

As in the case of the infinite square well, the stationary states of the harmonic oscillator are orthogonal:

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \delta_{mn}. \quad [2.68]$$

This can be proved using Equation 2.65, and Equation 2.64 twice—first moving a_+ and then moving a_- :

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_m^* (a_+ a_-) \psi_n dx &= n \int_{-\infty}^{\infty} \psi_m^* \psi_n dx \\ &= \int_{-\infty}^{\infty} (a_- \psi_m)^* (a_- \psi_n) dx = \int_{-\infty}^{\infty} (a_+ a_- \psi_m)^* \psi_n dx \\ &= m \int_{-\infty}^{\infty} \psi_m^* \psi_n dx. \end{aligned}$$

Unless $m = n$, then, $\int \psi_m^* \psi_n dx$ must be zero. Orthonormality means that we can again use Fourier's trick (Equation 2.34) to evaluate the coefficients, when we expand $\Psi(x, 0)$ as a linear combination of stationary states (Equation 2.16), and $|c_n|^2$ is again the probability that a measurement of the energy would yield the value E_n .

Example 2.5 Find the expectation value of the potential energy in the n th state of the harmonic oscillator.

Solution:

$$\langle V \rangle = \left\langle \frac{1}{2} m \omega^2 x^2 \right\rangle = \frac{1}{2} m \omega^2 \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n dx.$$

There's a beautiful device for evaluating integrals of this kind (involving powers of x or p): Use the definition (Equation 2.47) to express x and p in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-); \quad p = i \sqrt{\frac{\hbar m \omega}{2}} (a_+ - a_-). \quad [2.69]$$

In this example we are interested in x^2 :

$$x^2 = \frac{\hbar}{2m\omega} \left[(a_+)^2 + (a_+ a_-) + (a_- a_+) + (a_-)^2 \right].$$

So

$$\langle V \rangle = \frac{\hbar \omega}{4} \int \psi_n^* \left[(a_+)^2 + (a_+ a_-) + (a_- a_+) + (a_-)^2 \right] \psi_n dx.$$

ugly! primitive! boo!

But $(a_+)^2\psi_n$ is (apart from normalization) ψ_{n+2} , which is orthogonal to ψ_n , and the same goes for $(a_-)^2\psi_n$, which is proportional to ψ_{n-2} . So those terms drop out, and we can use Equation 2.65 to evaluate the remaining two:

$$\langle V \rangle = \frac{\hbar\omega}{4} (n + n + 1) = \frac{1}{2}\hbar\omega \left(n + \frac{1}{2}\right).$$

As it happens, the expectation value of the potential energy is exactly *half* the total (the other half, of course, is kinetic). This is a peculiarity of the harmonic oscillator, as we'll see later on.

*Problem 2.10

- (a) Construct $\psi_2(x)$.
 - (b) Sketch ψ_0 , ψ_1 , and ψ_2 .
 - (c) Check the orthogonality of ψ_0 , ψ_1 , and ψ_2 , by explicit integration. *Hint:* If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.
-

*Problem 2.11

- (a) Compute $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, and $\langle p^2 \rangle$, for the states ψ_0 (Equation 2.59) and ψ_1 (Equation 2.62), by explicit integration. *Comment:* In this and other problems involving the harmonic oscillator it simplifies matters if you introduce the variable $\xi \equiv \sqrt{m\omega/\hbar}x$ and the constant $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$.
 - (b) Check the uncertainty principle for these states.
 - (c) Compute $\langle T \rangle$ (the average kinetic energy) and $\langle V \rangle$ (the average potential energy) for these states. (No new integration allowed!) Is their sum what you would expect?
-

*Problem 2.12

Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, and $\langle T \rangle$, for the n th stationary state of the harmonic oscillator, using the method of Example 2.5. Check that the uncertainty principle is satisfied.

Problem 2.13

A particle in the harmonic oscillator potential starts out in the state

$$\Psi(x, 0) = A[3\psi_0(x) + 4\psi_1(x)].$$

- (a) Find A .
- (b) Construct $\Psi(x, t)$ and $|\Psi(x, t)|^2$.

- (c) Find $\langle x \rangle$ and $\langle p \rangle$. Don't get too excited if they oscillate at the classical frequency; what would it have been had I specified $\psi_2(x)$, instead of $\psi_1(x)$? Check that Ehrenfest's theorem (Equation 1.38) holds for this wave function.
 - (d) If you measured the energy of this particle, what values might you get, and with what probabilities?
-

Problem 2.14 A particle is in the ground state of the harmonic oscillator with classical frequency ω , when suddenly the spring constant quadruples, so $\omega' = 2\omega$, without initially changing the wave function (of course, Ψ will now *evolve* differently, because the Hamiltonian has changed). What is the probability that a measurement of the energy would still return the value $\hbar\omega/2$? What is the probability of getting $\hbar\omega$? [Answer: 0.943.]

