

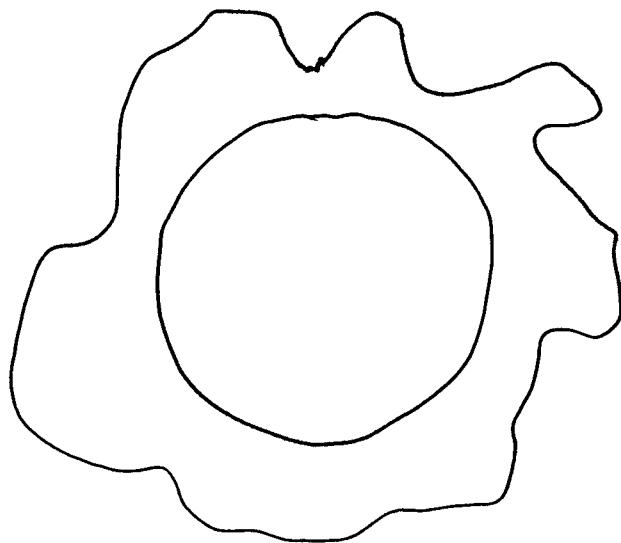
Lecture 14 Angular Momentum

Math: Spherical harmonics are a complete set of orthonormal functions on the sphere

Physics: Spherical harmonics describe the orbital angular momentum

MATH : SPHERICAL HARMONICS ARE
A COMPLETE ORTHONORMAL
SET OF BASIS FUNCTIONS
ON THE SPHERE.

PHYSICS : SPHERICAL HARMONICS DESCRIBE
THE ORBITAL ANGULAR MOMENTUM

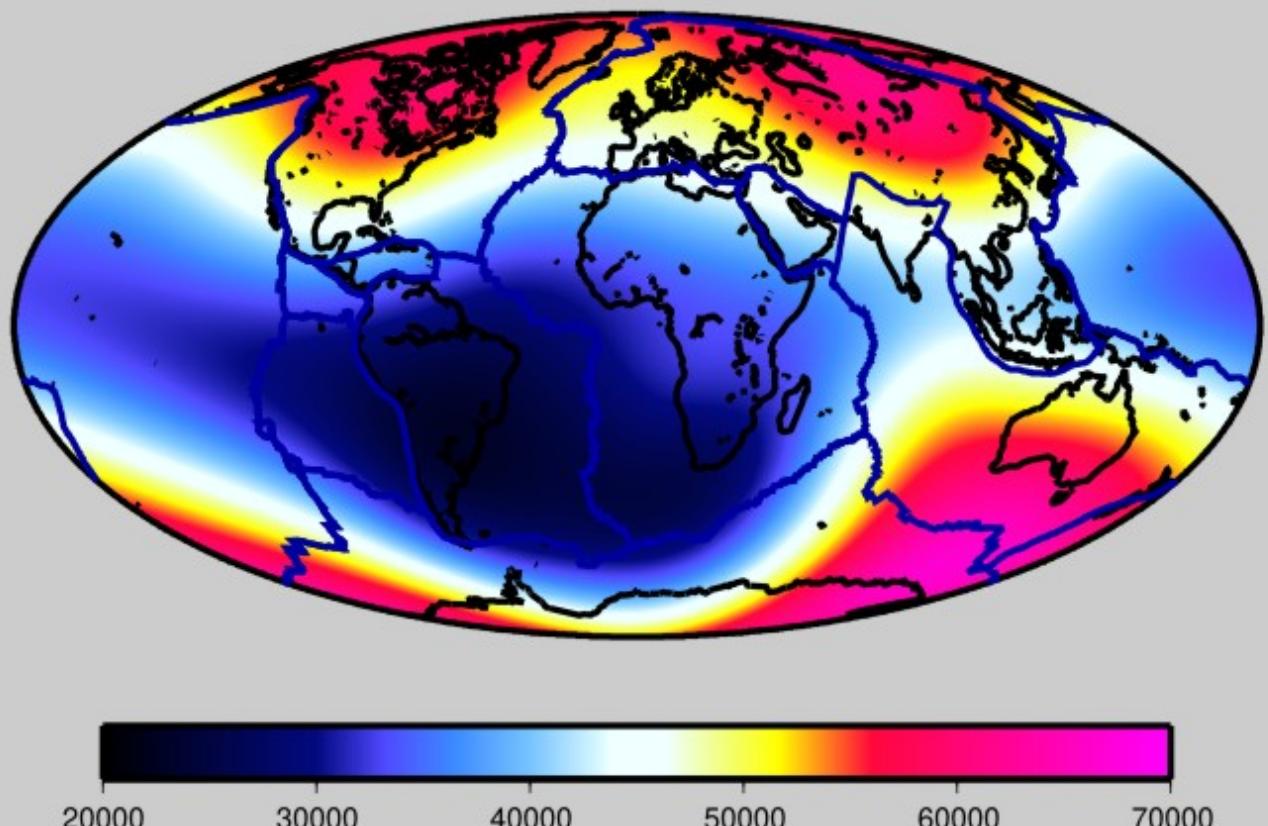


$$\text{ANY FCN}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta, \varphi)$$

$$a_{lm} = \int \text{ANY FCN}(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega$$

$$\langle \theta, \varphi | A \rangle = \sum_l \sum_m | l, m \rangle \langle l, m | A \rangle$$

