO $\Rightarrow$ unique eigenbasis

FINITE DIMENSIONAL $\longrightarrow$ INFINITE DIMENSIONAL

MATRICES	OPERATORS
VECTORS	FUNCTIONS
(DISCRETE EV'S)	(CONTINUOUS OR DISCRETE)

GENERALIZE ~~TO~~ INNER PRODUCT

DISCRETE:

$$\langle f \mid g \rangle = \sum_{i=1}^{n} f_i^* \, g_i$$

DEPEND ON BASIS

BUT SUM DOES NOT

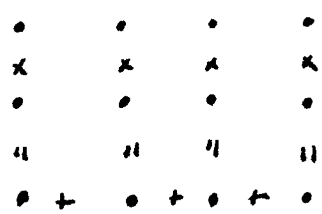
CONTINUOUS:

DEPEND ON BASIS

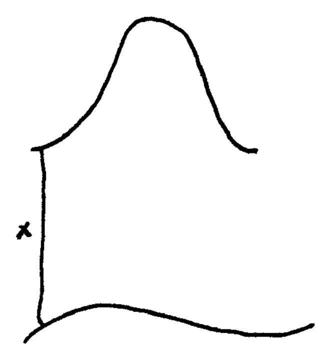
BUT INTEGRAL DOES NOT

$$\langle f \mid g \rangle = \int_a^b f^*(x) \, g(x) \, dx$$

DISCRETE



CONTINUOUS



INTEGRATE

MULTIPLY ~~POINT BY~~

COMPONENT BY COMPONENT

SUM

MULTIPLY

POINT BY POINT

INTEGRATE

DISCRETE CASE

$$\Omega \, | \omega_i \rangle = \omega_i \, | \omega_i \rangle$$

CONTINUOUS CASE

$$OP \left(\text{FUNCTION} \right) = \left(\text{NUMBER} \right) \left(\text{FUNCTION} \right)$$

$$\frac{d}{dx} \left(\text{FUNCTION} \right) = \left(\text{NUMBER} \right) \text{FUNCTION}$$

$$\frac{d}{dx} \left(e^{mx} \right) = \underbrace{(m)}_{ev} \underbrace{\left(e^{mx} \right)}_{ef}$$

$m \in \mathbb{C}$

m can be any complex number

"SPECTRUM IS CONTINUOUS"

DELTA FUNCTIONS

DISCRETE:

KRONECKER DELTA

$$\delta_{ij} = 1 \quad \text{if } i = j$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j$$

$$\langle e_i | e_j \rangle = \delta_{ij}$$

$$|v\rangle = \sum_i a_i \, |e_i\rangle$$

$$\langle e_j | v \rangle = \langle e_j | \sum_i a_i \, |e_i\rangle$$

$$= \sum_i a_i \underbrace{\langle e_j | e_i \rangle}_{\delta_{ji}}$$

$$= a_j$$

δ_{ij} PICKS OUT ONE OF THE EXPANSION COEFF'S

KRONECKER DELTA — DISCRETE BASIS

DIRAC DELTA — CONTINUOUS BASIS

A true story about the physcist/mathematician Paul Dirac.

Dirac was apparently a very hard person to get along with. Soon after he was awarded the Nobel Prize in Physics, Dirac went on a speaking tour of the country, visiting different universities and talking about his research. In those days, it was more convenient for him to travel by car, so he had a big car and a driver who took him from one speaking engagement to the next.

Dirac and his driver got to be very good friends after awhile and at one point, his driver remarked, "You know, I am so sick and tired of hearing the same lecture over and over again. I easily give it myself!"

Dirac thought about this for a moment, and then decided that his driver could give the next speaking engagement at U. Michigan in Ann Arbor, Michigan. Before reaching the university, Dirac and his driver switched clothes. When the reached the university, the driver went up to the podium and delivered Dirac's seminar flawlessly. After he was finished, an upstart graduate student asked a question, snottily pointing out a perceived mistake in the talk.