2.3.2 Analytic Method

We return now to the Schrödinger equation for the harmonic oscillator,

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi, \quad [2.70]$$

and solve it directly, by the series method. Things look a little cleaner if we introduce the dimensionless variable

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x; \quad [2.71]$$

in terms of ξ the Schrödinger equation reads

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi, \quad [2.72]$$

where K is the energy, in units of $(1/2)\hbar\omega$:

$$K \equiv \frac{2E}{\hbar\omega}. \quad [2.73]$$

Our problem is to solve Equation 2.72, and in the process obtain the “allowed” values of K (and hence of E).

To begin with, note that at very large ξ (which is to say, at very large x), ξ^2 completely dominates over the constant K , so in this regime

$$\frac{d^2\psi}{d\xi^2} \approx \xi^2\psi, \quad [2.74]$$

which has the approximate solution (check it!)

$$\psi(\xi) \approx Ae^{-\xi^2/2} + Be^{+\xi^2/2}. \quad [2.75]$$

elements of other operators between oscillator eigenstates. Consider for example $\langle 3 | X^3 | 2 \rangle$. In the X basis one would have to carry out the following integral:

$$\begin{aligned}\langle 3 | X^3 | 2 \rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{1}{2^3 3!} \cdot \frac{1}{2^2 2!}\right)^{1/2} \int_{-\infty}^{\infty} \left\{ \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \right. \\ &\quad \times H_3\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right] x^3 \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_2\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right] \left. \right\} dx\end{aligned}$$

whereas in the $|n\rangle$ basis

$$\begin{aligned}\langle 3 | X^3 | 2 \rangle &= \left(\frac{\hbar}{2m\omega}\right)^{3/2} \langle 3 | (a + a^\dagger)^3 | 2 \rangle \\ &= \left(\frac{\hbar}{2m\omega}\right)^{3/2} \langle 3 | (a^3 + a^2 a^\dagger + a a^\dagger a + a a^\dagger a^\dagger + a^\dagger a a + a^\dagger a a^\dagger + a^\dagger a^\dagger a + a^\dagger a^\dagger a^\dagger) | 2 \rangle\end{aligned}$$

Since a lowers n by one unit and a^\dagger raises it by one unit and we want to go up by one unit from $n = 2$ to $n = 3$, the only nonzero contribution comes from $a^\dagger a^\dagger a$, $a a^\dagger a^\dagger$, and $a^\dagger a a^\dagger$. Now

$$\begin{aligned}a^\dagger a^\dagger a | 2 \rangle &= 2^{1/2} a^\dagger a^\dagger | 1 \rangle = 2^{1/2} 2^{1/2} a^\dagger | 2 \rangle = 2^{1/2} 2^{1/2} 3^{1/2} | 3 \rangle \\ a a^\dagger a^\dagger | 2 \rangle &= 3^{1/2} a a^\dagger | 3 \rangle = 3^{1/2} 4^{1/2} a | 4 \rangle = 3^{1/2} 4^{1/2} 4^{1/2} | 3 \rangle \\ a^\dagger a a^\dagger | 2 \rangle &= 3^{1/2} a^\dagger a | 3 \rangle = 3^{1/2} N | 3 \rangle = 3^{1/2} 3 | 3 \rangle\end{aligned}$$

so that **beautiful! magnificent! bravo!**

$$\langle 3 | X^3 | 2 \rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2} [2(3^{1/2}) + 4(3^{1/2}) + 3(3^{1/2})]$$

What if we want not some matrix element of X , but the probability of finding the particle in $|n\rangle$ at position x ? We can of course fall back on Postulate III, which tells us to find the eigenvectors $|x\rangle$ of the matrix X [Eq. (7.4.32)] and evaluate the inner product $\langle x | n \rangle$. A more practical way will be developed in the next section.

