LECTURE 19   7-24-07

SOLVED TISE IN POSITION SPACE

$$H|n\rangle = E_n|n\rangle$$

OPERATOR IN X-SPACE

EIGENENERGIES

WAVE FUNCTIONS $\varphi_n(x)$ IN X-SPACE

FOUND $E_n = (n + \frac{1}{2})\hbar\omega$ $n = 0, 1, 2, 3, \ldots$

$$\varphi_n(y) = H_n(y)\, e^{-y^2/2}$$

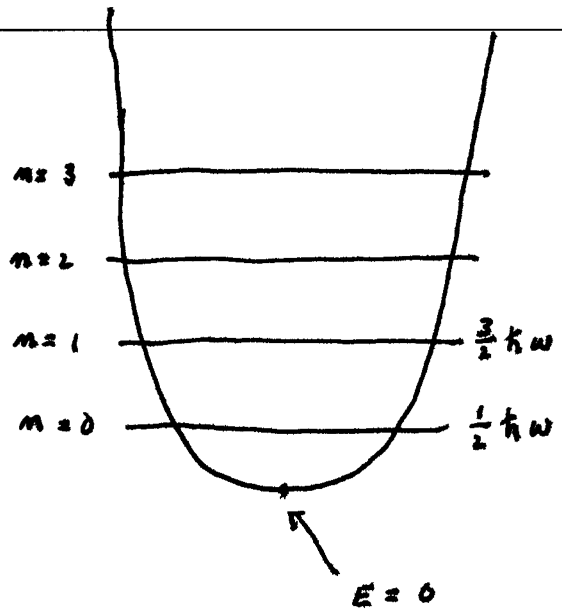
STATIONARY STATES

AKA

ENERGY EIGENFUNCTIONS

HERMITE POLYNOMIALS

GAUSSIAN



LADDER OF EQUALLY SPACED STATES

$E_m = (m + \frac{1}{2})\hbar\omega$

SPACING = $\hbar\omega$

GROUND STATE

$E_0 = \frac{1}{2}\hbar\omega$

ZERO POINT ENERGY

ZERO POINT MOTION

$\Delta x \, \Delta p \geq \frac{\hbar}{2}$

$\Rightarrow$ SYSTEM CAN NEVER STOP

FIND TWO LINEAR OPERATORS

a^+ = RAISING OPERATOR = CREATION OPERATOR

a = LOWERING OPERATOR = DESTRUCTION OP

IN FIELD THEORY

PHONONS

PHOTONS

ELECTRONS

⋮

FEYNMAN STORY

(2) OPERATOR METHOD

$$H \, |E_m\rangle = E_m \, |E_m\rangle$$

$$\left[\frac{p_{op}^2}{2m} + \frac{1}{2} m \omega^2 x_{op}^2 \right] |E_m\rangle = E_m \, |E_m\rangle$$

USE THE ALGEBRA OF THE OPERATORS

BASICALLY, REDUCE EVERYTHING BACK TO THE FUNDAMENTAL COMMUTATION RELATIONSHIP

$$[x_{op} , p_{op}] = i\hbar I = i\hbar$$

THIS IS BASIS INDEPENDENT!

x BASIS

$$x_{op} p_{op} - p_{op} x_{op} \rightarrow x \left(-i\hbar \frac{\partial}{\partial x} \right) - \left(-i\hbar \frac{\partial}{\partial x} \right) x$$

$$= i\hbar \left[\cancel{- x \frac{\partial}{\partial x}} + \frac{\partial x}{\partial x} + \cancel{x \frac{\partial}{\partial x}} \right]$$

$$= i\hbar$$

P BASIS

$$x_{op}\,p_{op} - p_{op}\,x_{op} \rightarrow \left(i\hbar \frac{\partial}{\partial p}\; p\right) - \left(p\; i\hbar \frac{\partial}{\partial p}\right)$$

$$= i\hbar \left[\frac{\partial p}{\partial p} + \cancel{p \frac{\partial}{\partial p}} - \cancel{p \frac{\partial}{\partial p}} \right]$$

$$= i\hbar$$

IDEA: FACTOR H

FACTORIZATION METHOD : OLD
LADDER OPERATOR METHOD : NEW

$$H = p^2 + x^2 = (x + i p)(x - i p)$$

INTRODUCE LADDER OPERATORS

$$a = \sqrt{\frac{m\omega}{2\hbar}}\; x_{op} + i \left(\frac{1}{2m\omega\hbar}\right)^{1/2} p_{op}$$

$$a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\; x_{op} - i \left(\frac{1}{2m\omega\hbar}\right)^{1/2} p_{op}$$

NOW x_{op} AND p_{op} ARE HERMITEAN,

BUT a AND a^{+} ARE NOT! $\Rightarrow$ a and a^{+} are not observable

$$x \longleftrightarrow p$$

$$m\omega \qquad\qquad \frac{1}{m\omega}$$

EXPRESS H IN TERMS OF a AND a^{+}

$$a^{+}a = \left[c\, x_{op} - i d\, p_{op}\right]\left[c\, x_{op} + i d\, p_{op}\right]$$

$$= c^2 x_{op}^2 - i^2 d^2 p_{op}^2 + i c d\,(x_{op} p_{op} - p_{op} x_{op})$$

$$= \frac{m\omega}{2\hbar} x_{op}^2 + \frac{1}{2m\omega\hbar} p_{op}^2$$

$$+ i \sqrt{\frac{m\omega}{2\hbar} \frac{1}{2m\omega\hbar}} \; \underbrace{\left[x_{op}, p_{op}\right]}_{i\hbar}$$

$$a^{+}a = \frac{H_{op}}{\hbar\omega} - \frac{1}{2}$$

$$H = \left(a^{+} a + \tfrac{1}{2}\right) \hbar \omega$$

define $\hat{H} = \dfrac{H}{\hbar \omega}$ dimensionless Hamiltonian

$$\hat{H} = a^{+} a + \tfrac{1}{2} \qquad E \to \epsilon = \frac{E}{\hbar \omega}$$

SOLVE eV, $e\vec{V}$ PROBLEM for $\hat{H}$, then translate back to H

$$\hat{H} \,|\, \epsilon_n \rangle = \epsilon_n \,|\, \epsilon_n \rangle$$

NOW WE CAN (and WILL) SOLVE THIS USING ONLY THE ALGEBRA OF THE COMMUTATORS

need 3 commutators

$$[a, a^{+}] = ?$$

$$[a, \hat{H}] = ?$$

$$[a^{+}, H] = ?$$

reduce each one to $[x_{op}, p_{op}]$

FIRST ONE

$$[a, a^+] = a a^+ - a^+ a$$

$$= (cx + idp)(cx - idp) - (cx - idp)(cx + idp)$$

$$= \left[c^2 x^2 + d^2 p^2 - icd\ (xp - px)\right]$$

$$- \left[c^2 x^2 + d^2 p^2 + icd\ (xp - px)\right]$$

$$[a, a^+] = -2i\ cd\ [x, p]$$

$$= -2i \sqrt{\frac{m\omega}{2\hbar} \frac{1}{2m\omega\hbar}}\ i\hbar$$

$$\boxed{[a, a^+] = I = 1}$$

SECOND ONE

$$[a, \hat{H}] = [a, a^+ a + 1/2]$$

$$= [a, a^+ a]$$

$$= \underbrace{a\ a^+}a - \underbrace{a^+ a}\, a = [a, a^+]\, a$$

$$\boxed{[a, \hat{H}] = a}$$

THIRD ONE

$$[a^\dagger, \hat{H}] = [a^\dagger, a^\dagger a + 1/2]$$

$$= [a^\dagger, a^\dagger a]$$

$$= a^\dagger \underbrace{a^\dagger a} - a^\dagger \underbrace{a a^\dagger}$$

$$= a^\dagger \underbrace{[a^\dagger, a]}_{-1}$$

$$\boxed{[a^\dagger, \hat{H}] = -a^\dagger}$$

WHAT GOOD ARE a AND $a^\dagger$?