The Driver gave the student a long look of contempt and then exclaimed, "That question is so stupid that even my driver could answer it!", and Dirac stepped forward and proceeded to do so.

Today, if you go to U Mich and see a picture on the wall of Dirac and his driver, you would have to know this story to realize that the two are switched.

Path Integral Formulation
Sum over Histories Formulation
Lagrangian Formulation
Amplitude Formulation

Feynman (1941; age 23)

The probability to go from a to b is the square of an amplitude

$$P(b,a) = | \, Amp(b,a) \, |^2$$

The amplitude is the weighted sum over all possible ways to go to b from a

$$Amp(b,a) = \text{constant} \sum_{\text{all paths}} \exp(\mathrm{i}\mathrm{S}/\hbar)$$

S is the classical action

I went to a beer party in the Nassau Tavern in Princeton. There was a gentleman, newly arrived from Europe (Herbert Jehle) who came and sat next to me. Europeans are much more serious than we are in America because they think a good place to discuss intellectual matters is a beer party. So he sat by me and asked, "What are you doing" and so on, and I said, "I'm drinking beer." Then I realized that he wanted to know what work I was doing and I told him I was struggling with this problem, and I simply turned to him and said "Listen, do you know any way of doing quantum mechanics starting with action--where the action integral comes into the quantum mechanics?" "No," he said, "but Dirac has a paper in which the Lagrangian, at least, comes into quantum mechanics. I will show it to you tomorrow."

Next day we went to the Princeton Library (they have little rooms on the side to discuss things) and he showed me this paper. Dirac's short paper in the *Physikalische Zeitschrift der Sowjetunion* claimed that a mathematical tool which governs the time development of a quantal system was "analogous" to the classical Lagrangian.

Professor Jehle showed me this; I read it; he explained it to me, and I said, "What does he mean, they are analogous; what does that mean, *analogous?* What is the use of that?" He said, "You Americans! You always want to find a use for everything!" I said that I thought that Dirac must mean that they were equal. "No," he explained, "he doesn't mean they are equal." "Well," I said, "let's see what happens if we make them equal."

So, I simply put them equal, taking the simplest example . . . but soon found that I had to put a constant of proportionality *A* in, suitably adjusted. When I substituted . . . and just calculated things out by Taylor-series expansion, out came the Schrödinger equation. So I turned to Professor Jehle, not really understanding, and said, "Well you see Professor Dirac meant that they were proportional." Professor Jehle's eyes were bugging out -- he had taken out a little notebook and was rapidly copying it down from the blackboard and said, "No, no, this is an important discovery."

Feynman's thesis advisor, John Archibald Wheeler (age 30), was equally impressed. He believed that the amplitude formulation of quantum mechanics--although mathematically equivalent to the matrix and wave formulations--was so much more natural than the previous formulations that it had a chance of convincing quantum mechanics's most determined critic. Wheeler writes:

Visiting Einstein one day, I could not resist telling him about Feynman's new way to express quantum theory. "Feynman has found a beautiful picture to understand the probability amplitude for a dynamical system to go from one specified configuration at one time to another specified configuration at a later time. He treats on a footing of absolute equality every conceivable history that leads from the initial state to the final one, no matter how crazy the motion in between. The contributions of these histories differ not at all in amplitude, only in phase. And the phase is nothing but the classical action integral, apart from the Dirac factor h. This prescription reproduces all of standard quantum theory. How could one ever want a simpler way to see what quantum theory is all about!

Doesn't this marvelous discovery make you willing to accept the quantum theory, Professor Einstein?"

Einstein replied in a serious voice, "I still cannot believe that God plays dice. But maybe", he smiled, "I have earned the right to make my mistakes."

John Wheeler

Thirty-one years ago, Dick Feynman told me about his "sum over histories" version of quantum mechanics.