Consider a remarkable feature of the above solution to the eigenvalue problem of H . Usually we work in the X basis and set up the eigenvalue problem (as a differential equation) by invoking Postulate II, which gives the action of X and P in the X basis ($X \rightarrow x$, $P \rightarrow -i\hbar d/dx$). In some cases (the linear potential problem), the P basis recommends itself, and then we use the Fourier-transformed version of Postulate II, namely, $X \rightarrow i\hbar d/dp$,

BRAVO !!! ENCORE !!!

http://quantummechanics.ucsd.edu/ph130a/130_notes/130_notes.html

The expectation value of \hat{x} in eigenstate

We can compute the expectation value of \hat{x} simply.

$$\begin{aligned}\langle u_n | \hat{x} | u_n \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle u_n | A + A^\dagger | u_n \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\langle u_n | Au_n \rangle + \langle u_n | A^\dagger u_n \rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n} \langle u_n | u_{n-1} \rangle + \sqrt{n+1} \langle u_n | u_{n+1} \rangle) = 0\end{aligned}$$

We should have seen that coming. Since each term in the \hat{x} operator changes the eigenstate, the dot product with the original (orthogonal) state must give zero.

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$$\begin{aligned}< n | x | n > &= C < n | a^* + a | n > \\ &= C < n | a^* | n > + C < n | a | n > \\ &= C' < n | n+1 > + C'' < n | n-1 > = 0\end{aligned}$$

$$\begin{aligned}< 3 | x | 3 > &= C < 3 | a^* + a | 3 > \\ &= C < 3 | a^* | 3 > + C < 3 | a | 3 > \\ &= C' < 3 | 4 > + C'' < 3 | 2 > = 0\end{aligned}$$

$$\begin{aligned}C' &= \text{sqrt}(n+1) C & C' &= \text{sqrt}(4) C \\ C'' &= \text{sqrt}(n) C & C'' &= \text{sqrt}(3) C\end{aligned}$$