GIVEN ANY EIGENSTATE OF $\hat{H}$,

a AND $a^\dagger$ GENERATE THE ADJACENT STATES

$a^\dagger$ goes up the ladder

a goes down the ladder

TO SEE THIS, CONSIDER AN $\vec{ev}$ OF $\hat{H}$

$$\hat{H} \mid \epsilon \rangle = \epsilon \mid \epsilon \rangle$$

CALCULATE

$$a \mid \epsilon \rangle = \ ?$$

$$a^{+} \mid \epsilon \rangle = \ ?$$

$$\hat{H} \left[a \mid \epsilon \rangle \right] = \left(a^{+} a + \tfrac{1}{2} \right) \left[a \mid \epsilon \rangle \right]$$

$$[a, H] = aH - Ha = a \implies Ha = aH - a$$

$$H a \mid \epsilon \rangle = \left[aH - a \right] \mid \epsilon \rangle$$

$$= \left[a \epsilon - a \right] \mid \epsilon \rangle$$

$$= (\epsilon - 1) \left[a \mid \epsilon \rangle \right]$$

$\implies a \mid \epsilon \rangle$ is an $\vec{ev}$ OF H with ev $\epsilon - 1$!

$$a \mid \epsilon \rangle \propto \mid \epsilon - 1 \rangle$$

$$a \mid \epsilon \rangle = C(\epsilon) \mid \epsilon - 1 \rangle \quad \text{norm factor}$$

NORM COEFF

$$C(E) = \sqrt{n}$$

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$$

IN JUST THE SAME WAY

$$[a^+, H] = a^+H - Ha^+ = -a^+$$

$$Ha^+ = a^+H + a^+$$

$$a^+|\epsilon\rangle = D(\epsilon)\,|\epsilon+1\rangle$$

WE WILL SHOW

$$D(\epsilon) = \sqrt{n+1}$$

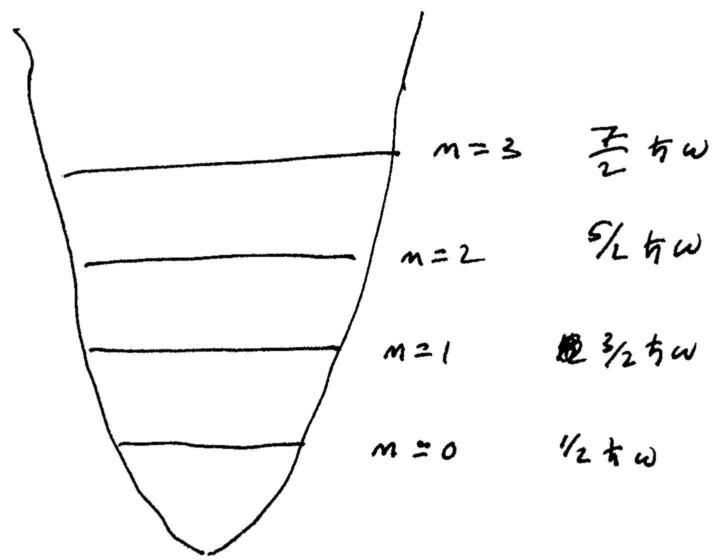
$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

NOTATION:

$|n\rangle$ HAS EIGENVALUE $(n + 1/2)$

$$\hat{H}\, |n\rangle = (n + 1/2)\, |n\rangle$$

$$H\, |n\rangle = (n + 1/2)\hbar\omega\, |n\rangle$$



∞ number of states

$\frac{1}{2}\hbar\omega$

g.s. energy $\neq 0$!   ZERO POINT ~~MOTION~~ ENERGY

$\Rightarrow$ ZERO POINT MOTION

TO COMPLETE THE SOLUTION, WE NEED TO FIND THE GROUND STATE

$$a\,|gs\rangle = a\,|\epsilon_0\rangle = \vec{0}$$

$$a^+\,\vec{0} = \vec{0}$$

$$a^+ a\,|\epsilon_0\rangle = \vec{0}$$

$$\left[\hat{H} - \tfrac{1}{2}\right]|\epsilon_0\rangle = \vec{0}$$

$$\hat{H}\,|\epsilon_0\rangle = \tfrac{1}{2}\,|\epsilon_0\rangle$$

ϵ_0 HAS EIGENVALUE $+\frac{1}{2}$!

NOW WE CAN FIND ALL THE REST OF THE STATES BY APPLYING a^+

$$a^+\,|\epsilon_0\rangle \sim |\epsilon_0 + 1\rangle$$

$$a^+ a^+\,|\epsilon_0\rangle \sim |\epsilon_0 + 2\rangle$$

$$(a^+)^m\,|\epsilon_0\rangle \sim |\epsilon_0 + m\rangle$$

WANT A NORMALIZED SET OF EIGENKETS

$$\left.\begin{array}{l} |0\rangle \\ |1\rangle \\ |2\rangle \end{array}\right\} \Rightarrow \langle n | m \rangle = \delta_{nm}$$ ORTHO NORMAL

CONSIDER

$$a|n\rangle = c_n |n-1\rangle$$

ADJOINT

$$\langle n| a^\dagger = \langle n-1| c_n^*$$

MAKE SANDWICH

$$\langle n | a^\dagger a | n \rangle = \langle n-1 | c_n^* c_n | n-1 \rangle$$

$$= |c_n|^2 \langle n-1 \overset{1}{\cancel{|}} n-1 \rangle$$

$$\langle n | \hat{H} - \tfrac{1}{2} | n \rangle = \langle n | (n+\tfrac{1}{2}) - \tfrac{1}{2} | n \rangle$$

$$|c_n|^2 = n$$ CHOOSE $c_n = \sqrt{n}$

LADDER OPERATORS

$$a\,|m\rangle = \sqrt{m}\,|m-1\rangle \qquad \text{DOWN}$$

$$a^{+}|m\rangle = \sqrt{m+1}\,|m+1\rangle \qquad \text{UP}$$

TRANSLATE ALL THIS INTO MATRIX LANGUAGE

$$a\,|m\rangle = \sqrt{m}\,|m-1\rangle$$

$$a = \begin{bmatrix} 0 & \sqrt{1} & & & 0 \\ 0 & 0 & \sqrt{2} & & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ & & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \begin{array}{l} m=0 \\ m=1 \\ m=2 \\ \\ \\ \end{array}$$

MATRIX ELEMENTS

$$\langle m'|a|m\rangle = \langle m'|\sqrt{m}\,|m-1\rangle$$

$$= \sqrt{m}\;\delta_{m',m-1}$$

$$a^{+} = \begin{bmatrix} 0 & & & & \\ \sqrt{1} & 0 & & 0 & \\ 0 & \sqrt{2} & 0 & & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$\langle n' | a^{+} | n \rangle = \sqrt{n+1}\ \delta_{n', n+1}$$

$$x_{op} = \sqrt{\frac{\hbar}{2m\omega}}\ (a + a^{+})$$

$$x_{op} = \begin{bmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ & & & \ddots & \end{bmatrix} \sqrt{\frac{\hbar}{2m\omega}}$$

$$P_{op} = i\sqrt{\frac{m\hbar\omega}{2}}\left[a^{+} - a\right]$$

$$P_{op} = i\sqrt{\frac{m\hbar\omega}{2}}\begin{bmatrix} 0 & -\sqrt{1} & & & \\ \sqrt{1} & 0 & -\sqrt{2} & & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \\ & & & & \\ & & & & \end{bmatrix}$$

$$H = \begin{bmatrix} 1/2 & & & & \\ & 3/2 & & & \\ & & 5/2 & & \\ & & & 7/2 & \\ & & & & \ddots \end{bmatrix}\hbar\omega$$

SUMMARY

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^{+}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$H = \left(a^{+}a + \tfrac{1}{2}\right)\hbar\omega$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^{+})^{n}\,|0\rangle$$

→ generating fcn

NUMBER OPERATOR

$$a^{+}a = N$$

$$N = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$

TO FIND $\psi_0(x)$

$$a\,|0\rangle = 0$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\; X_{op} + i\sqrt{\frac{1}{2m\omega\hbar}}\; P_{op}$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\; x + i\sqrt{\frac{1}{2m\omega\hbar}}\left(-i\hbar\frac{\partial}{\partial x}\right)$$

CHANGE VARIABLES

$$y = \sqrt{\frac{m\omega}{2\hbar}}\; x$$

$$dy = \sqrt{\frac{m\omega}{2\hbar}}\; dx$$

$$a \rightarrow \frac{1}{\sqrt{2}}\left[y + \frac{d}{dy}\right]$$

$$a\,|0\rangle = 0 \implies \frac{1}{\sqrt{2}}\left[y + \frac{d}{dy}\right]\psi_0(y) = 0$$

FIRST ORDER DIFF EQ!

$$\frac{d\,\psi_0(y)}{\psi_0(y)} = -y\,dy$$

$$\ln \psi_0 = -\frac{y^2}{2} + C$$

$$\psi_0(y) = e^{C}\, e^{-y^2/2} = A_0\, e^{-y^2/2}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

TO CALCULATE HIGHER STATES

$$a^{+} = \frac{1}{\sqrt{2}}\left[y - \frac{d}{dy}\right]$$

$$\psi_n(y) = \frac{(a^{+})^n}{\sqrt{n!}}\,\psi_0(y)$$

$$\psi_n(y) = H_n(y)\, e^{-y^2/2}$$

Gaussian

Hermite polynomials

1. Quantitative Aspects of the Harmonic Oscillator

Consider a particle moving in a simple harmonic oscillator well with the zero-time state vector

$$\mid \psi(t=0) > = \; N \left[\mid n=3 > \; + \; \mid n=4 > \right].$$

(a) Calculate the normalization constant N. Write down the equation for the normalized zero-time state vector $\mid \psi(0) >$ in terms of the energy eigenkets $\mid n >$. Using your equation for $\mid \psi(0) >$ in terms of the energy eigenkets $\mid n >$ write down the equation for the corresponding normalized time-dependent state vector $\mid \psi(t) >$ in terms of the energy eigenkets $\mid n >$. Convert your equation for the time-dependent state vector $\mid \psi(t) >$ in terms of the energy eigenkets $\mid n >$ into the corresponding equation for the time-dependent position-space wavefunction $\psi(x,t)$ in terms of the position-space stationary states $\psi_n(x)$.