"The electron does anything it likes," he said. "It just goes in any direction at any speed, . . . however it likes, and then you add up the amplitudes and it gives you the wave function."

I said to him, "You're crazy." But he wasn't.

Freeman Dyson

Thirty-one years ago, Dick Feynman told me about his "sum over histories" version of quantum mechanics.

"The electron does anything it likes," he said. "It just goes in any direction at any speed, . . . however it likes, and then you add up the amplitudes and it gives you the wave function."

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Freeman Dyson

Lecture 3

GEOMETRY OF HILBERT SPACES

$\Rightarrow$ ALGEBRA OF HILBERT SPACES

DIRAC NOTATION

$$\langle \quad | \; | \quad \rangle$$

BRA KET

BRACKET

	3d	HILBERT	
VECTORS	$\vec{v}$, $\vec{w}$ BOLD	$\vert v\rangle$ $\langle w\vert$	TWO FLAVORS no arrow not bold
ADDITION	$\vec{v} + \vec{w}$	$\vert v\rangle + \vert w\rangle$	
SUBTRACTION	$\vec{v} - \vec{w}$	$\vert v\rangle - \vert w\rangle$	
MULTIPLY BY A SCALER	$a\vec{v}$	$a\vert v\rangle$	
DOT PRODUCT INNER PRODUCT	$\vec{v}\cdot\vec{w}$ real number	$\langle v\vert w\rangle$ complex number	
OUTER PRODUCT		$\vert v\rangle\langle w\vert$ MATRIX OPERATOR	

which implies

$$|V_i|^2 \cdot |V_j|^2 \geq |\langle V_i | V_j \rangle|^2 \qquad \text{Q.E.D.}$$

Retracing our steps, we find that the inequality is saturated (i.e., becomes an equality) if and only if $\mathbf{V}$ vanishes, which in turn means $\mathbf{V}_i$ is some scalar λ times $\mathbf{V}_j$, exactly as in the case of $\mathbb{V}^3(R)$. This is a very useful result and will be recalled later.

Theorem 4 (Triangle Inequality).

$$|V_i + V_j| \leq |V_i| + |V_j|$$

Proof. This theorem also says something obvious in $\mathbb{V}^3(R)$: if two arrows are added, the length of the sum is less than or equal to the sum of the individual lengths. (The equality results if the vectors are parallel. See Fig. 1.1.) In the general case,

$$\begin{aligned} |V_i + V_j|^2 &= \langle V_i + V_j | V_i + V_j \rangle \\ &= |V_i|^2 + |V_j|^2 + \langle V_i | V_j \rangle + \langle V_j | V_i \rangle \end{aligned}$$

Now

$$\langle V_j | V_i \rangle = \langle V_i | V_j \rangle^*$$

so that

$$\langle V_i | V_j \rangle + \langle V_j | V_i \rangle = 2\text{Re}\langle V_i | V_j \rangle$$

where "Re" means the real part of the complex number that follows. So we have now

$$|V_i + V_j|^2 = |V_i|^2 + |V_j|^2 + 2\text{Re}\langle V_i | V_j \rangle$$

Now, for any complex number $z = a + ib$, it is obviously true that

$$a = \text{Re}\, z \leq |z| = (a^2 + b^2)^{1/2}$$

the equality sign applying when z is real and positive. So

$$|V_i + V_j|^2 \leq |V_i|^2 + |V_j|^2 + 2|\langle V_i | V_j \rangle|$$

From the previous theorem,

$$|\langle V_i | V_j \rangle| \leq |V_i| \cdot |V_j|$$

the equality sign applying only when $\mathbf{V}_i = \lambda \mathbf{V}_j$. Consequently

$$|V_i + V_j|^2 \leq (|V_i|^2 + |V_j|^2 + 2|V_i| \cdot |V_j|) = (|V_i| + |V_j|)^2$$

Q.E.D.