http://quantummechanics.ucsd.edu/ph130a/130_notes/130_notes.html

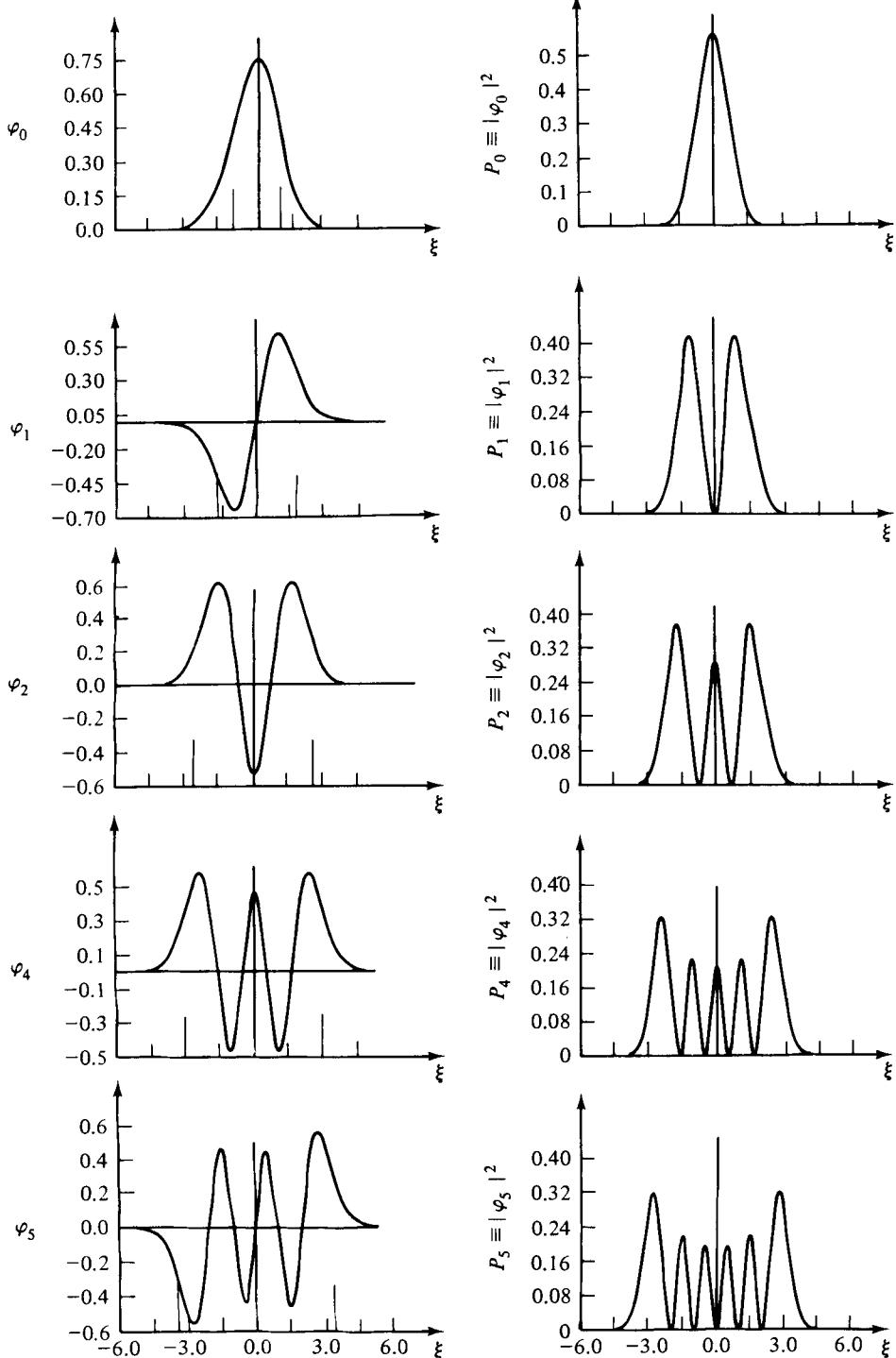


FIGURE 7.10 The first few eigenstates of the simple harmonic oscillator and corresponding probability densities. Turning points, $\xi_0^{(n)} = \sqrt{1 + 2n}$, are denoted by vertical marks.

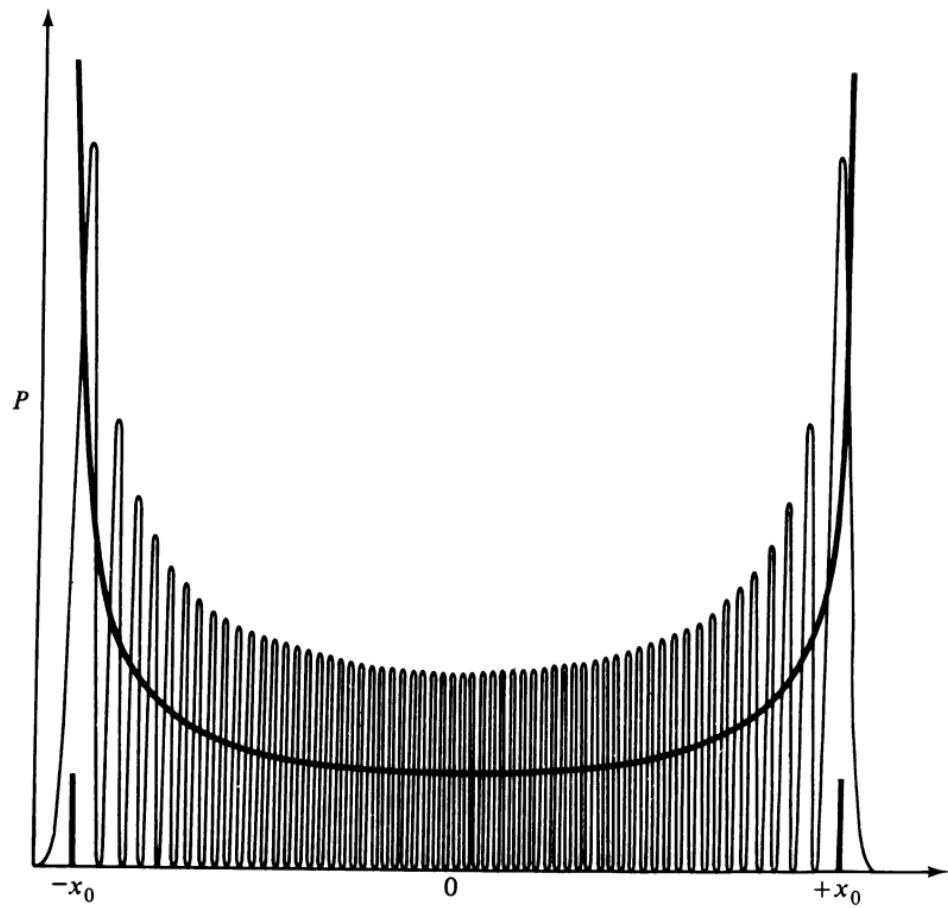


FIGURE 7.11 Classical probability density

The expectation value of $\frac{1}{2}m\omega^2x^2$ in eigenstate

The expectation of x^2 will have some nonzero terms.

$$\begin{aligned}\langle u_n | x^2 | u_n \rangle &= \frac{\hbar}{2m\omega} \langle u_n | \cancel{AA} + AA^\dagger + A^\dagger A + \cancel{A^\dagger A^\dagger} | u_n \rangle \\ &= \frac{\hbar}{2m\omega} \langle u_n | AA^\dagger + A^\dagger A | u_n \rangle\end{aligned}$$

$$\begin{aligned}< n | a a^* | n > &= \text{sqrt}(n+1) < n | a | n+1 > = n+1 < n | n > = n+1 \\ < n | a^* a | n > &= \text{sqrt}(n) < n | a^* | n-1 > = n < n | n > = n\end{aligned}$$