(b) If you measure E at $t=0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< E(t) >$. Calculate the time-dependent uncertainty $\Delta E(t)$. Explain the time-dependence, or lack thereof, of $< E(t) >$ and $\Delta E(t)$.

(c) If you measure x at $t=0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< x(t) >$. Calculate the time-dependent uncertainty $\Delta x(t)$. Simulate the time evolution of this system using http://falstad.com. Do your expressions for $< x(t) >$ and $\Delta x(t)$ agree with your simulation?

(d) If you measure p at $t=0$ what are the possibilities and what are the probabilities? Calculate the time-dependent expectation value $< p(t) >$. Calculate the time-dependent uncertainty $\Delta p(t)$. Simulate the time evolution of this system using http://falstad.com. Do your expressions for $< p(t) >$ and $\Delta p(t)$ agree with your simulation?

(e) Sketch the $t=0$ probability density distributions $P(E,0)$, $P(x,0)$, and $P(p,0)$. Add your calculated expectation values and uncertainties to your sketches. Do they agree?

Three Ways to Solve:

(1) Use Matrix Method

(2) Do the integrals in x-space

(3) Use Dirac Notation

Matrix Method

1) Simple harmonic Oscillator:

$$|\psi(t=0)\rangle = N\left[|N=3\rangle + |N=4\rangle\right]$$

a) $$\langle \psi(0)|\psi(0)\rangle = (0\ 0\ 0\ 1\ 1\ 0\ 0\ \ldots)\, N \cdot N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = 1$$

$$- |N|^2 (1 + 1) = 2|N|^2$$

$$\Rightarrow \boxed{N = \frac{1}{\sqrt{2}}}$$

$$- \boxed{|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|N=3\rangle + |N=4\rangle\right]}$$

$$E_N = (N + 1/2)\hbar\omega$$

$$\Rightarrow E_N = \frac{7}{2}\hbar\omega \text{ or } \frac{9}{2}\hbar\omega$$

$$- \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-7i\omega t/2}|3\rangle + e^{-9i\omega t/2}|4\rangle\right]}$$

$$- \langle x|\psi(t)\rangle = \psi(x,t)$$

$$= \frac{1}{\sqrt{2}}\left(\langle x|3\rangle e^{-7i\omega t/2} + \langle x|4\rangle e^{-9i\omega t/2}\right)$$

$$= \boxed{\frac{1}{\sqrt{2}}\left(\psi_3(x)\, e^{-7i\omega t/2} + \psi_4(x)\, e^{-9i\omega t/2}\right)}$$

1b) ✓ E at $t=0$, possibilities & probabilities ?

possibilities : $E_N = \frac{7}{2}\hbar\omega,\ \frac{9}{2}\hbar\omega$ → only possible eigenvalues @ $t=0$.

Probabilities :

$$P(7/2\,\hbar\omega) = \left|\langle N=3|\Psi(t=0)\rangle\right|^2 = \left|(0\,0\,0\,1\,0\ldots)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\\1\\0\\\vdots\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(1\cdot1 + 0\cdot1\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \boxed{\frac{1}{2}}$$

$$P(9/2\,\hbar\omega) = \left|\langle N=4|\ \Psi(t=0)\rangle\right|^2 = \left|(0\,0\,0\,0\,1\,0\ldots)\ \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\\1\\0\\\vdots\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(0\cdot1 + 1\cdot1)\right|^2 = \left|\frac{1}{\sqrt{2}}\right| = \boxed{\frac{1}{2}}$$

- $\langle E(t)\rangle = \langle\Psi(t)|\mathcal{H}|\Psi(t)\rangle$

$$= \langle\Psi(t)|\ \begin{array}{c|c} 0 & \\ 1 & \\ 2 & \\ 3 & \\ 4 & \end{array}\begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 5/2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 7/2 & 0 & \\ 0 & 0 & 0 & 0 & 9/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\hbar\omega\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\e^{-i\cdot7\omega t/2}\\e^{-i\cdot9\omega t/2}\\\vdots\end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\left(0\quad0\quad0\quad \cancel{e}^{\cancel{7i\omega t/2}}\quad \cancel{e}^{\cancel{9i\omega t/2}}\ \ldots\right)\begin{pmatrix}0\\0\\0\\7/2\ \cancel{e^{-i7\omega t/2}}\\9/2\ \cancel{e^{-i9\omega t/2}}\\\vdots\end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\left[7/2 + 9/2\right] = \boxed{4\hbar\omega}$$

1b) cont

$$\Delta E(t) = \left[\langle \psi(t) | \mathcal{H}^2 | \psi(t) \rangle - \langle \psi(t) | \mathcal{H} | \psi(t) \rangle^2 \right]^{1/2}$$

– 1st calculate $\mathcal{H}^2$: $\langle \mathcal{H} \rangle = \left(a^{+}a + \frac{1}{2}\right)\hbar\omega \left(a^{+}a + \frac{1}{2}\right)\hbar\omega$

$$= \left(a^{+}a\,a^{+}a + \frac{1}{2}a^{+}a + a^{+}a\frac{1}{2} + \frac{1}{4}\right)\hbar^2\omega^2 \qquad a^{+}a = N$$

$$= \boxed{\left(n^2 + n + 1/4\right)\hbar^2\omega^2}$$

– $\langle \psi(t) | \mathcal{H}^2 | \psi(t) \rangle = \frac{\hbar^2\omega^2}{2}\left[\langle 3| e^{7i\omega t/2} + \langle 4| e^{9i\omega t/2}\right] \cdot$

$$\left(n^2 + N + 1/4\right) \cdot \left[|3\rangle e^{-7i\omega t/2} + |4\rangle e^{-9i\omega t/2}\right]$$

$$= \frac{\hbar^2\omega^2}{2}\left[\langle 3| e^{7i\omega t/2} + \langle 4| e^{9i\omega t/2}\right] \cdot \left[\left(3^2 + 3 + 1/4\right)|3\rangle e^{-7i\omega t/2} +\right.$$

$$\left.\left(4^2 + 4 + 1/4\right)|4\rangle e^{-9i\omega t/2}\right] = \frac{\hbar^2\omega^2}{2}\left[\langle 3| \cancel{e^{7i\omega t/2}} + \langle 4| \cancel{e^{9i\omega t/2}}\right]$$

$$\cdot \left[\frac{49}{4}|3\rangle \cancel{e^{-7i\omega t/2}} + \frac{81}{4}|4\rangle \cancel{e^{-9i\omega t/2}}\right]$$

$$= \frac{\hbar^2\omega^2}{2}\left[\cancelto{1}{\langle 3|3\rangle} \cancelto{1}{e^{0}} \cdot \frac{49}{4} + \frac{81}{4}\cancelto{1}{\langle 4|4\rangle} \cancelto{1}{e^{0}}\right] = \boxed{\frac{65}{4}\hbar^2\omega^2}$$

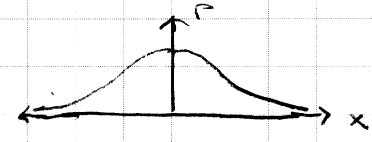
$$\Delta E(t) = \left[\frac{65}{4}\hbar^2\omega^2 - (4\hbar\omega)^2\right]^{1/2} = \left[\frac{65}{4}\hbar^2\omega^2 - \frac{64}{4}\hbar^2\omega^2\right]^{1/2}$$

$$= \boxed{\frac{\hbar\omega}{2}}$$

– Both $\langle E(t) \rangle$ + $\Delta E(t)$ should be free of time dependence, because of the symmetrical Bra/Ket relation for the Hamiltonian + the orthonormality of all energy eigenvectors.