If we take the hint from $\mathbb{V}^3(R)$ and try to saturate the bound by choosing $\mathbf{V}_j = \lambda \mathbf{V}_i$, we will find that λ has to be real and positive. Generally one refers to two vectors in $\mathbb{V}^n(C)$ as being parallel if $\mathbf{V}_j = \lambda \mathbf{V}_i$ for any λ. Thus the condition for saturating the inequality of this theorem is something more than just parallel vectors: one must be a *real positive multiple of the other*.

*Exercise 1.2.1.** By going through the derivation of Theorem 3, show that the inequality becomes an equality if $\mathbf{V}_i = \lambda \mathbf{V}_j$, where λ is an arbitrary scalar.

*Exercise 1.2.2.** Show likewise by analyzing the proof of Theorem 4 that the inequality becomes an equality if $\mathbf{V}_i = \lambda \mathbf{V}_j$, where λ is a real positive scalar. (There are two inequalities here, $\text{Re}\langle V_i | V_j \rangle \leq |\langle V_i | V_j \rangle|$, and $|\langle V_i | V_j \rangle| \leq |V_i| \cdot |V_j|$, both of which must become equalities.)

1.3. The Dirac Notation

We are now ready to get acquainted with a notation invented by Dirac that is particularly suited for quantum mechanics. Although you might wonder at first what we stand to gain by a mere change in notation, you will be more than convinced of its utility before this course ends.

Let us begin with the observation that a vector is completely specified by its components in a given basis. If we choose for convenience an orthonormal basis,

$$\mathbf{V} = \sum_{i=1}^{n} v_i \mathbf{e}_i \tag{1.3.1}$$

all vector operations—addition, scalar multiplication, and inner product—are carried out in terms of these v_i. Since the v_i are unique, there exists a one-to-one correspondence: to each vector there is a unique n-tuple of components and to each ordered n-tuple, there is a unique vector in $\mathbb{V}^n$. We represent this correspondence as follows, by collecting the n-tuple into a column vector:

$$\mathbf{V} \underset{\substack{\text{in the}\\ \text{given}\\ \text{basis}}}{\longleftrightarrow} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{1.3.2}$$

To every abstract operation performed on the vectors, there is a corresponding image in terms of the components: if, for example, we add two of them to get a third, the n-tuples (images) corresponding to the first two

DIRAC NOTATION

ABSTRACT VECTORS

(BASIS DEPENDENT REPRESENTATION)

$\vec{v}$ $\quad \mathbb{V}$

$|v\rangle$

no arrow

no bold

$$\vec{v} = \sum_i \alpha_i \hat{e}_i$$

$$\vec{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix}$$

ADDITION

$\vec{v} + \vec{w}$

$|v\rangle + |w\rangle$

$$\vec{v} = \sum \alpha_i \hat{e}_i$$

$$\vec{w} = \sum \beta_i \hat{e}_i$$

$$\vec{v} + \vec{w} = \sum (\alpha_i + \beta_i) \hat{e}_i$$

$$\vec{v} + \vec{w} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \vdots \\ \alpha_m \beta_m \end{pmatrix}$$

INNER PRODUCT

$$\langle v | w \rangle = \alpha_1^* \beta_1 + \alpha_2^* \beta_2 + \cdots + \alpha_m^* \beta_m$$

$$= (\alpha_1^*, \alpha_2^*, \alpha_3^*, \ldots, \alpha_m^*) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}$$

