Expressing the field intensity E, the current density J, and the charge density ρ in similar fashion, we find that

$$E = \frac{4}{3}\frac{V_0}{s}\left(\frac{x}{s}\right)^{1/3}, \tag{4-212}$$

$$J = \frac{4\epsilon_0}{9}\left(\frac{2e}{m}\right)^{1/2}\left(\frac{V_0^{3/2}}{s^2}\right), \tag{4-213}$$

$$= 2.335 \times 10^{-6}\frac{V^{3/2}}{s^2} \quad \text{(amperes/meter}^2\text{)}. \tag{4-214}$$

Thus

$$\rho = \frac{4\epsilon_0}{9s^2}V_0\left(\frac{x}{s}\right)^{-2/3}. \tag{4-215}$$

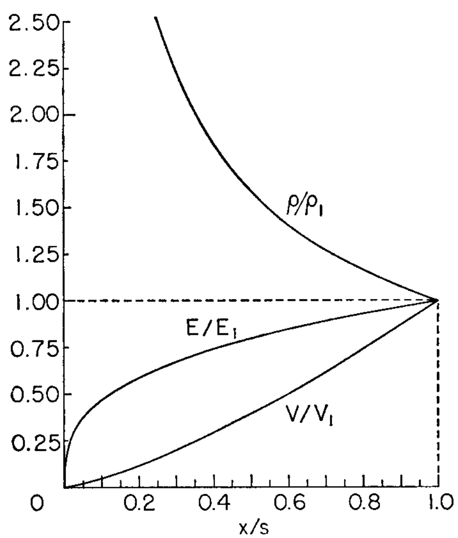

Figure 4-28. *The space charges density ρ, the electric field intensity E, and the electrostatic potential V as functions of the distance from the cathode in a plane-parallel infinite diode. The index 1 refers to the value at the anode. The distance between cathode and anode is s.*

Equation *4-214*, which is known as the *Child-Langmuir law*, is valid only for the plane parallel diode and for electrons emitted with zero velocity. Figure 4-28 shows the distribution of potential V, electric field intensity E, and charge density ρ in the plane parallel diode. One can show,* however, that, in general, no matter what the geometry of the diode may be, the current is related to the potential difference between cathode and anode by the relation

$$J = KV^{3/2}, \tag{4-216}$$

where K is a constant.

In an actual diode, electrons are emitted with finite velocities, and the equilibrium field intensity E_c at the cathode is negative. In this case, a potential minimum is established at a small distance in front of the cathode and only electrons with velocities greater than a critical value can get past the potential minimum, which is a potential energy maximum for electrons.

4.8. Summary

In this chapter we have dealt with electrostatic problems which cannot easily be solved by direct integration from Coulomb's law or by application of Gauss's law. We have sought solutions of Poisson's equation

* K. R. Spangenberg, *Fundamentals of Electron Devices* (McGraw-Hill, 1957), p. 169.

$$\nabla^2 V = -\rho/\epsilon_0 \tag{4-1}$$

or, more often, of Laplace's equation

$$\nabla^2 V = 0. \tag{4-2}$$

The solutions of these equations must always be consistent with certain *boundary conditions* which necessarily prevail between different media.

(1) For charge distributions of finite extent the potential V must go to zero at infinity; it must be constant throughout a conductor; and it must be continuous across any physical boundary.

(2) The normal component of the displacement vector $\mathbf{D}$ differs on the two sides of a boundary by the free charge density σ residing on the boundary.

(3) The tangential component of the field intensity $\mathbf{E}$ is continuous across a boundary.

We showed that a potential V which satisfies both Poisson's equation and the pertinent boundary conditions is the only possible potential. Thus, any potential we can devise, whether by intuitive or formal methods, is the correct one. This is the *uniqueness theorem.*

We discussed the method of *images*, in which an electrostatic problem is converted into an equivalent problem which is simpler to solve. This method is particularly appropriate for point charges near conductors; we used it for the case of a point charge and a conducting plane, for the case of a point charge and a conducting sphere, and for the case of a sphere near a conducting plane. We showed that the electrostatic forces calculated for the image problem are the same as for the equivalent arrangement of charges and conductors which the images replace. We also showed how to find the field intensities by image methods in the case of a point charge near a dielectric slab.