Strength of the magnetic field in 2006



NOAA/NGDC

Gauss' 1838 spherical harmonic expansion of the earth's magnetic field

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G. D. Garland

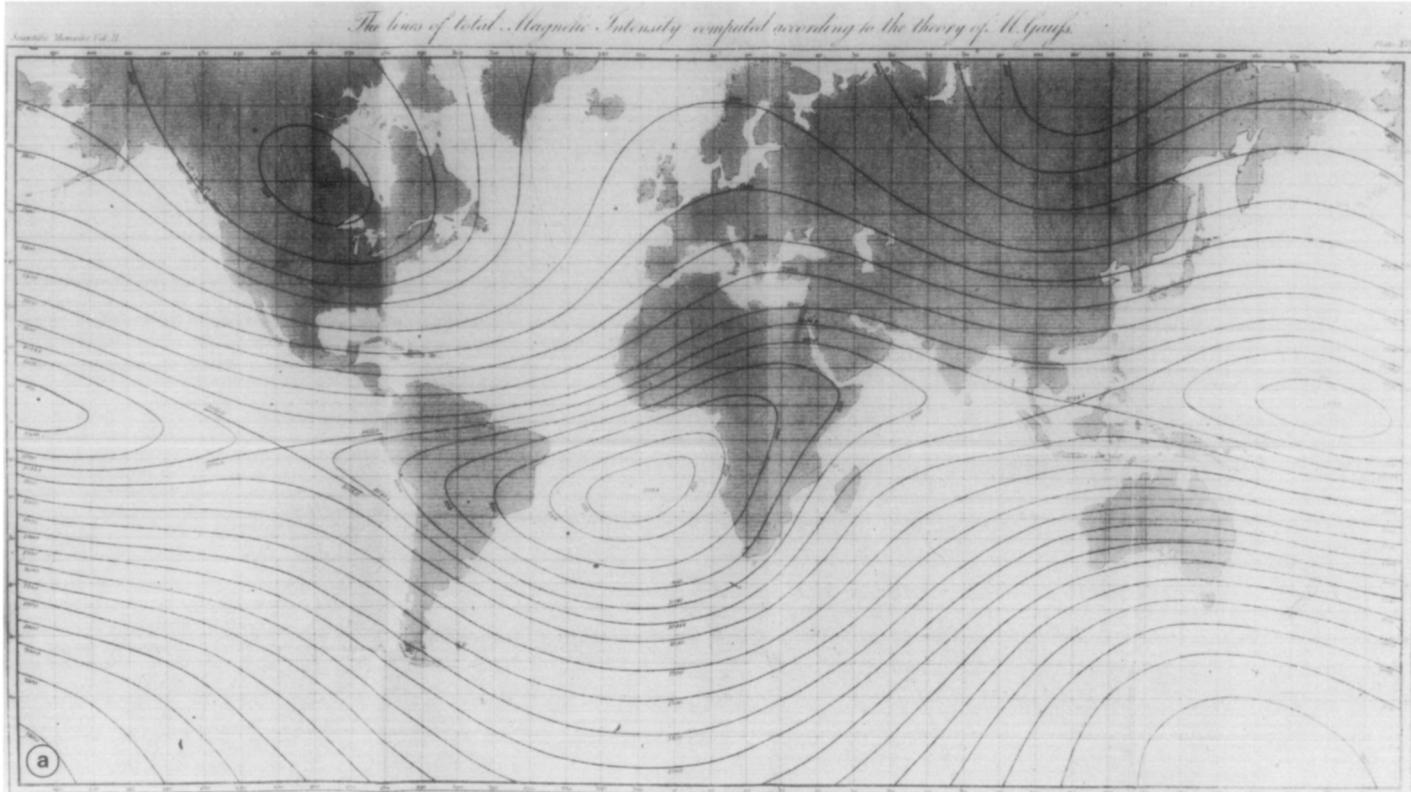
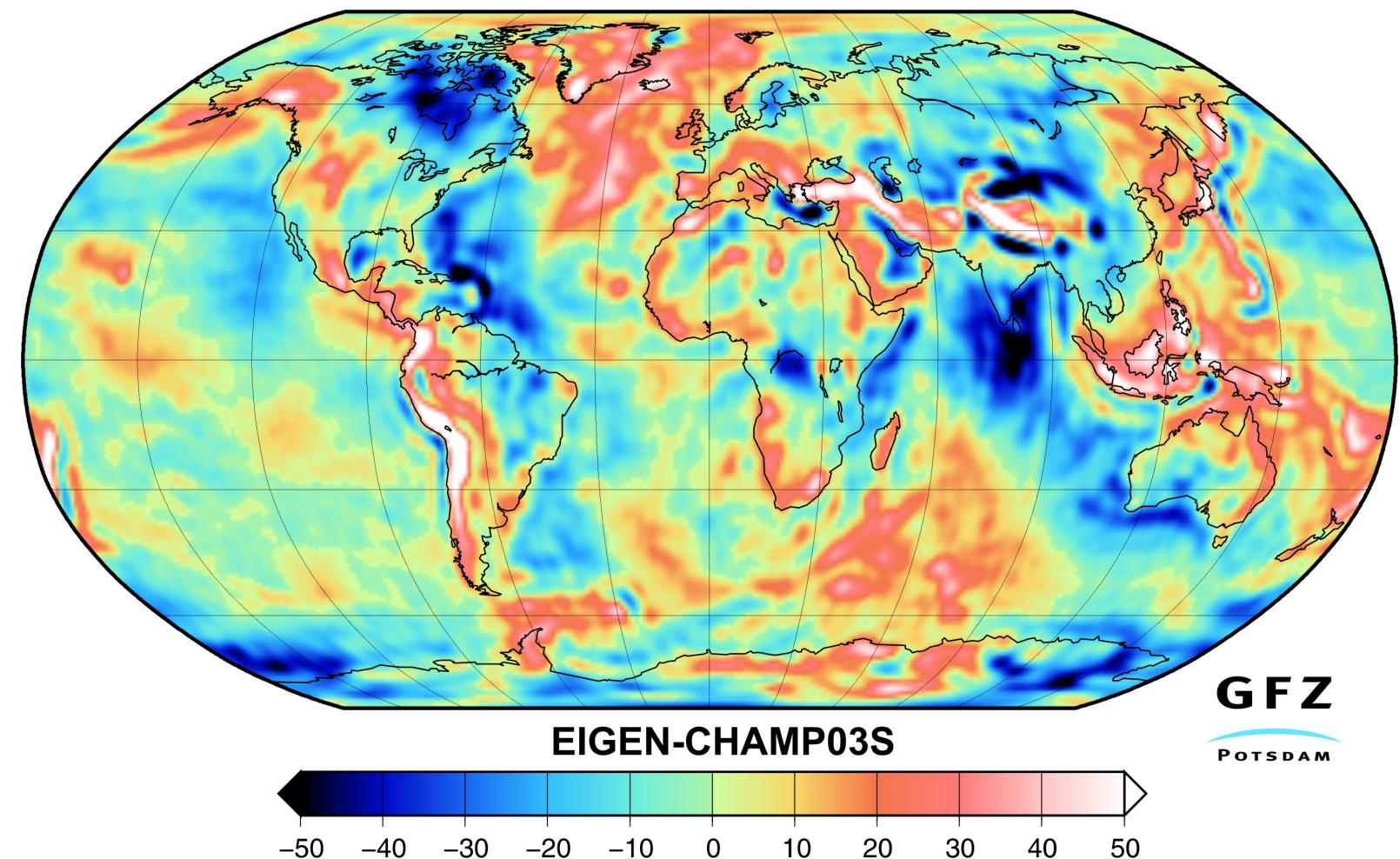