We could drop the AA term and the $A^\dagger A^\dagger$ term since they will produce 0 when the dot product is taken.

$$\begin{aligned}\langle u_n | x^2 | u_n \rangle &= \frac{\hbar}{2m\omega} (\langle u_n | \sqrt{n+1} A u_{n+1} \rangle + \langle u_n | \sqrt{n} A^\dagger u_{n-1} \rangle) \\ &= \frac{\hbar}{2m\omega} (\langle u_n | \sqrt{n+1} \sqrt{n+1} u_n \rangle + \langle u_n | \sqrt{n} \sqrt{n} u_n \rangle) \\ &= \frac{\hbar}{2m\omega} ((n+1) + n) = \left(n + \frac{1}{2} \right) \frac{\hbar}{m\omega}\end{aligned}$$

$$< n | x^2 | n > = C [(n+1) + n] = (n+1/2) 2C$$

With this we can compute the expected value of the potential energy.

$$\langle u_n | \frac{1}{2}m\omega^2 x^2 | u_n \rangle = \frac{1}{2}m\omega^2 \left(n + \frac{1}{2} \right) \frac{\hbar}{m\omega} = \frac{1}{2} \left(n + \frac{1}{2} \right) \hbar\omega = \frac{1}{2}E_n$$

The expectation value of x in the state $\frac{1}{\sqrt{2}}(u_0 + u_1)$. ($|0\rangle + |1\rangle$)

$$\langle x(0) \rangle \sim (\langle 0 | + \langle 1 |) (a^* + a) (|0\rangle + |1\rangle)$$

$$\begin{aligned} \frac{1}{2} \langle u_0 + u_1 | x | u_0 + u_1 \rangle &= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \langle u_0 + u_1 | A + A^\dagger | u_0 + u_1 \rangle \\ &= \sqrt{\frac{\hbar}{8m\omega}} \langle u_0 + u_1 | Au_0 + Au_1 + A^\dagger u_0 + A^\dagger u_1 \rangle \\ &= \sqrt{\frac{\hbar}{8m\omega}} \langle u_0 + u_1 | 0 + \sqrt{1}u_0 + \sqrt{1}u_1 + \sqrt{2}u_2 \rangle \\ &= \sqrt{\frac{\hbar}{8m\omega}} (\sqrt{1}\langle u_0 | u_0 \rangle + \sqrt{1}\langle u_0 | u_1 \rangle \\ &\quad + \sqrt{2}\langle u_0 | u_2 \rangle + \sqrt{1}\langle u_1 | u_0 \rangle \\ &\quad + \sqrt{1}\langle u_1 | u_1 \rangle + \sqrt{2}\langle u_1 | u_2 \rangle) \\ &= \sqrt{\frac{\hbar}{8m\omega}} (1 + 1) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \end{aligned}$$

$$\langle x(t) \rangle \sim \{ \langle 0 | \exp(i(1/2)w t) + \langle 1 | \exp(i(3/2)w t) \}$$

$$\{ a^* + a \}$$

$$\{ \exp(-i(1/2)w t) | 0 \rangle + \exp(-i(3/2)w t) | 1 \rangle \}$$

Time Development Example

Start off in the state at $t = 0$.

$$|\psi(0)\rangle = N(|1\rangle + |2\rangle)$$

$$\psi(t=0) = \frac{1}{\sqrt{2}}(u_1 + u_2)$$

Now put in the simple time dependence of the energy eigenstates, $e^{-iEt/\hbar}$.

$$\psi(t) = \frac{1}{\sqrt{2}}(u_1 e^{-i\frac{3}{2}\omega t} + u_2 e^{-i\frac{5}{2}\omega t}) = \frac{1}{\sqrt{2}}e^{-i\frac{3}{2}\omega t}(u_1 + e^{-i\omega t}u_2)$$

$$|\psi(t)\rangle = N\{|1\rangle \exp(-i(3/2)\omega t) + |2\rangle \exp(-i(5/2)\omega t)\}$$

We can compute the expectation value of \mathcal{P} .