We next found general solutions of Laplace's equation, such solutions being known as harmonic functions. We looked for solutions by the process of variable separation, first in rectangular coordinates. In this process we seek solutions of the form

$$V = X(x)\,Y(y)\,Z(z), \tag{4-73}$$

where $X(x)$, $Y(y)$, and $Z(z)$ are functions only of x, y, and z, respectively. We showed that such solutions exist and that by taking linear combinations of them we can satisfy arbitrary boundary conditions. We showed how to find the solutions for two grounded, semi-infinite, parallel electrodes terminated by a plane electrode at a potential V_0, as well as for a pair of parallel plates extending to infinity in one direction and terminated by electrodes at arbitrary potentials on the other two sides. The series of sine and cosine terms which we use in rectangular coordinates to fit arbitrary boundary conditions are known as

Fourier series. The coefficients of the various terms in the series can be evaluated through the orthogonality of the sine and cosine functions. For example,

$$\int_0^b C_n \sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}\,dy = \begin{cases} 0 & \text{if } p \neq n, \\ C_n\dfrac{b}{2} & \text{if } p = n. \end{cases} \tag{4-95}$$

In spherical coordinates Laplace's equation was solved by the method of variable separation. We restricted ourselves to problems of axial symmetry, in which the potential is independent of the azimuth angle ϕ. In seeking solutions of the form

$$V(r, \theta) = R(r)\,\Theta(\theta), \tag{4-110}$$

we were led to two ordinary differential equations:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)r = 0, \tag{4-115}$$

and Legendre's equation,

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\theta}\right] + n(n+1)\Theta = 0, \tag{4-121}$$

where $\mu = \cos\theta$.

The first equation is readily solved by functions of the type

$$R(r) = Ar^n + \frac{B}{r^{n+1}}. \tag{4-116}$$

The solutions of Legendre's equation are called *Legendre polynomials*, which we denote by $P_n(\cos\theta)$, there being a different polynomial for each value of the index n. We built up a set of these polynomials by making use of a property of harmonic functions, namely, that the derivative of such a function with respect to a rectangular coordinate is also a solution. The general form of the solutions we found by this method is

$$P_n(\cos\theta) = \frac{1}{2^n n!}\frac{\partial^n}{\partial(\cos\theta)^n}(\cos^2\theta - 1)^n. \tag{4-138}$$

The general solution of Laplace's equation in spherical coordinates, if we assume axial symmetry, is then

$$V = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \tag{4-139}$$

The individual terms of this equation constitute a complete set of functions; any arbitrary boundary value of the potential having axial symmetry can be satisfied with such a series. The coefficients in the series can be determined by using the specified potentials on the boundaries and by using the orthogonality property of the Legendre functions:

$$\int_{-1}^{+1} P_m(\cos\theta)P_n(\cos\theta)\, d(\cos\theta) = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n. \end{cases} \qquad (4\text{-}140)$$

We used the general equation *4-139* to solve several typical problems: a conducting sphere in a uniform electrostatic field, which we examined from several points of view, a dielectric sphere in a uniform electrostatic field, and a uniformly charged ring.

We finally discussed the solution of Poisson's equation for the parallel-plate vacuum diode. This leads to the *Child-Langmuir law* relating the current density J to the potential difference V_0 between the plates:

$$J = \frac{4\epsilon_0}{9}\left(\frac{2e}{m}\right)^{1/2}\left(\frac{V_0^{3/2}}{s^2}\right). \qquad (4\text{-}213)$$

Problems

4-1. Two infinite conducting planes intersect at right angles, the line of intersection being the x-axis and the planes being the xz- and xy-planes. A charge Q is placed in the yz-plane at a distance a from the y-axis and b from the z-axis. Use the method of images to find the field intensity **E** at the surface of each conductor.

Compute the surface charge density σ.

Find the force **F** on the charge Q.

4-2. A conducting sphere of radius R bearing a charge Q is at a distance $d = 3R$ from an infinite, grounded, conducting plane. Determine the potential of the sphere within one percent.

4-3. The centers of two conducting spheres are separated by 25 centimeters. The radius of the first is 5 centimeters, and that of the second is 10 centimeters. The potential of the first is 10 volts; the second is grounded. What is the charge on each sphere, within one percent?

4-4. A grounded metal sphere of radius R is under the influence of an external point charge Q at point P. What fraction of the induced charge on the sphere can be seen from P?

4-5. A thin conducting spherical shell of radius a contains within it a point charge Q at a distance r from the center. Find, by the method of images, the charge density induced on the outside surface of the sphere.

Find also the force **F** on the charge Q.

Is the equilibrium point at the center stable, unstable, or neutral?

4-6. A charge Q is situated between two horizontal parallel conducting plates separated by a distance s. The charge Q is at a distance x above the lower plate. Calculate, the force due to the image charges, in the form of an infinite series.

Find an approximate value for the force when Q is situated (a) near one of the plates and (b) near the position $x = s/2$.