DIRAC NOTATION

ASSOCIATE COLUMN VECTORS WITH $|v\rangle$

KET VECTORS

ASSOCIATE COMPLEX-CONJUGATED ROW VECTORS WITH $\langle v|$

BRA VECTORS

KET SPACE

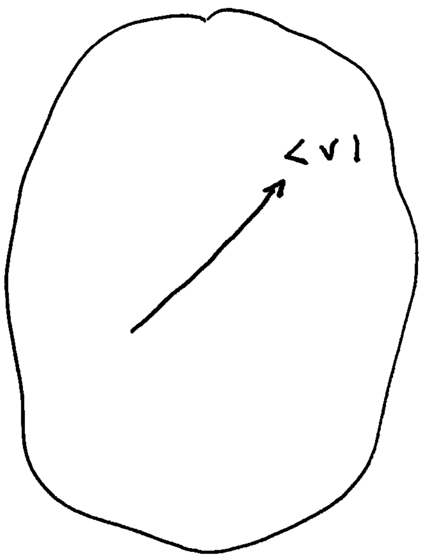
$$|v\rangle \xrightarrow[\text{in a basis}]{} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix}$$

BRA SPACE

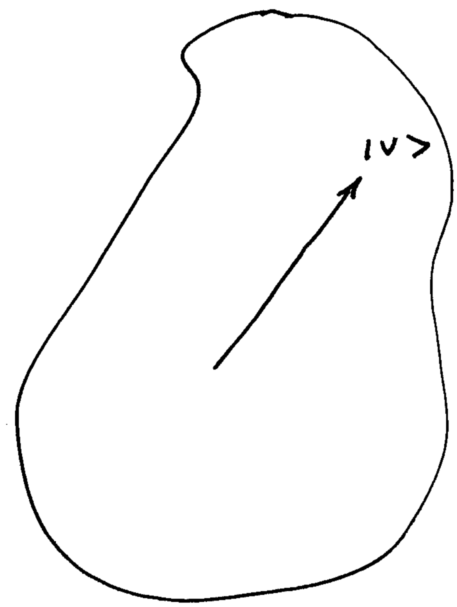
$$\langle v| \xrightarrow[\text{in a basis}]{} (\alpha_1^*, \alpha_2^*, \alpha_3^*, \ldots, \alpha_m^*)$$

PAIR OF
ISOMORPHIC DUAL SPACES

BRA SPACE



KET SPACE



THEY HAVE
EXACTLY THE SAME GEOMETRY, ALGEBRA, ...

DIRAC'S PUN

BRACKET

$$\langle v \,|\, w \rangle$$

BRA - KET

OUTER PRODUCT

$|v\rangle\langle w|$ is an operator

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix} \begin{pmatrix} \beta_1^* & \beta_2^* & \beta_3^* & \cdots \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1^* & \alpha_1\beta_2^* & \alpha_1\beta_3^* & \cdots \\ \alpha_2\beta_1^* & \alpha_2\beta_2^* & \alpha_2\beta_3^* & \cdots \\ \vdots & & & \end{pmatrix}$$

ORDER MATTERS!

$$\underbrace{|a\rangle}_{\text{vector}}\;\underbrace{\langle b|c\rangle}_{\text{inner product = scaler}}$$

$$\underbrace{|a\rangle\langle b|}_{\text{outer product}}\;\underbrace{|c\rangle}_{\text{vector}}$$

"

MATRIX

ORTHONORMAL BASIS VECTORS

$\hat{e}_i$

$\hat{x}\ \hat{y}\ \hat{z}$

$\hat{i}\ \hat{j}\ \hat{k}$

$\hat{r}\ \hat{\theta}\ \hat{\varphi}$

$$|e_i\rangle = |i\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ place}$$