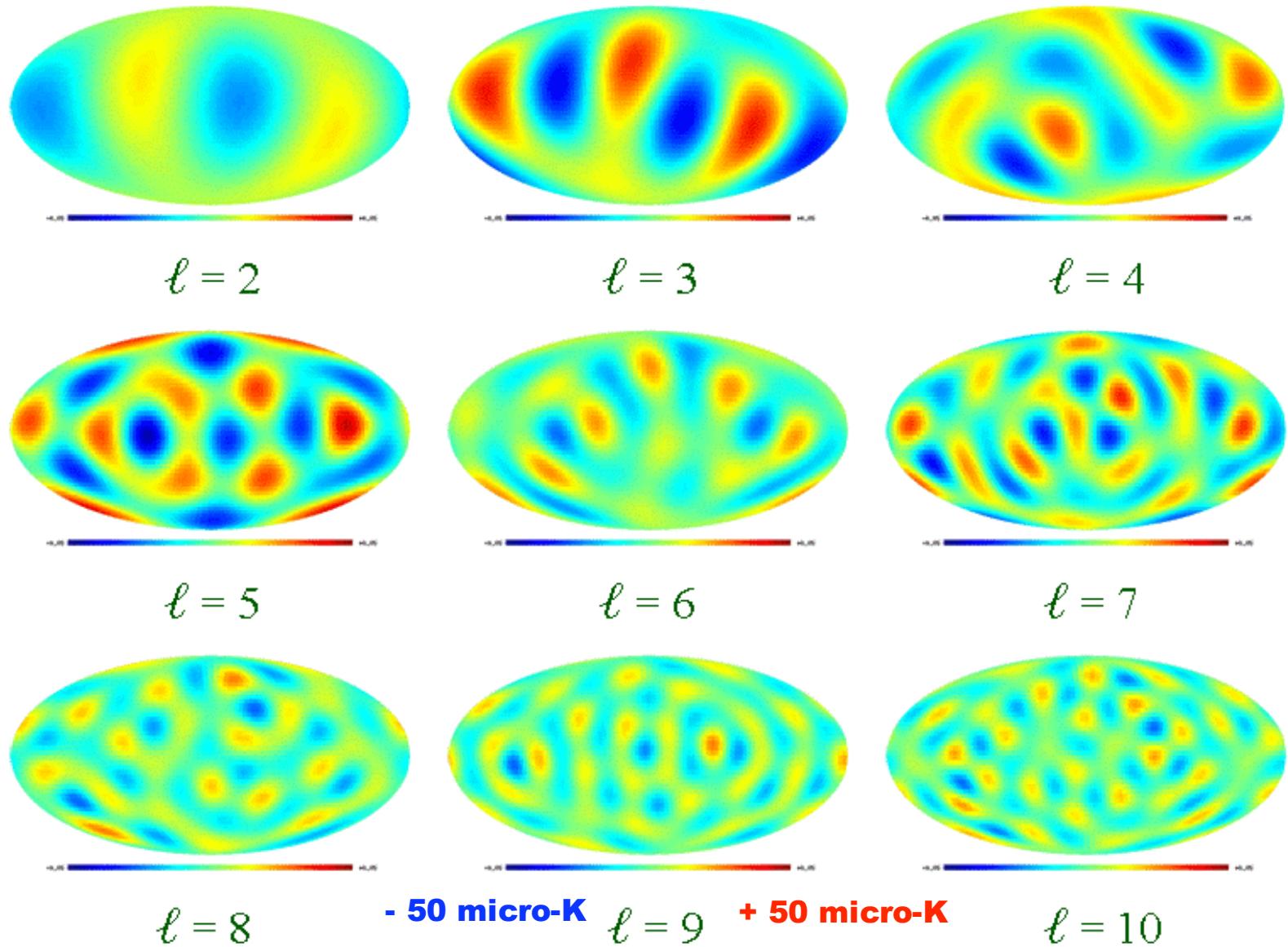
Figure 6: (a) The total intensity field of the earth, as synthesized by Gauss from his spherical harmonic expansion to the fourth degree. It may be compared with Figure 6 (b), a modern chart (the International Geomagnetic Reference Field) of the total intensity.