1c) ✓ The possible values for x @ $t=0$ are any and all values of x between $-\infty$ + ∞. This is due to the continuous nature of position in the universe. The probabilities of the position values will be governed by the gaussian distribution, given by the square of the wave fxn, however, to obtain a single precise value of x will cause the probability to go to zero, as shown by the equation below, because are $dx \to 0$.

$$N \cdot \int_a^a |\Psi(x,t)|^2 dx = 0 \Rightarrow$$ No possible probability except zero for a single position measurement.



$$\langle x(t)\rangle = \langle \Psi(t)|X|\Psi(t)\rangle$$

$$= \langle\Psi(t)| \left(\frac{\hbar}{2m\omega}\right)^{1/2} \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \end{array} & \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{array} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i7\omega t/2} \\ e^{-i9\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \langle\Psi(t)| \frac{1}{\sqrt{2}} \left(\frac{\hbar}{2m\omega}\right)^{1/2} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{4}\, e^{-i7\omega t/2} \\ \sqrt{4}\, e^{-i9\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \frac{\cancel{2}}{\cancel{2}} \left(\frac{\hbar}{2m\omega}\right)^{1/2} \cdot \begin{pmatrix} 0 & 0 & 0 & e^{+i7\omega t/2} & e^{+i9\omega t/2} & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i9\omega t/2} \\ e^{-i9\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \left[\frac{\hbar}{2m\omega}\right]^{1/2} \left(e^{i7\omega t/2} \cdot e^{-i9\omega t/2} + e^{i9\omega t/2} \cdot e^{-i7\omega t/2}\right) = \left[\frac{\hbar}{2m\omega}\right]^{1/2} \frac{\left(e^{i\omega t} + e^{-i\omega t}\right)}{2} \cdot 2$$

$$= \boxed{\left[\frac{2\hbar}{m\omega}\right]^{1/2} \cos(\omega t)}$$

1c)cont $\Delta x(t) = \left[\langle \psi(t) | x^2 | \psi(t) \rangle - \left(\langle \psi(t) | x | \psi(t) \rangle \right)^2 \right]^{1/2}$

$$x^2 = x \cdot x = \left(\frac{\hbar}{2m\omega} \right) \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \\ \hline 0 & 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 1 & 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ 2 & \sqrt{2} & 0 & 5 & 0 & \sqrt{12} & \cdots \\ 3 & 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 4 & 0 & 0 & \sqrt{12} & 0 & 9 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

– from HW 5, ch.8 soln's

$$\langle \psi(t) | x^2 | \psi(t) \rangle = \frac{1}{2} \cdot \frac{\hbar}{2m\omega} \langle \psi(t) | \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 & \sqrt{6} & 0 \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} \\ 0 & \sqrt{6} & 0 & 7 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i7\omega t/2} \\ e^{-i9\omega t/2} \end{pmatrix}$$

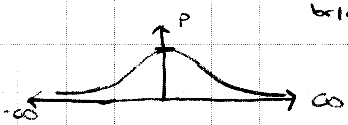
$$= \frac{\hbar}{4m\omega} \left(0 \quad 0 \quad 0 \quad \cancel{e^{7i\omega t/2}} \quad \cancel{e^{9i\omega t/2}} \; \cdots \right) \begin{pmatrix} 0 \\ \sqrt{6}\, e^{-7i\omega t/2} \\ \sqrt{12}\, e^{-9i\omega t/2} \\ 7\cancel{e^{-i\omega 7t/2}} \\ 9\cancel{e^{-9i\omega t/2}} \end{pmatrix} = \boxed{\frac{4\hbar}{m\omega}}$$

$$\Delta x(t) = \left[\frac{4\hbar}{m\omega} - \left(\left[\frac{2\hbar}{m\omega} \right]^{1/2} \cos(\omega t) \right)^2 \right] = \boxed{\left(\frac{2\hbar}{m\omega} \left[2 - \cos^2(\omega t) \right] \right)^{1/2}}$$

– according to http://falstad.com, my values of $\langle x(t) \rangle$ + $\Delta x(t)$ are consistent w/ the simulation.

1d) 8

similiar to the position possibilities + probabilities, measurement of p @ time t=0 could have an infinite number of possibilities, in the interval $-\infty$ to ∞. (for p) If we were able to pinpoint one value for p then the corresponding probability will go to zero for the reasons listed in part 1c. This all boils down to p being directly proportional to $\dot{x}$ (ie velocity), and x's continuous nature in the real spectrum. P will still have a gaussian probability centered at zero + falling off exponential as shown below



$$- \langle p(t) \rangle = \langle \Psi(t) | P | \Psi(t) \rangle$$

$$= \frac{1}{2} \langle \Psi(t) | \, i \left(\frac{m\omega\hbar}{2}\right)^{1/2} \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} & \begin{pmatrix} 0 & i & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & i\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & i\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & i\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-7i\omega t/2} \\ e^{-9i\omega t/2} \\ \vdots \end{pmatrix}$$

$$= \frac{i}{2}\left(\frac{m\omega\hbar}{2}\right)^{1/2} \left(\cancel{0} \;\; 0 \;\; \cancel{0} \;\; \cancel{e^{7i\omega t/2}} \;\; e^{9i\omega t/2} \right) \begin{pmatrix} \cancel{0} \\ \cancel{0} \\ i\cancel{\sqrt{3}}e^{-9i\omega t/2} \\ 2i\,e^{-9i\omega t/2} \\ 2e^{-7i\omega t/2} \end{pmatrix}$$

$$= \frac{i}{\cancel{2}}\left(\frac{m\omega\hbar}{2}\right)^{1/2} \left(e^{-i\omega t} + e^{i\omega t}\right) \cdot \frac{2}{2}$$

$$= \boxed{\cancel{\pm}\left(\frac{+m\omega\hbar}{2}\right)^{1/2} \cos(\omega t) \;\; 2\sin(\omega t)}$$

$$- \Delta p(t) = \left[\langle \Psi(t) | P^2 | \Psi(t) \rangle - \left(\langle \Psi(t) | P | \Psi(t) \rangle \right)^2 \right]^{1/2}$$

1d) cont

$$p^2 = p \cdot p = i^2\left(\frac{m\omega\hbar}{2}\right) \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \end{matrix} & \begin{pmatrix} 0 & i & 0 & 0 & 0 & \cdots \\ 1 & 0 & \sqrt{2}i & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & i\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 2i & \cdots \\ 0 & 0 & 0 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array} \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2}i & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3}i & 0 \\ 0 & 0 & \sqrt{3} & 0 & 2i \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 3i & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 5i & 0 & -2\sqrt{3} \\ 0 & -\sqrt{3}\sqrt{2} & 0 & 7i & 0 \\ 0 & 0 & 2\sqrt{3} & 0 & 4i \end{pmatrix}$$

$$- \quad \langle \psi(t) | p^2 | \psi(t) \rangle = \frac{1}{2} \cdot \frac{-m\omega\hbar}{2} \langle \psi(t) | \begin{pmatrix} i & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 3i & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 5i & 0 & -2\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 7i & 0 \\ 0 & 0 & 2\sqrt{3} & 0 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-7i\omega t/2} \\ e^{-9i\omega t/2} \end{pmatrix}$$

$$= \frac{-m\omega\hbar}{4} \langle \cancel{0}\ \cancel{0}\ \cancel{0}\ e^{7i\omega t/2}\ e^{9i\omega t/2} | \begin{pmatrix} \cancel{0} \\ \cancel{-\sqrt{3}\sqrt{2}\, e^{-7i\omega t/2}} \\ \cancel{-2\sqrt{3}}\, e^{-9i\omega t/2} \\ 7i\, \cancel{e^{-7i\omega t/2}} \\ 4i\, \cancel{e^{-9i\omega t/2}} \end{pmatrix}$$

$$= \frac{-m\omega\hbar}{4}\left(7i + 4i\right)$$

$$\Rightarrow \Delta p(t) = \left[-\frac{11}{4} i m\omega\hbar - \left(i \left(\frac{m\omega\hbar}{2}\right)^{1/2} \cos(\omega t) \right)^2 \right]^{1/2}$$

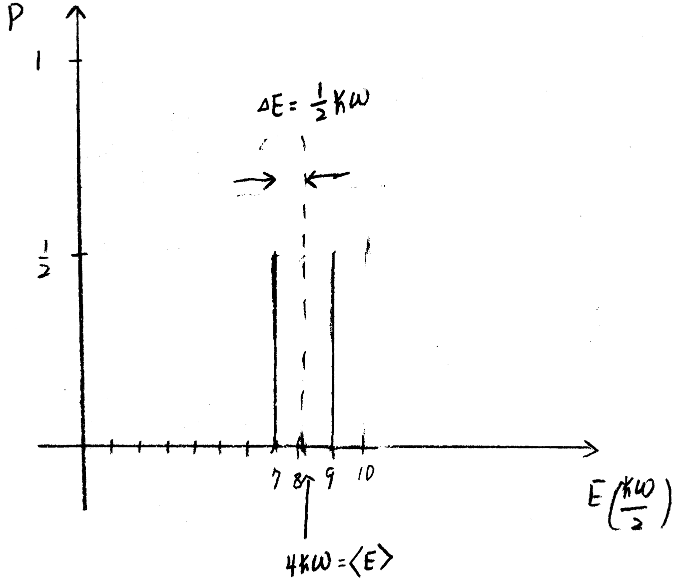
$$= \left[-\frac{11}{4} i m\omega\hbar + \frac{m\omega\hbar}{2} \cos^2(\omega t) \right]^{1/2} = \boxed{\left(\frac{m\omega\hbar}{2}\right)^{1/2} \left(\cos(\omega t) - \frac{11}{2} i\right)^{1/2}}$$

$$\sqrt{2m\omega\hbar}\ (1 + \cos^2 \omega t)^{1/2}$$

- After checking http://falstad.com, my expressions for $\Delta p(t)$ + $\langle p(t) \rangle$ appear to agree w/ the simulations.