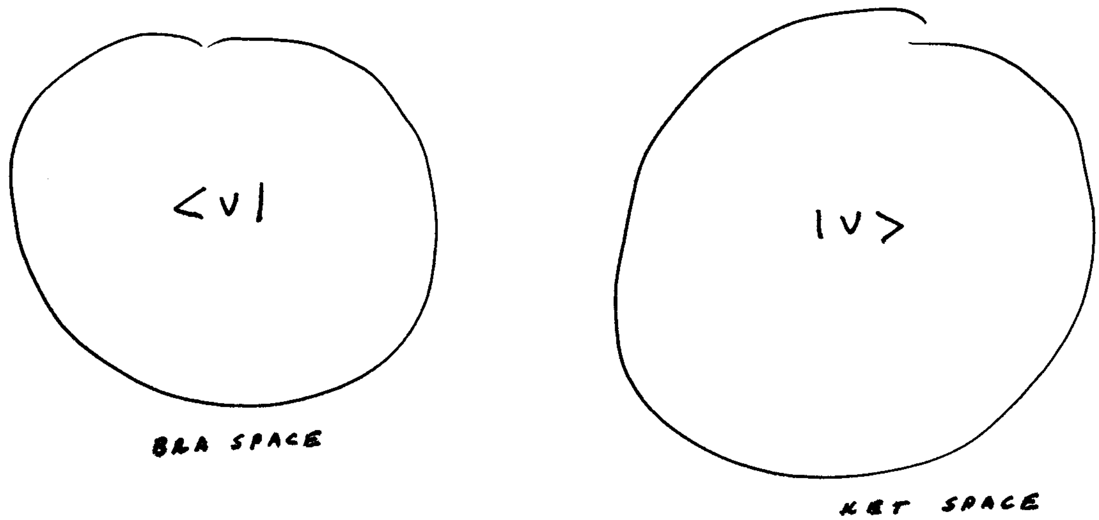
$$\langle e_i| = \langle i| = (0\ 0\ 0\ \ldots\ \underset{\substack{\uparrow \\ i^{th} \text{ place}}}{1}\ \ldots\ 0\ 0\ 0\ \ldots\ 0)$$

EXPANSION IN AN ORTHONORMAL BASIS

$$|v\rangle = \sum v_i\, |i\rangle$$

$$\langle v| = \sum v_i^*\, \langle i|$$

WANT GEOMETRY EXACTLY THE SAME



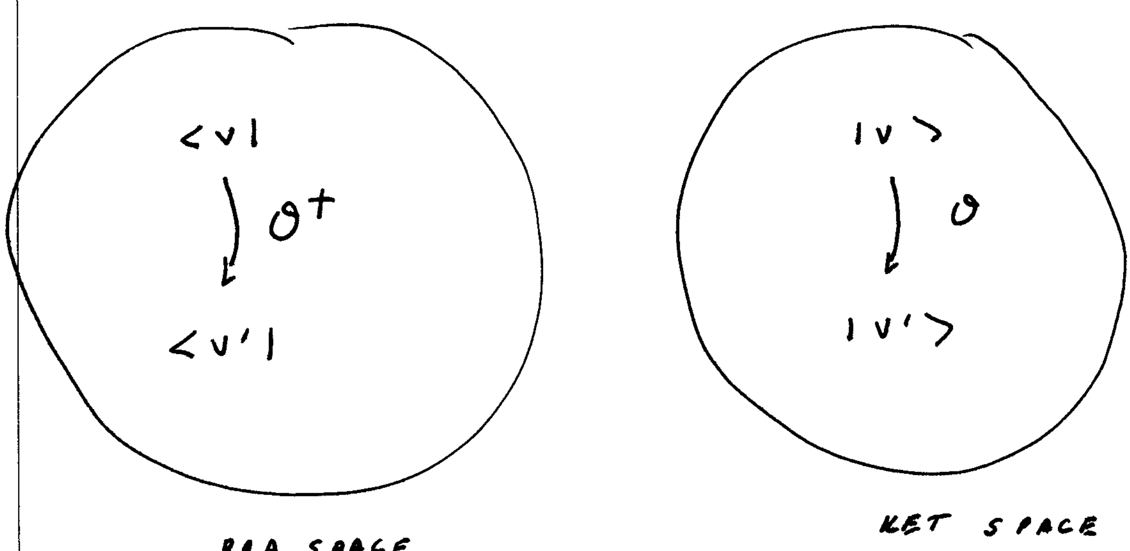
THIS DETERMINES WHAT HAPPENS TO

SCALARS $\quad \alpha = a + b i \xrightarrow{\text{COMPLEX CONJUGATION}} \alpha^* = a - b i$