HM6

Gravitational Field



CMB cosmic microwave background spherical harmonic expansion



<http://www.orbitals.com/orb/orbtable.htm>

http://taras-zavedy.narod.ru/PROGRAMMS/ATOM_ORBITALS_v_1_5_ENG/Atom_Orbitals_v_1_5_ENG.html

WHERE WE ARE GOING

SOLVE $\hat{e}_r, \hat{e}_{\theta}, \hat{e}_{\phi}$ PROBLEM FOR ANGULAR MOMENTUM

$$\hat{L}^2 |l, m\rangle = l(l+1) \hbar^2$$

$$\hat{L}_z |l, m\rangle = m \hbar |l, m\rangle$$

$$\hat{L}_{\pm} |l, m\rangle = \sqrt{l(l+1)-m(m\pm 1)} \hbar |l, m\pm 1\rangle$$

THREE REPRESENTATIONS

DIRAC REP (IN THE HILBERT SPACE)

MATRIX REP (IN ANG MOMENTUM SPACE)

FUNCTION REP (IN POSITION SPACE θ, ϕ)

$$\ell = 1$$

$$L^2 = \ell(\ell+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

every vector is an $e\vec{v}$

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e\vec{v} \Rightarrow \frac{i}{\sqrt{2}} \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$L_4 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$e^{\hat{L}_4 t} \Rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