✓ (e)

(1) $P(E,0)$

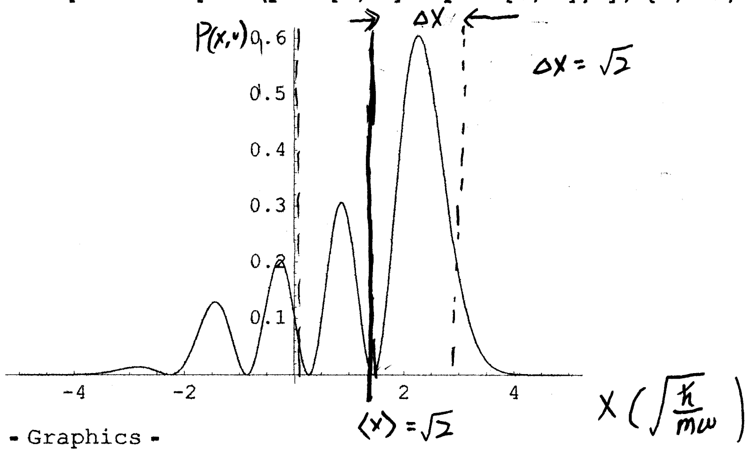


P
1
$\frac{1}{2}$
$\Delta E = \frac{1}{2}\hbar\omega$
7
8
9
10
$E\left(\frac{\hbar\omega}{2}\right)$
$4\hbar\omega = \langle E \rangle$

- **(2.) P(x,0): (assume $\frac{m\omega}{\text{hBar}}$ =1)**

In[1]:= $\text{phi1[x\_, n\_]} = \left(\frac{1}{\pi\,(2\hat{}(2\,n))\,((n!)\hat{}2)}\right)^{0.25} \text{E}\hat{}\left(-\frac{x\hat{}2}{2}\right) \text{HermiteH[n, x]};$

```
In[2]:= Plot[Evaluate[0.5 (phi1[x, 3] + phi1[x, 4])^2], {x, -5, 5}]
```

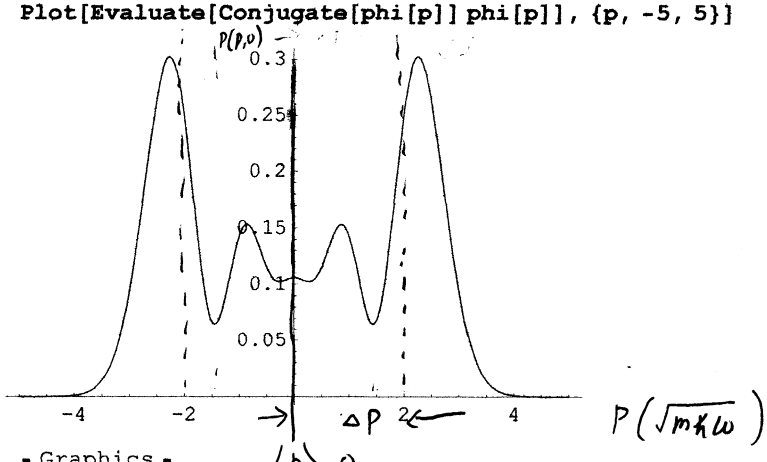


Out[2]= - Graphics -

- **(2.) P(p,0): (assume (m ω hBar) =1)**

In[10]:= phi3[p_] = FourierTransform[phi1[x, 3], x, p];
phi4[p_] = FourierTransform[phi1[x, 4], x, p];
$\text{phi[p\_]} = \frac{1}{2\hat{}0.5}\,(\text{phi3[p] + phi4[p]});$

```
In[15]:= Plot[Evaluate[Conjugate[phi[p]] phi[p]], {p, -5, 5}]
```



50 - 0 = 50

Out[15]= - Graphics -

⟨p⟩ = 0

ΔP = 2

Yes, they agree

Do the integrals

c.) The possible locations for a particle are unlimited, although ~~the~~ you are
8 very unlikely to measure a particle far outside the potential well.
The probability is given by $\int_{x_1}^{x_2} \psi^*(x,0)\, \psi(x,0)\, dx$.

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx \qquad d\xi = \sqrt{\frac{m\omega}{\hbar}}\, dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{x}{2^3 \cdot 3!} H_3^2(\xi) e^{-\xi^2} + \frac{x}{2^4 \cdot 4!} H_4^2(\xi) e^{-\xi^2} + \left[e^{\frac{i(E_3-E_4)t}{\hbar}} + e^{\frac{-i(E_3-E_4)t}{\hbar}}\right] x H_3(\xi) H_4(\xi) \frac{e^{-\xi^2}\, d\xi}{\sqrt{2^3 2^4 3!\, 4!}}$$

$$= \frac{1}{2\sqrt{\pi}} \left[\cancel{2\sqrt{\pi}} + \int_{-\infty}^{\infty} x \cos(\omega t)\, H_3(\xi) H_4(\xi) \frac{e^{-\xi^2}}{\sqrt{2^3 2^4 3!\, 4!}}\, d\xi \right] =

$$\langle x(t)\rangle = \frac{1}{\sqrt{\pi}\,2}\int_{-\infty}^{\infty} \frac{\zeta}{2^3 3!} H_3^2(\zeta) e^{-\zeta^2} + \frac{\zeta}{2^4\cdot 4!} H_4^2(\zeta) e^{-\zeta^2} + \frac{2\zeta\cos(\omega t) H_3(\zeta) H_4(\zeta)}{\sqrt{2^3 2^4\cdot 3!\cdot 4!}} e^{-\zeta^2}\, d\zeta$$

$$= \frac{1}{2\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\sqrt{\frac{\hbar}{m\omega}}\left[\underbrace{\frac{\zeta}{2^3 3!}\left(8^2\zeta^6 - 16\cdot 12\zeta^4 + 12^2\zeta^2\right)e^{-\zeta^2}}_{\to 0} + \underbrace{\frac{\zeta}{2^4 4!}\left(16^2\zeta^8 - 2\cdot 16\cdot 48\zeta^6 + 2\cdot 16\cdot 12\zeta^4 + 48^2\zeta^4 - 2\cdot 12\cdot 48\zeta^2 \ldots\right)e^{-\zeta^2}}_{\to 0}\right.\right.$$

$$\sqrt{\frac{\hbar}{m\omega}}\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\cos(\omega t)\left[\frac{1}{\sqrt{2^7\cdot 3!\cdot 4!}}\left(8\cdot 16\zeta^8 - 8\cdot 48\zeta^6 - \cancel{12\cdot 16\zeta^5} + 12\cdot 8\zeta^4 + \cancel{12\cdot 48\zeta^3} - \cancel{12\cdot 12\zeta}\right)e^{-\zeta^2}\, d\zeta\right]$$

odd terms are 0

$$= \frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!\cdot \pi}}\cdot\frac{1}{2}\cdot\sqrt{\frac{\hbar}{m\omega}}\left[8\cdot 105\sqrt{\pi} + 15\cdot 48\sqrt{\pi} + 6\cdot 12\sqrt{\pi}\right]$$

← UGLY

$\sqrt{\frac{2\hbar}{m\omega}}$ ⇒ Why didn't you use a and a^+?