$$\begin{aligned} \langle\psi|p|\psi\rangle &= -i\sqrt{\frac{m\hbar\omega}{2}}\frac{1}{2}\langle u_1 + e^{-i\omega t}u_2 | A - A^\dagger | u_1 + e^{-i\omega t}u_2 \rangle \\ &= \sqrt{\frac{m\hbar\omega}{2}}\frac{1}{2i}(\langle u_1 | A | u_2 \rangle e^{-i\omega t} - \langle u_2 | A^\dagger | u_1 \rangle e^{i\omega t}) \\ &= \sqrt{\frac{m\hbar\omega}{2}}\frac{1}{2i}(\sqrt{2}e^{-i\omega t} - \sqrt{2}e^{i\omega t}) \\ &= -\sqrt{m\hbar\omega} \sin(\omega t) \end{aligned}$$

LECTURE 12: ANGULAR MOMENTUM

TODAY: MOTIVATION

BACK GROUND

LADDER OP

NEXT TIME: DIFF EQ SOLN

SPHERICAL HARMONICS

LEGENDRE POLYNOMIALS

MOTIVATION: We want to solve hydrogen problem

1d \rightarrow 3dClassical analogue: planetary motion
 \vec{L} conserved

SHO $H = p^2 + kx^2$

$$H = p_x^2 + p_y^2 + p_z^2 + V(\vec{x})$$

$$\vec{a} = \vec{x} - i\vec{p}$$

$$= \left[\frac{p_\theta^2}{2m} + V(\vec{r}) \right] + \frac{\vec{L}^2}{2I}$$

$$\vec{a}^\pm = \vec{x} + i\vec{p}$$

$$\begin{matrix} \swarrow & \searrow \\ L_x & L_z \end{matrix}$$

$$L^\pm = L_x \pm iL_y$$

ORBITAL ANGULAR MOMENTUM

$$l = 0, 1, 2, 3, \dots$$

$$m = -l, \dots, 0, \dots, l$$

$$L^2 |l, m\rangle = l(l+1) \hbar^2 |l, m\rangle$$

$$L_z |l, m\rangle = m \hbar |l, m\rangle$$

$$L_{\pm} |l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)} \hbar |l, m \pm 1\rangle$$

SPECIAL CASES:

$$l = 0 \Rightarrow m = 0$$

$$\underline{\hspace{1cm}} |0, 0\rangle$$

$$L^2 |0, 0\rangle = 0 \hbar^2 |0, 0\rangle$$

$$L_z |0, 0\rangle = 0 \hbar |0, 0\rangle$$

Spherical Coordinates

SPHERICAL COORDINATES

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0$$

$$\Phi(r, \theta, \varphi) = R(r)P(\theta)Q(\varphi)$$

$$\frac{1}{r^2 R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{r^2 P \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \frac{1}{r^2 Q \sin^2 \theta} \frac{d^2 Q}{d\varphi^2} = 0$$

multiply with $r^2 \sin^2 \theta$:

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{P} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) = -\frac{1}{Q} \frac{d^2 Q}{d\varphi^2}$$

The left-hand side depends only on r and θ , while the right-hand side depends only on φ . Thus the two sides must be a constant, m^2 .

$$\frac{d^2 Q}{d\varphi^2} + m^2 Q = 0 \quad ; \quad Q(\varphi) \sim e^{\pm im\varphi} \quad ; \quad m = 0, 1, 2, \dots$$

Note: If the physical problem limits φ to a restricted range m can be a non-integer.

Now we return to the left-hand side and rearrange the terms:

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = -\frac{1}{P \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \frac{m^2}{\sin^2 \theta}$$

The new left-hand side depends only on r and the right-hand side on only θ . Thus, they must be a constant, $l(l+1)$.

CLASSICALLY

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$= (y p_z - z p_y) \hat{i} - (x p_z - z p_x) \hat{j} + (x p_y - y p_x) \hat{k}$$

QM

$$L_{x \text{ op}} = y_{\text{op}} p_{z \text{ op}} - z_{\text{op}} p_{y \text{ op}}$$

$$L_{y \text{ op}} = - (x_{\text{op}} p_{z \text{ op}} - z_{\text{op}} p_{x \text{ op}})$$

$$L_{z \text{ op}} = x_{\text{op}} p_{y \text{ op}} - y_{\text{op}} p_{x \text{ op}}$$

POSITION SPACE

$$x_{op} \rightarrow x \quad p_{xop} \rightarrow -i\hbar \frac{\partial}{\partial x}$$

$$y_{op} \rightarrow y \quad p_{yop} \rightarrow -i\hbar \frac{\partial}{\partial y}$$

$$z_{op} \rightarrow z \quad p_{zop} \rightarrow -i\hbar \frac{\partial}{\partial z}$$