VECTORS $\quad |v\rangle \xrightarrow{\text{ADJOINT}} \langle v|$ $\qquad$ TRANSPOSE CONJUGATE

$\qquad\qquad \langle v| \longrightarrow |v\rangle$

OPERATORS $\quad \mathcal{O} \xrightarrow{\text{ADJOINT}} \mathcal{O}^+$



$\langle v'| = \langle \mathcal{O} v| = \langle v| \mathcal{O}^+$ $\qquad$ $|v'\rangle = |\mathcal{O} v\rangle = \mathcal{O}|v\rangle$

ALGEBRA

$$|v\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \end{pmatrix}$$

is Ω Hermitean?

NO! $\Omega \neq \Omega^{+} = (\Omega^{T})^{*}$

$$\Omega\,|v\rangle = |w\rangle \qquad |w\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\langle v|\,\Omega = (1\ 0\ 0\ 0 \cdots) \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\langle v|\,\Omega \neq \langle w|$$

transpose & conjugate

$$\langle v|\,\Omega^{+} = (1\ 0\ 0\ 0 \cdots) \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \end{pmatrix} = (0\ 1\ 0\ 0 \cdots)$$

$$\langle v|\,\Omega^{+} = \langle w|$$

When $\Omega = \Omega^{+}$, left action and right action are the same!

ANALOGY BETWEEN OPERATORS AND COMPLEX NUMBERS

COMPLEX CONJUGATION

$$\alpha = a + bi$$
$$\alpha^* = a - bi$$

ADJOINT OPERATION

$$\Omega = H + A$$
$$\Omega^+ = H^+ + A^+$$
$$= H - A$$

H IS HERMITEAN

A IS ANTI-HERMITEAN

REAL NUMBER COMPONENT $\quad \mathrm{Re}\,\alpha \quad \mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \alpha^*)$

IMAGINARY NUMBER COMPONENT $\quad \mathrm{Im}(\alpha) = \frac{1}{2}(\alpha - \alpha^*)$

HERMITEAN OPERATOR $\quad H = H^+ \quad H = \frac{1}{2}(\Omega + \Omega^+)$

ANTI-HERMITEAN OPERATOR $\quad A = \frac{1}{2}(\Omega - \Omega^+)$

$$A = -A^+$$

COMPLEX NUMBER ON THE UNIT CIRCLE

$$\beta = e^{i\varphi}$$
$$|\beta| = 1$$

UNITARY OPERATOR

$$UU^+ = U^+U = I$$

$$U = \begin{pmatrix} e^{i\varphi_1} & & \\ & e^{i\varphi_2} & \\ & & \ddots \end{pmatrix}$$

EXAMPLE

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

IS Ω HERMITEAN?

DOES $\Omega = \Omega^{+}$ YES

ev and $e\vec{v}$ OF Ω

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

NORMALIZE

$$N \begin{pmatrix} 1 & 1 \end{pmatrix}^{*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \qquad 2N = 1 \qquad N = \frac{1}{\sqrt{2}}$$

$ev = 1$ $\quad e\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$ev = -1$ $\quad e\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

PROJECTION ONTO THE $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ SUBSPACE

$|i\rangle\langle i|$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}^* = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

ONTO THE $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ SUBSPACE

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}^* = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

COMPLETE $\Rightarrow$ $\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{I}$

$$\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

PROJECT $\underbrace{|i\rangle\langle i|}A\rangle$ OUTER PRODUCT FIRST

$$\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a+b\\a+b\end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}(a+b)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a-b\\-a+b\end{pmatrix} \rightarrow \cancel{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}}$$

$$\frac{1}{\sqrt{2}}(a-b)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

PROJECT $|i\rangle\langle i|A\rangle$ INNER PRODUCT FIRST

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}^{*}\begin{pmatrix}a\\b\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(a+b)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}^{*}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(a-b)$$