PRETTY!

$$= \frac{816}{\sqrt{18432}}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t) = \frac{17}{\cancel{282}}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t)$$

$$\langle x^2(t)\rangle = \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty}\frac{\zeta^2 e^{-\zeta^2}}{2^3\cdot 3!}H_3^2(\zeta) + \frac{\zeta^2 e^{-\zeta^2}}{2^4 4!}H_4^2(\zeta) + \frac{\zeta^2 e^{-\zeta^2}}{\sqrt{2^7\cdot 3!\cdot 4!}}H_3(\zeta)H_4(\zeta)\overset{\cos(\omega t)}{\wedge}d\zeta$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\left[\int_{-\infty}^{\infty}\frac{e^{-\zeta^2}}{2^3\cdot 3!}\left(8^2\zeta^8 - 16\cdot 12\zeta^6 + 12^2\zeta^4\right) + \frac{e^{-\zeta^2}}{2^4\cdot 4!}\left(16^2\zeta^{10} - 2\cdot 16\cdot 48\zeta^8 + 2\cdot 16\cdot 12\zeta^6 + 48^2\zeta^6 - 2\cdot 12\cdot 48\zeta^4\right)d\zeta\right.$$

$$\left. + \int_{-\infty}^{\infty}\frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!}}e^{-\zeta^2}\left(8\cdot 16\cancel{\zeta^9} - 8\cdot 48\cancel{\zeta^7} - 12\cdot 16\zeta^6 + 12\cdot 8\cancel{\zeta^5} + 12\cdot 48\zeta^4 - 12\cdot 12\zeta^2\right)d\zeta\right]$$

odd terms drop out

$$= \frac{1}{2\cancel{\sqrt{\pi}}}\sqrt{\frac{\hbar}{m\omega}}\left[\overset{\frac{35}{4}\sqrt{\pi}}{\frac{2\cdot 105\cdot 8^2\cdot\sqrt{\pi}}{2^3\cdot 3!\cdot 32}} - \overset{-15/2\sqrt{\pi}}{\frac{\cancel{16}\cdot\cancel{12}\cdot\cancel{2}\cdot 15}{\cancel{16}\cdot 2^2\cdot\cancel{3!}}}\cancel{\sqrt{\pi}} + \overset{9/4\sqrt{\pi}}{\frac{2\cdot 12^2\cdot 3}{8\cdot 2^3\cdot 3!}}\cancel{\sqrt{\pi}} + \frac{1}{2^4 4!}\left(\overset{315/16\sqrt{\pi}}{\frac{16^2\cdot 2\cdot 945}{64}}\cancel{\sqrt{\pi}} - \overset{-105/4\sqrt{\pi}}{\frac{4\cdot 16\cdot 48\cdot 105}{32}}\cancel{\sqrt{\pi}} + \overset{\frac{15}{8}\sqrt{\pi}}{\frac{4\cdot 16\cdot 12\cdot 15}{16}}\cancel{\sqrt{\pi}}\right.\right.$$

$$\left.\left. + \overset{45/4\sqrt{\pi}}{\frac{2\cdot 48^2\cdot 15}{16}}\cancel{\sqrt{\pi}} - \overset{9/4\sqrt{\pi}}{\frac{4\cdot 12\cdot 48\cdot 3}{8}}\cancel{\sqrt{\pi}}\right) + \frac{\cos(\omega t)}{\sqrt{2^7\cdot 3!\cdot 4!}}\left(-2\cdot 12\cdot 16\cdot\frac{15}{16}\cancel{\sqrt{\pi}} + 12\cdot 48\cdot 2\cdot\frac{3}{8}\cancel{\sqrt{\pi}} - \frac{12^2\cdot 2}{4}\cancel{\sqrt{\pi}}\right)\right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{197}{16} + \cos(\omega t)\,(0)\right) = \frac{197}{32}\sqrt{\frac{\hbar}{m\omega}}$$

$$\Delta x(t) = \sqrt{\frac{197}{32}\sqrt{\frac{\hbar}{m\omega}} - \frac{289}{8}\cdot\frac{\hbar}{m\omega}\cos^2(\omega t)}$$

Both seem to match falstad.com.

$$\sqrt{\frac{2\hbar}{m\omega}}\left(1 + \sin^2\omega t\right)^{1/2}$$

again ladder operators are more beautiful and easier!

d) If you measure p at ~~any~~ time zero the value of p could be anywhere between $-\infty$ and ∞ with probability $\int_{P_1}^{P_2} \hat{\psi}^*(p,0)\,\hat{\psi}(p,0)\,dp$.

$$\langle p(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, -i\hbar\frac{\partial}{\partial x}\psi(x,t)\,dx$$

$$= \frac{-i\hbar}{2}\int_{-\infty}^{\infty} \underbrace{\psi_3^*(x)\frac{\partial}{\partial x}\psi_3(x)}_{\text{part one}} + \underbrace{\psi_4^*(x)\frac{\partial}{\partial x}\psi_4(x)}_{\text{part two}} + \underbrace{e^{\frac{i(E_4-E_3)t}{\hbar}}\psi_4^*(x)\frac{\partial}{\partial x}\psi_3(x)}_{\text{part three}} + \underbrace{e^{\frac{-i(E_4-E_3)t}{\hbar}}\psi_3^*(x)\frac{\partial}{\partial x}\psi_4(x)}_{\text{part four}}\,dx$$

$A = \frac{m\omega}{\hbar}$

$$\text{part one} = \frac{-i\hbar}{2}\int_{-\infty}^{\infty}\frac{A}{\sqrt{\pi}}\cdot\frac{1}{2^3 3!}\left[(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2^4\,3!}\frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\left[(24Ax^2-12)e^{-\frac{Ax^2}{2}} - Ax(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2^4\,3!}\frac{A}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\left[\cancel{8\cdot24A^2x^5} - \cancel{8\cdot12Ax^3} - \cancel{8^2A^3x^7} + \cancel{8\cdot12A^2x^5} - \cancel{12\cdot24Ax^3} + \cancel{12^2x} + \cancel{12\cdot8A^2x^5} - \cancel{12^2Ax^3}\right]dx$$

odd terms drop out

$$= \cancel{\frac{-i\hbar A^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}x^4e^{-Ax^2}dx = \frac{-i\hbar A^2}{A^{5/2}}\cdot\frac{3}{4} = \frac{-i\hbar\cdot3}{A^{1/2}\cdot4}} = 0$$

$$\text{part two} = \frac{-i\hbar}{2}\int_{-\infty}^{\infty}\frac{A^{1/2}}{\sqrt{\pi}}\frac{1}{2^4 4!}\left[(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}dx\right]$$

$$= \frac{-i\hbar A^{1/2}}{2^5\,4!\sqrt{\pi}}\int_{-\infty}^{\infty}(16A^2x^4-48Ax^2+12)e^{-Ax^2}\left[(64A^2x^3-96Ax) - Ax(16A^2x^4-48Ax^2+12)\right]dx$$

$$= \frac{-i\hbar A^{1/2}}{2^8\cdot3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\left[16\cdot64A^4x^7 - 16\cdot96A^3x^5 - 16^2A^4x^9 + 16\cdot48A^3x^7 + 12\cdot16A^3x^5 \ldots\right] = 0$$

$$\text{part three} = \frac{-i\hbar}{2}e^{\frac{i(E_4-E_3)t}{\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2^7 3!\,4!}}\frac{A}{\sqrt{\pi}}\left[(16A^2x^4-48Ax^2+12)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}(8Ax^3-12x)e^{-\frac{Ax^2}{2}}\right]dx$$

$$= \frac{-i\hbar}{2\cdot3\sqrt{2^{11}}}\frac{A}{\sqrt{\pi}}e^{i\omega t}\int_{-\infty}^{\infty}(16A^2x^4-48Ax^2+12)e^{-Ax^2}\left[\underbrace{(3\cdot8Ax^2-12)}_{24Ax^2-12} - \underbrace{Ax(8Ax^3-12x)}_{-8A^2x^4+12Ax^2}\right]dx$$

$$= \frac{-i\hbar A}{2\cdot3\cdot\sqrt{\pi}}e^{i\omega t}\int_{-\infty}^{\infty}e^{-Ax^2}\big(16\cdot24A^3x^6 - 12\cdot16A^2x^4 - 8\cdot16\cdot A^4x^8 + 12\cdot16A^3x^6 - 48\cdot24A^2x^4 + 12\cdot48Ax^2 + 8\cdot48A^3x^6 - 12\cdot48A^2x^4$$
$$+ 12\cdot24Ax^2 - 12^2 - 8\cdot12A^2x^4 + 12^2Ax^2\big)\,dx$$

$$= \frac{-i\hbar A}{2^{13/2}\cdot3\cdot\sqrt{\pi}}e^{i\omega t}\sqrt{\pi}\left[\frac{-8\cdot16\cdot A^4\cdot2\cdot105}{32\cdot A^{9/2}} + \frac{2(16\cdot24\cdot A^3 + 12\cdot16A^3 + 8\cdot48A^3)\cdot15}{16\,a^{7/2}} + \frac{2\cdot3}{8\,a^{5/2}}\left(-12\cdot16A^2 - 48\cdot24A^2 - 12\cdot48A^2 - 8\cdot12A^2\right)\right.$$

$$+ \frac{1}{2A^{3/2}}\left(12\cdot48A + 12\cdot24A + 12^2A\right) + \frac{-12^2}{A^{1/2}}\Bigg]$$

$$= \frac{-i\hbar A^{1/2}}{2^{13/2}\cdot 3}e^{i\omega t}\left[-840 + 1800 + -1512 + 504 - 144\right] = i\hbar A^{1/2}\frac{\sqrt{2}}{2}e^{i\omega t} = \frac{i\hbar A^{1/2}}{\sqrt{2}}e^{i\omega t} = i\sqrt{\frac{m\omega\hbar}{2}}\,e^{i\omega t}$$