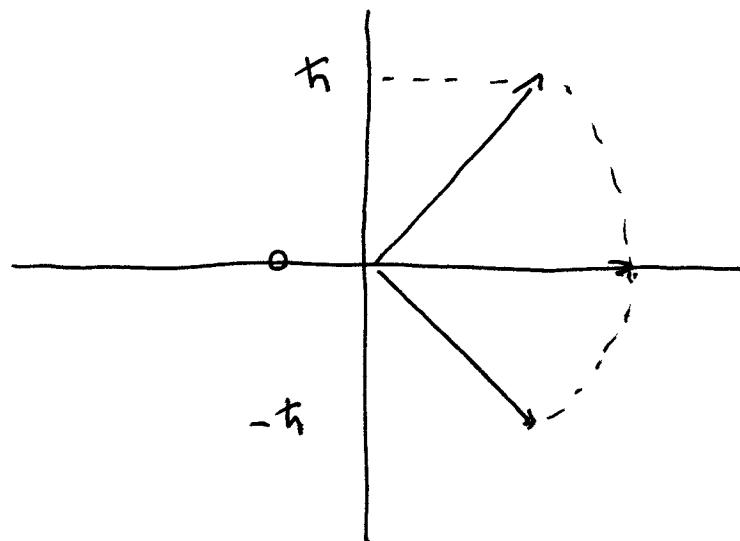
$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_0(x) \\ \psi_-(x) \end{pmatrix}$$

PROB AMP $S_2 = 1$
 PROB AMP $S_2 = 0$
 PROB AMP $S_2 = -1$

$|l, l\rangle$
 $|l, 0\rangle$
 $|l, -l\rangle$

$$\langle l^2 \rangle = l(l+1)\hbar^2 = 2\hbar^2$$

length = $\sqrt{2}\hbar$



$$\langle l^2 \rangle = l(l+1)\hbar^2 = 2\hbar^2$$

$$\text{LENGTH} = \sqrt{2}\hbar$$

$$\langle s_z \rangle = m\hbar$$

$$\langle s_x \rangle = 0$$

$$\langle s_y \rangle = 0$$

NEXT STEP ... go into real space

$$|l, m\rangle$$

$$\langle \theta, \varphi | l, m \rangle = Y_{lm}(\theta, \varphi)$$

$$\langle l, m | \theta, \varphi \rangle = Y_{lm}^*(\theta, \varphi)$$

$$\langle l', m' | l, m \rangle = \delta_{ll'} \delta_{mm'}$$

$$\int Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d(\cos \theta) d\varphi = \delta_{ll'} \delta_{mm'}$$

Q: HOW TO FIND ALL THE Y_{lm} 'S?

A: solve the differential equation ... OR

JUST LIKE SHO, WE HAVE TWO CHOICES

$$L^2 |\ell, m\rangle = \ell(\ell+1) \hbar^2 |\ell, m\rangle$$

$$L_z |\ell, m\rangle = m \hbar |\ell, m\rangle$$

$$-\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{\ell m}(\theta, \varphi) = \ell(\ell+1) \hbar^2 Y_{\ell m}(\theta, \varphi)$$

$$\ell(\ell+1) \hbar^2 Y_{\ell m}(\theta, \varphi)$$

$$-i \hbar \frac{\partial}{\partial \varphi} Y_{\ell m}(\theta, \varphi) = m \hbar Y_{\ell m}(\theta, \varphi)$$

OR

$$L_+ Y_{\ell \ell}(\theta, \varphi) = 0$$

$$\Rightarrow Y_{\ell \ell}$$

$\downarrow L_-$

$$Y_{\ell, \ell-1}$$

$\downarrow L_-$

$$\text{or } L_- Y_{\ell-1, \ell}(\theta, \varphi) = 0 \Rightarrow Y_{\ell-1, \ell}$$

next step find $F_{em}(\theta)$'s

$$L + Y_{ee}(\theta, \varphi) = 0$$

~~$$k e^{i\varphi} \left[\frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right] [F_{ee}(\theta) e^{i\ell\varphi}] = 0$$~~

~~$$\frac{d F_{ee}}{d\theta} e^{i\ell\varphi} + i \cot\theta F_{ee} i\ell e^{i\ell\varphi} = 0$$~~

$$\left[\frac{d}{d\theta} - \ell \cot\theta \right] F_{ee}(\theta) = 0$$

try $F_{ee} = A (\sin\theta)^\ell$

$$\frac{d}{d\theta} A (\sin\theta)^\ell = A (\sin\theta)^{\ell-1} \cos\theta$$

$$A (\sin\theta)^{\ell-1} \cos\theta - \ell \frac{\cos\theta}{\sin\theta} A (\sin\theta)^\ell \stackrel{?}{=} 0$$

yep!