MOMENTUM SPACE

$$x_{op} \rightarrow i\hbar \frac{\partial}{\partial p_x} \quad p_{xop} \rightarrow p_x$$

$$y_{op} \rightarrow i\hbar \frac{\partial}{\partial p_y} \quad p_{yop} \rightarrow p_y$$

$$z_{op} \rightarrow i\hbar \frac{\partial}{\partial p_z} \quad p_{zop} \rightarrow p_z$$

DEEP IDEA HERE

SYMMETRY \longrightarrow CONSERVATION LAW

TRANSLATIONAL INvariance \longrightarrow LINEAR MOMENTUM CONSERVATION

ROTATIONAL INvariance \longrightarrow ANGULAR MOMENTUM CONSERVATION

TIME TRANSLATION INvariance \longrightarrow ENERGY CONSERVATION

UNITARY OPERATORS

TRANSLATION BY $\vec{a} = (a_x, a_y, a_z)$

$$T(\vec{a}) = e^{-i \vec{p} \cdot \vec{a} / \hbar}$$

$$T(\vec{b}) T(\vec{a}) = T(\vec{a} + \vec{b})$$

$$[p_i, p_j] = \delta_{ij}$$

ROTATIONS

$$R(\vec{\theta}) = e^{-i \vec{L} \cdot \vec{\theta} / \hbar}$$

$$R(\theta_z) = e^{-i L_z \cdot \theta_z / \hbar}$$

in coordinates

$$L_z = x_{op} p_{yop} - y_{op} p_{xop}$$

$$= -i \hbar \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]$$

very messy to calculate ...

TIME TRANSLATION

$$U(t) = e^{-iHt/\hbar}$$

ROTATIONAL SYMMETRY \Rightarrow WORK IN SPHERICAL COORDINATES

$$(x, y, z) \rightarrow (r, \theta, \varphi)$$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$L_x = i\hbar \left[\sin \varphi \frac{\partial}{\partial \theta} + \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} \right]$$

$$L_y = i\hbar \left[-\cos \varphi \frac{\partial}{\partial \theta} + \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} \right]$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]$$

$$L^2 = -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]$$

LADDER OPERATORS $L_{\pm} = L_x \pm iL_y$

$$L_{\pm} = \pm \hbar e^{\pm i\varphi} \left[\frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \varphi} \right]$$

EIGENVALUE PROBLEM

$$L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$L_z |l, m\rangle = m\hbar |l, m\rangle$$

IF H IS ROTATIONALLY INVARIANT

$$[H, L_i] = 0$$

$$[H, L^2] = 0$$

$$[H, L_z] = 0$$

(H eigenfunc) = (radial eigenfun)(spherical harmonic)

LADDER OPERATOR SOLUTION

$$L^2 = \vec{L} \cdot \vec{L} = L_x^2 + L_y^2 + L_z^2$$

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

EINSTEIN
SUMMATION
OVER K.

ϵ_{ijk} CALLED TOTALLY ANTI SYMMETRIC TENSOR

1 2 3

2 1 3

any two equal

2 3 1

1 3 2

3 1 2

3 2 1

$\epsilon_{ijk} = +1$

$\epsilon_{ijk} = -1$

$\epsilon_{ijk} = 0$

1 2 3

x 4 z

ANOTHER VIEW : $\epsilon_{123} = +1$

change sign when interchange
two indices

$$[L_x, L_y] = i \hbar L_z$$

$$[L_y, L_z] = i \hbar L_x$$

$$[L_z, L_x] = i \hbar L_y$$

$$\vec{L} \times \vec{L} = i \hbar \vec{L} =$$

$$\begin{vmatrix} i & 1 & \hat{k} \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix}$$

http://en.wikipedia.org/wiki/Levi-Civita_symbol

CROSS PRODUCTS

$$\vec{c} = \vec{a} \times \vec{b}$$

$$c_i = \epsilon_{ijk} a_j b_k$$

EINSTEIN SUM OVER j, k .

SINCE L_x, L_y, L_z DO NOT COMMUTE...