-192

$$\text{part four} = -\frac{i\hbar}{2}\frac{e^{-i\omega t}A^{1/2}}{\sqrt{2^{11}\cdot 3^2}\sqrt{\pi}}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\frac{\partial}{\partial x}\left(16A^2x^4 - 48Ax^2 + 12\right)e^{-\frac{Ax^2}{2}}\,dx$$

$$= \frac{-i\hbar A^{1/2}}{2^{13/2}\cdot 3\sqrt{\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-Ax^2}\Big[\left(16\cdot4A^2x^3 - 48\cdot2Ax\right) - Ax\left(16A^2x^4 - 48Ax^2 + 12\right)\Big]dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\cdot 3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\Big[8\cdot16\cdot4A^{7/2}x^6 - 8\cdot48\cdot2A^{5/2}x^4 - 8\cdot16\cdot A^{9/2}x^8 + 8\cdot48\cdot A^{7/2}x^6 - 12\cdot8\cdot A^{5/2}x^4 - 12\cdot16\cdot4A^{5/2}x^4$$
$$+ 12\cdot48\cdot2A^{3/2}x^2 + 12\cdot16A^{7/2}x^6 - 12\cdot48A^{5/2}x^4 + 12^2A^{3/2}x^2\Big]dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\,3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\left[-128A^{9/2}x^8 + 1088A^{7/2}x^6 - 2208A^{5/2}x^4 + 1296A^{3/2}x^2\right]dx$$

$$= \frac{-i\hbar A^{1/2}e^{-i\omega t}}{2^{13/2}\cdot3\cdot\cancel{\sqrt{\pi}}}\,2\cdot\cancel{\sqrt{\pi}}\left[-420 + 1020 - 828 + 324\right] = -i\sqrt{\frac{m\omega\hbar}{2}}\,e^{-i\omega t}$$

Again LADDER OP's!

$$\langle p(t)\rangle = \sqrt{\frac{m\omega\hbar}{2}}\left[ie^{i\omega t} - ie^{-i\omega t}\right] = \sqrt{2m\omega\hbar}\left[-\frac{e^{i\omega t} + e^{-i\omega t}}{2i}\right] = -\sqrt{2m\omega\hbar}\,\sin(\omega t)$$

$$\langle p^2(t)\rangle = \int_{-\infty}^{\infty}-\hbar^2\,\Psi^*(x,t)\frac{\partial^2}{\partial x^2}\Psi(x,t)\,dx$$

$$= -\frac{\hbar^2}{2}\int_{-\infty}^{\infty}\left(\Psi_3^*(x,t) + \Psi_4^*(x,t)\right)\frac{\partial^2}{\partial x^2}\left(\Psi_3(x,t) + \Psi_4(x,t)\right)dx$$

$$= -\frac{\hbar^2}{2}\int_{-\infty}^{\infty}\Psi_3^*(x,t)\frac{\partial^2}{\partial x^2}\Psi_3(x,t) + \Psi_4^*(x,t)\frac{\partial^2}{\partial x^2}\Psi(x,t) + \Psi_3^*(x,t)\frac{\partial^2}{\partial x^2}\Psi_4(x,t) + \Psi_4^*(x,t)\frac{\partial^2}{\partial x^2}\Psi_3(x,t)\,dx$$

$$\int_{-\infty}^{\infty}\Psi_3^*(x,t)\frac{\partial^2}{\partial x^2}\Psi_3(x,t)\,dx = \int_{-\infty}^{\infty}\frac{A^{1/2}}{2^3\,3!\sqrt{\pi}}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\frac{\partial^2}{\partial x^2}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\,dx$$

$$= \frac{A^{1/2}}{2^3\,3!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\Big[\left(8A^{3/2}x^3 - 12A^{1/2}x\right)\left[8A^{7/2}x^5 - 68A^{5/2}x^3 + 84A^{3/2}x\right]\Big]dx$$

$$= \frac{A^{1/2}}{2^3\,3!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-Ax^2}\left[64A^{5}x^8 - 640A^4x^6 + 1488A^3x^4 - 1008A^2x^2\right]dx$$

$$= \frac{A^{1/2}\, 2\cdot\sqrt{\pi}}{2^3 3! \sqrt{\pi}}\left[A^{1/2}\,210 - A^{1/2}\,600 + A^{1/2}\,558 - A^{1/2}\,252\right] = A\left[-\tfrac{7}{2}\right]$$

$$\int_{-\infty}^{\infty} \psi_4^*(x,t)\, \frac{\partial^2}{\partial x^2}\psi_4(x,t) = \frac{A^{1/2}}{2^4 4!}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\left(16A^2x^4 - 48Ax^2 + 12\right)\frac{\partial^2}{\partial x^2}\left(16A^2x^4 - 48Ax^2 + 12\right) dx$$

$$= \frac{A^{1/2}}{2^4 4! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-Ax^2}\left[256A^6x^{10} - 3840A^5x^8 + 18624A^4x^6 - 31680A^3x^4 + 12096A^2x^2 - 1296A\right] dx$$

$$= \frac{A}{2^3 4!}\left[3780 - 12600 + 17460 - 11880 + 3024 - 648\right] = A\left(\frac{-9}{2}\right)$$

$$\int_{-\infty}^{\infty} \psi_3^*(x,t)\, \frac{\partial^2}{\partial x^2}\psi_4(x,t)\, dx = \frac{A^{1/2}}{\sqrt{2^7 3! 4! \pi}} e^{-i\omega t}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)e^{-\frac{Ax^2}{2}}\frac{\partial^2}{\partial x^2}\left(16A^2x^4 - 48Ax^2 + 12\right)e^{-\frac{Ax^2}{2}} dx$$

$$= \frac{A^{1/2} e^{-i\omega t}}{\sqrt{2^7 3! 4! \pi}}\int_{-\infty}^{\infty}\left(8A^{3/2}x^3 - 12A^{1/2}x\right)\left(16A^4x^6 + -192A^3x^4 + 576A^2x^2 - 108A\right) dx = 0$$

odd terms drop out

Likewise $\int_{-\infty}^{\infty} \psi_4^*(x,t)\, \frac{\partial^2}{\partial x^2}\left(\psi_3(x,t)\right) dx = 0$

AGAIN LADDER OP'S!

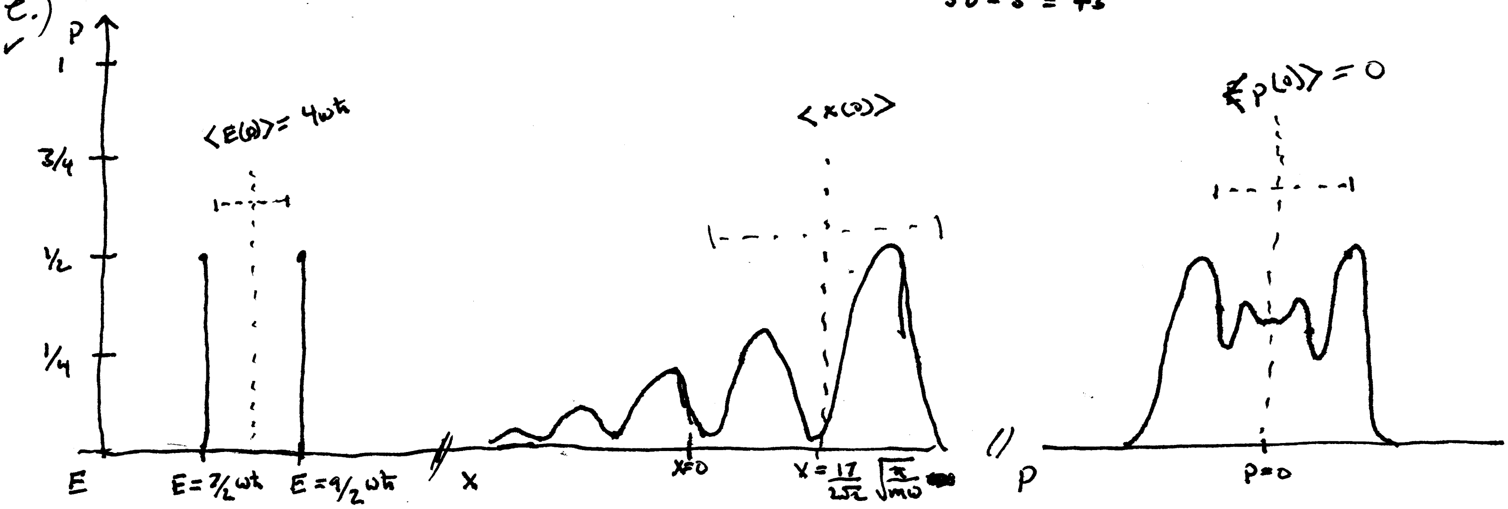
$$\langle p^2(t)\rangle = -\frac{\hbar^2}{2}\left[-8A\right] = 4m\omega\hbar$$

$$\Delta p(t) = \sqrt{\langle p^2(t)\rangle - \langle p(t)\rangle^2} = \sqrt{4m\omega\hbar - 2m\omega\hbar \sin^2(\omega t)} = \sqrt{2m\omega\hbar\left(2 - \sin^2(\omega t)\right)}$$

The expectation of p is 90° out of phase with $\langle x(t)\rangle$ exactly as the simulation on falstad.com shows. The uncertainty seems to match as well.

e.)