$$\Rightarrow Y_{ee}(\theta, \varphi) = A_\ell (\sin\theta)^\ell e^{i\ell\varphi}$$

TO GET THE REST

$$L_- Y_{ee} = a Y_{e e-1}$$

+ then normalize using

$$\int Y_{em}^* Y_{em} d\Omega = 1$$

$$L_- = k e^{-i\varphi} \left(\frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right)$$

SPIRICAL HARMONICS

$$l=0$$

$$Y_{00}(\theta, \varphi) = \langle \theta, \varphi | 0, 0 \rangle = \frac{1}{\sqrt{4\pi}}$$

$$\int \left(\frac{1}{\sqrt{4\pi}} \right)^* \frac{1}{\sqrt{4\pi}} d\Omega = 1 \quad \checkmark$$

$$l=1$$

$$\left. \begin{aligned} Y_{11}(\theta, \varphi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} \\ Y_{10}(\theta, \varphi) &= \sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_{1-1}(\theta, \varphi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi} \end{aligned} \right\} \text{note symmetry}$$

$$l=2$$

$$Y_{2\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} (\sin \theta)^2 e^{\pm 2i\varphi}$$

$$Y_{1\pm 1}(\theta, \varphi) = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\varphi}$$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

ORBITAL ANGULAR MOMENTUM

$$l = 0, 1, 2, 3, \dots$$

$$m = -l, \dots, 0, \dots, l$$

$$L^2 |l, m\rangle = l(l+1) \hbar^2 |l, m\rangle$$

$$L_z |l, m\rangle = m \hbar |l, m\rangle$$

$$L_{\pm} |l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)} \hbar |l, m \pm 1\rangle$$

SPECIAL CASES:

$$l = 0 \Rightarrow m = 0$$

$$\underline{\hspace{1cm}} |0, 0\rangle$$

$$L^2 |0, 0\rangle = 0 \hbar^2 |0, 0\rangle$$

$$L_z |0, 0\rangle = 0 \hbar |0, 0\rangle$$

$$l=1 \quad m = -1, 0, +1$$

$$\text{---} \quad |1,1\rangle$$

$$\text{---} \quad |1,0\rangle$$

$$\text{---} \quad |1,-1\rangle$$

$$\mathcal{L}^2 |1,m\rangle = i(i+1) \hbar^2 |1,m\rangle = 2 \hbar^2 |1,m\rangle$$

$$\mathcal{L}_z |1,m\rangle = m \hbar |1,m\rangle$$

$$l=2 \quad m = -2, -1, 0, +1, +2$$

$$\text{---} \quad |2,2\rangle$$

$$\text{---} \quad |2,1\rangle$$

$$\text{---} \quad |2,0\rangle$$

$$\text{---} \quad |2,-1\rangle$$

$$\text{---} \quad |2,-2\rangle$$

$$\mathcal{L}^2 |2,m\rangle = 6 \hbar^2 |2,m\rangle$$

$$\mathcal{L}_z |2,m\rangle = m \hbar |2,m\rangle$$

Spherical Harmonics

The Meaning of the Spherical Harmonics

http://infovis.uni-konstanz.de/research/projects/SimSearch3D/images/harmonics_img.jpg