CHOOSE L^L, L_z

RAISING AND LOWERING OPERATORS

$$L^+ = L_x + i L_y$$

$$L^- = L_x - i L_y$$

WORK OUT COMMUTATORS

$$[L^L, L_i] = 0$$

$$[L^L, L_\pm] = 0$$

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

PROCEEDED AS BEFORE

$$L^L | \alpha, \beta \rangle = \alpha | \alpha, \beta \rangle$$

$$L_z | \alpha, \beta \rangle = \beta | \alpha, \beta \rangle$$

$$[L_z, L_+] = \hbar L_+$$

$$L_z L_+ - L_+ L_z = \hbar L_+$$

$$L_z [L_+ | \alpha, \beta \rangle] = (\hbar L_+ + L_+ L_z) | \alpha, \beta \rangle$$

$$= (\beta + \hbar) [L_z | \alpha, \beta \rangle]$$

CONCLUDE

$L_+ | \alpha, \beta \rangle$ is even of L_z with even $\beta + \hbar$

$L_- | \alpha, \beta \rangle$ is odd of L_z with odd $\beta - \hbar$

STEP SIZE = \hbar

HOW ABOUT L^2

$$[L^2, L_+] = 0$$

$$L^2 L_+ - L_+ L^2 = 0$$

$$L^2 [L_+ | \alpha, \beta \rangle] = L_+ [L^2 | \alpha, \beta \rangle]$$

$$= \alpha L_+ | \alpha, \beta \rangle$$

$L_+ | \alpha, \beta \rangle$ is \vec{e}_1 of L^2 with an α

$$L_- | \alpha, \beta \rangle$$

α

NEW FEATURE!

This ladder has a top and a bottom

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 - L_z^2 = L_x^2 + L_y^2$$

$$\langle \alpha, \beta | L^2 - L_z^2 | \alpha, \beta \rangle = \langle \alpha, \beta | L_x^2 + L_y^2 | \alpha, \beta \rangle$$
$$\alpha - \beta^2$$

$$(\alpha - \beta^2) \langle \alpha, \beta | \alpha, \beta \rangle = \langle \alpha, \beta | L_x^2 + L_y^2 | \alpha, \beta \rangle \geq 0$$

$$\beta^2 \leq \alpha$$

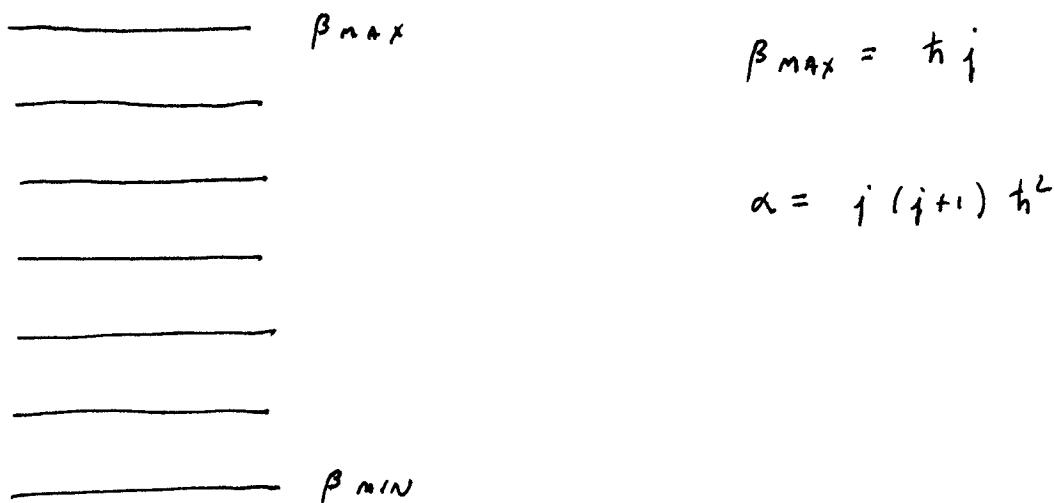
Physically, given a total angular momentum,
there is a maximum z projection

TOP AND BOTTOM TO LADDER

$$L+ |\alpha, \beta_{\max} \rangle = 0$$

$$L- |\alpha, \beta_{\min} \rangle = 0$$

next show $\beta_{\max} = -\beta_{\min}$



$$\beta_{\max} = \pm j$$

$$\alpha = j(j+1) \hbar^2$$

OH, MY GOSH! WE ACCIDENTALLY SOLVED A
MORE GENERAL PROBLEM!

ORBITAL ANGULAR MOMENTUM L^2 , L_z CAN
ONLY HAVE INTEGRAL L !

TOTAL ANGULAR MOMENTUM J^2 , J_z CAN
HAVE INTEGER OR HALF INTEGER $\pm J$...

SPIN ANGULAR MOMENTUM S^2 , S_z CAN
HAVE INTEGER OR HALF INTEGER $\pm S$

$$\vec{J} = \vec{L} + \vec{S}$$

NEXT QUARTER \vec{J}_1 and \vec{S}_1 :
adding angular momentum

CLEBSCH-GORDON

CORFF'S

GRIFFITH'S QUOTE...

LET'S CONSIDER SOME EXAMPLES