50 - 5 = 45



Dirac Notation

2. Quantitative aspects of SHO

✓ (a)

(1) we have $\langle\psi(0)|\psi(0)\rangle = N^*N(\langle 4| + \langle 3|)(|3\rangle + |4\rangle) = 2N^2 = 1$

$\Rightarrow N$ can be $\frac{1}{\sqrt{2}}$

(2) the normalized wave function at $t=0$ is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle) \quad \text{———} \quad ①$$

(3) Assuming the ground state is $|0\rangle$, we have the time-dependent state vector

$$|\psi(t)\rangle \text{ be : } |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|3\rangle e^{-\frac{iE_3 t}{\hbar}} + |4\rangle e^{-\frac{iE_4 t}{\hbar}}\right) \text{, where} \quad \text{———} \quad ②$$

$$E_3 = (3+\tfrac{1}{2})\hbar\omega = \tfrac{7}{2}\hbar\omega$$
$$E_4 = (4+\tfrac{1}{2})\hbar\omega = \tfrac{9}{2}\hbar\omega \quad \#$$

(4) Using $\langle x|\psi(t)\rangle = \psi(x,t)$ we have

$$\psi(x,t) = \langle x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\langle x|3\rangle e^{-\frac{iE_3 t}{\hbar}} + \langle x|4\rangle e^{-\frac{iE_4 t}{\hbar}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\psi_3(x) e^{-\frac{iE_3 t}{\hbar}} + \psi_4(x) e^{-\frac{iE_4 t}{\hbar}}\right) \#$$

✓ (b) (1) Since $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$ contains only eigenstates corresponding to E_3 and E_4,

the possible energy measurements are only :

$E_3 = \frac{7}{2}\hbar\omega$, with probability $|\langle 3|\psi(0)\rangle|^2 = \frac{1}{2}$

$E_4 = \frac{9}{2}\hbar\omega$, - - - - $|\langle 4|\psi(0)\rangle|^2 = \frac{1}{2}$

(2) $\langle E(t)\rangle = \langle\psi|H|\psi\rangle = \langle\psi(t)|\left(\frac{1}{\sqrt{2}}E_3|3\rangle e^{-\frac{iE_3t}{\hbar}} + \frac{1}{\sqrt{2}}E_4|4\rangle e^{-\frac{iE_4t}{\hbar}}\right)$

$=$ (using $e^{ix}\cdot e^{-ix}=1$ and $\langle n|m\rangle=\delta_{nm}$ $\quad \frac{1}{2}E_3+\frac{1}{2}E_4 = \frac{1}{2}\left(\frac{7}{2}+\frac{9}{2}\right)\hbar\omega$

$= 4\hbar\omega$ ※

(3) $\Delta E(t)^2 = \langle H^2\rangle - \langle H\rangle^2 = \langle\psi(t)|\left(\frac{1}{\sqrt{2}}E_3^2|3\rangle e^{-\frac{iE_3t}{\hbar}} + \frac{1}{\sqrt{2}}E_4^2 e^{-\frac{iE_4t}{\hbar}}|4\rangle\right) - 16\hbar^2\omega^2$

$= \frac{1}{2}E_3^2 + \frac{1}{2}E_4^2 - 16(\hbar\omega)^2 = \frac{1}{4}\hbar^2\omega^2$

$\Delta E(t) = \frac{1}{2}\hbar\omega$ ※

(4) Since the energy E is the eigenvalue of the stationary states of the system, we expect that $\langle E\rangle$ and ΔE will not change with t. Our calculation in (2), (3) verify this.

✓ (c)

(1) Since the potential energy is not infinite (unless at $x \to \pm\infty$), the possible measurement of position can be $x \in [-\infty, \infty]$, with probability density be

$$|\langle x|\psi(0)\rangle|^2 = \frac{1}{2}\left(\psi_3(x) + \psi_4(x)\right)^2, \text{ where}$$

$\psi_n(x)$ is the stationary energy eigenstate in position space

(2) using $X = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a + a^+)$, where a, a^+ are the lowering/ raising operators,

we have

$$\langle x(t)\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\cancel{\left(\frac{1}{2}\right)}\left(\cancel{2}e^{\frac{it}{\hbar}(E_3 - E_4)} + \cancel{2}e^{\frac{it}{\hbar}(E_4 - E_3)}\right)$$

$$= \left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\cos\left(\frac{t}{\hbar}(E_4 - E_3)\right) = \left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\cos(\omega t)$$

(3) We have $\langle X^2\rangle = \frac{\hbar}{2m\omega}\langle\psi(t)|(a+a^+)^2|\psi(t)\rangle = \frac{\hbar}{2m\omega}\langle(a+a^+)\psi(t)|(a+a^+)\psi(t)\rangle$

$$= \frac{\hbar}{2m\omega}\left|\frac{1}{\sqrt{2}}e^{\frac{-iE_3t}{\hbar}}(\sqrt{3}|2\rangle+2|4\rangle)+\frac{1}{\sqrt{2}}e^{\frac{-iE_4t}{\hbar}}(2|3\rangle+\sqrt{5}|5\rangle)\right|^2$$

$$= \frac{\hbar}{4m\omega}(7+9) = \frac{4\hbar}{m\omega}$$

Thus, $\Delta X^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{4\hbar}{m\omega} - \frac{2\hbar}{m\omega}\cos^2(\omega t)$

$$= \frac{2\hbar}{m\omega}(1+\sin^2\omega t)$$

$$\Rightarrow \Delta X = \sqrt{\frac{2\hbar}{m\omega}}(1+\sin^2\omega t)^{\frac{1}{2}}$$ ※

(4) The simulations do agree with my calculation (both $\langle x\rangle$, ΔX are periodic, with the frequency of ΔX is twice as frequency of $\langle x\rangle$.

✓ (d)

(1) From the symmetric of X, P in the Hamiltonian of H we know that $\psi_n(P) = \langle p | n \rangle$ is only a translation of $\psi_n(x)$ to $\psi_n\left(\frac{P}{m\omega}\right)$. Thus, from (c) we know that the possible measurement of momentum can be $P \in [-\infty, \infty]$, with the probability density be $|\langle P | \psi(0) \rangle|^2 = \frac{1}{2}\left(\hat{\psi}_3\left(\frac{P}{m\omega}\right) + \hat{\psi}_4\left(\frac{P}{m\omega}\right)\right)^2$, where

$\psi_n(x)$ is the ~~n~~ nth eigenstates in position space

(2) using $P = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}(a^{\dagger} - a)$, we have

$$\langle P(t) \rangle = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} \langle \psi(0) | (a^{\dagger} - a) | \psi(0) \rangle$$

$$= i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} \left(e^{i(E_4 - E_3)\frac{t}{\hbar}} - e^{-i(E_4 - E_3)\frac{t}{\hbar}} \right)$$

$$= 2i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} i \sin(\omega t) = -2\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}} \sin(\omega t) \ast$$

(3) for $\langle p^2\rangle$, Using $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ we have

$$\langle H\rangle = 4\hbar\omega = \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle$$

$$= \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}m\omega^2\frac{4\hbar}{m\omega} = \frac{1}{2m}\langle p^2\rangle + 2\hbar\omega$$

$$\Rightarrow \langle p^2\rangle = 2m(4\hbar\omega - 2\hbar\omega) = 4m\hbar\omega$$

Thus, we have $\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = 4m\hbar\omega - 2m\hbar\omega\sin^2\omega t$

$$= 2m\hbar\omega(1+\cos^2\omega t)$$

$$\Rightarrow \Delta p = \sqrt{2m\hbar\omega}\,(1+\cos^2\omega t)^{\frac{1}{2}}$$

(4) The simulation agrees with my expressions of $\langle p(t)\rangle$ and $\Delta p(t)$

(both $\langle p\rangle$, Δp are periodic, with frequency $\omega_{(\Delta p)} = 2\,\omega_{(\langle p\rangle)}$.

There is a $\frac{\pi}{2}$ phase difference between $\langle p\rangle$ and $\langle x\rangle$, which is consistent to the $\sin\omega t$ and $\cos\omega t$ expressions