The Spherical Harmonics

<http://oak.ucc.nau.edu/jws8/dpgraph/Yellm.html>

<http://www.bpreid.com/applets/poasDemo.html>

<http://www.du.edu/~jcalvert/math/harmonic/harmonic.htm>

Encyclopedia

http://en.wikipedia.org/wiki/Spherical_harmonics

http://en.wikipedia.org/wiki/Table_of_spherical_harmonics

<http://mathworld.wolfram.com/SphericalHarmonic.html>

Applications of Spherical Harmonics

<http://www.falstad.com/qmrotator/>

<http://www.falstad.com/qmatom/>

<http://www.falstad.com/qmatomrad/>

<http://www.falstad.com/qm2dosc/>

<http://www.falstad.com/qm3dosc/>

Legendre Polynomials

The Meaning of the Legendre Polynomials

<http://physics.unl.edu/~tgay/content/multipoles.html>

Encyclopedia

http://en.wikipedia.org/wiki/Legendre_polynomials

<http://mathworld.wolfram.com/LegendrePolynomial.html>

Wolfram Demonstrations

<http://demonstrations.wolfram.com/SphericalHarmonics/>

<http://demonstrations.wolfram.com/VisualizingAtomicOrbitals/>

<http://demonstrations.wolfram.com/HydrogenOrbitals/>

<http://demonstrations.wolfram.com/PlotsOfLegendrePolynomials/>

<http://demonstrations.wolfram.com/PolarPlotsOfLegendrePolynomials/>

<http://demonstrations.wolfram.com/DipoleAntennaRadiationPattern/>

Magnetism

http://cgc.rncan.gc.ca/geomag/nmp/early_nmp_e.php?p=1

<http://www.agu.org/history/mf/articles/Geomag19.html>

http://www.nationalatlas.gov/articles/geology/a_geomag.html

http://geomag.org/info/Declination/magnetic_lines.avi

<http://www.geomag.us/models/HDGM.html>

<http://geomag.usgs.gov/observatories/>

<http://geomag.usgs.gov/>

http://www.intermagnet.org/lmomap_e.php

http://www.intermagnet.org/lmophoto_e.php

Spherical Harmonic Lighting

<http://www.youtube.com/watch?v=HHHoemb3C3Y>

<http://www.youtube.com/watch?v=QTkuK0-sdY8>

<http://www.yasrt.org/shlighting/>

<http://www.youtube.com/watch?v=NeE3ndibBj0>

<http://vimeo.com/13456070>

Page 1

Epoch:

2005.0

Data-range:

2000.5 2009.3

IMF-By correlated fields in GSM (IMF-By * Y1,1 and IMF-By * Y1,-1):

-0.0953 -0.2317

SM-Factor for Est/Ist field:

0.7930

SM-External field:

2							
7.5722	0.0000	0.0000					
0.0000	0.4810	1.7378	0.0000	0.0000			

GSM-External field:

2							
12.8965	0.1091	-0.0336					
0.1239	-0.3959	-0.0674	-0.1529	0.1491			

Internal field:

Degree: static 1st-deriv 2nd-deriv-before-epoch 2nd-deriv-after-epoch

n	m	60	16	16	16	g_lm	h_lm	gd	hd	gdd-	hdd-	gdd+	hdd+
1	0	-29554.0320		0.0000	12.3043	0.0000		0.1774	0.0000	-0.3205	0.0000		
1	1	-1668.5468	5078.5517	14.4093	-22.1252	1.2383	-0.3087	1.0990	-1.7348				
2	0	-2337.3374		0.0000	-12.4496	0.0000		0.7286	0.0000	0.1715	0.0000		
2	1	3047.6077	-2594.2170	-5.1900	-22.3939	-0.3400	0.1400	0.3164	0.0802				
2	2	1658.0393	-514.6903	-0.1791	-10.7573	1.1047	0.4603	1.1523	-0.4873				
3	0	1336.5186		0.0000	0.0847	0.0000		0.1726	0.0000	0.2058	0.0000		
3	1	-2306.2089	-199.3190	-4.1021	5.8252	-0.2815	0.1320	-0.0064	0.6915				
3	2	1245.8632	269.7782	-2.0657	-5.4032	-0.4235	-0.2088	-0.4705	0.7741				
3	3	672.6386	-524.5811	-8.3661	-3.6868	-0.0817	1.2137	0.5442	0.6745				
4	0	920.7152		0.0000	-1.9877	0.0000		0.1953	0.0000	0.1858	0.0000		
4	1	797.9619	282.3571	2.5402	2.3638	0.1306	0.1953	-0.1081	-0.6271				
4	2	210.6560	-225.1683	-8.2170	2.1830	-0.2051	0.3687	-0.2767	0.2532				
4	3	-379.8818	144.8311	4.9933	4.6098	0.1030	-0.2390	-0.2143	-0.3514				
4	4	100.2480	-305.4853	-2.6737	-0.6528	-0.1730	-0.1448	0.3722	-0.0065				
5	0	-226.3028		0.0000	-1.5372	0.0000		0.1069	0.0000	0.3285	0.0000		
5	1	354.5269	42.3736	0.5608	0.0283	-0.0288	0.0948	-0.0134	0.1512				
5	2	208.9643	180.2456	-2.4455	2.2564	0.1185	0.2641	0.3249	-0.2303				
5	3	-136.5622	-123.4001	-1.2698	1.6466	-0.0206	-0.1399	0.1096	-0.2708				
5	4	-168.0214	-19.5401	0.3752	4.3029	0.1152	0.1299	0.2603	-0.1362				
5	5	-13.4622	103.9000	0.1599	-0.5521	0.1775	-0.0072	0.4828	-0.0100				
6	0	73.8148		0.0000	0.0201	0.0000	-0.0716	0.0000	-0.0581	0.0000			
6	1	69.4836	-20.2061	0.0244	-0.3892	-0.1310	0.0918	-0.0854	0.1336				
6	2	76.6932	54.7768	0.1785	-1.9035	-0.1208	-0.0516	-0.1295	-0.0986				
6	3	-151.3175	63.6634	1.8513	-0.4189	-0.0476	-0.0419	0.0611	0.0039				
6	4	-14.6077	-63.5725	-1.7468	-0.5801	0.0350	-0.0591	0.0282	0.0026				
6	5	14.5603	0.3299	-0.5578	0.1319	-0.0560	0.0944	0.1002	0.1953				
6	6	-86.3087	50.9758	1.3193	1.5722	0.2451	0.0974	0.2008	-0.3718				
7	0	79.8914		0.0000	0.0361	0.0000	-0.0539	0.0000	0.0378	0.0000			
7	1	-74.3646	-61.1191	-0.0821	0.7750	0.0066	0.0559	-0.0181	-0.0671				
7	2	-1.6481	-22.5693	-0.5438	0.2373	-0.0974	-0.0615	-0.0527	0.0177				
7	3	38.7218	6.8116	1.1351	0.0587	0.0241	-0.0215	0.0731	-0.0374				
7	4	12.2839	25.3445	0.6437	0.0914	0.0105	-0.0824	-0.1302	-0.0780				
7	5	9.3729	10.9127	0.5113	-0.8383	0.0115	-0.0234	-0.1340	0.0218				
7	6	5.3839	-26.3044	-0.5382	-0.1942	-0.0787	0.0159	-0.0966	-0.0410				
7	7	1.9634	-4.6967	0.7897	0.3422	0.0647	0.0501	-0.0686	-0.0488				

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60	6	-0.1037	0.0714	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	7	0.0576	0.0137	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	8	-0.0840	-0.1270	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	9	0.0243	0.0297	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	10	0.1207	-0.0575	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	11	-0.2090	0.0729	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	12	-0.1990	0.0025	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	13	-0.1107	0.0186	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	14	-0.0302	0.0739	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	15	0.0677	0.1226	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	16	0.0789	-0.0645	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	17	0.0522	-0.0372	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	18	0.1482	0.0105	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	19	0.0258	-0.0515	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	20	0.0991	-0.0766	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	21	-0.0223	-0.1903	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	22	-0.0631	0.0071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	23	-0.0242	-0.0194	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	24	-0.0296	0.0439	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	25	0.0804	0.0263	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	26	-0.0220	0.1279	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	27	-0.1007	-0.0159	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	28	-0.1359	-0.0412	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	29	-0.0021	0.1878	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	30	0.1639	0.0242	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	31	-0.0382	0.1888	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	32	0.0748	-0.0050	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	33	0.0193	-0.0524	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	34	0.0024	0.0072	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	35	0.0682	0.0081	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	36	0.1223	0.0812	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	37	-0.0088	-0.0281	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	38	-0.1437	0.0689	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	39	0.0152	-0.0367	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	40	0.0357	-0.0412	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	41	-0.0289	0.0457	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	42	0.0664	-0.0401	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	43	-0.0066	-0.0854	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	44	-0.0674	-0.0140	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	45	0.0131	0.0246	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	46	-0.0464	-0.0434	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	47	-0.0305	-0.0064	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	48	-0.0463	-0.0710	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	49	-0.1221	-0.0297	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	50	0.0428	-0.0408	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	51	0.0076	0.0043	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	52	0.0729	-0.0620	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	53	0.0473	-0.0544	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	54	-0.0346	-0.0224	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	55	0.0081	-0.0146	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	56	-0.1153	-0.0254	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	57	0.0772	-0.0471	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	58	0.1031	0.0662	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	59	0.1217	0.0133	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	60	0.0764	-0.0833	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Other Examples

The Earth's Magnetic Field

http://cgc.rncan.gc.ca/geomag/nmp/early_nmp_e.php?p=1
http://en.wikipedia.org/wiki/Earth%27s_magnetic_field
<http://www.ngdc.noaa.gov/geomag/WMM/DoDWMM.shtml>
http://www.geomag.us/info/Declination/magnetic_lines_2010.gif

The Earth's Gravitational Field

<http://en.wikipedia.org/wiki/Geoid>
http://www.esri.com/news/arcuser/0703/graphics/geoid1_lg.gif
<http://www.geomag.us/models/pomme5.html>
<http://earth-info.nga.mil/GandG/images/ww15mgh2.gif>
<http://op.gfz-potsdam.de/champ/>
http://www.gfy.ku.dk/~pditlev/annual_report/matematiker.jpg

The Universe

http://abyss.uoregon.edu/~js/21st_century_science/lectures/lec27.html
http://wmap.gsfc.nasa.gov/media/080997/080997_5yrFullSky_WMAP_4096B.tif

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Art

<http://www.math.hawaii.edu/~dale/bleecker/bleecker.html>
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