4.6. Solutions of Laplace's Equation in Spherical Coordinates. Legendre's Equation. Legendre Polynomials

Although electrostatic fields can usually be calculated in Cartesian coordinates, certain cases of symmetry are best treated in spherical polar coordinates. Laplace's equation then takes the form

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0. \qquad (4\text{-}108)$$

The solutions of this equation are known as *spherical harmonic functions.*

We shall restrict ourselves here to problems with axial symmetry, that is, to problems in which V is independent of the angle ϕ. Equation *4-108* then reduces to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right) = 0. \qquad (4\text{-}109)$$

As in Cartesian coordinates, we seek solutions in which the variables are separated and set

$$V(r,\theta) = R(r)\,\Theta(\theta), \qquad (4\text{-}110)$$

where R is a function only of r and Θ is a function only of θ. Substituting $V = R\Theta$ into Eq. *4-109*, we obtain

$$\Theta\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) = 0, \qquad (4\text{-}111)$$

and dividing through by $R\Theta$ gives

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = 0. \qquad (4\text{-}112)$$

We have written total instead of partial derivatives, since the functions to be differentiated in each case are functions of a single variable.

Since the second term of Eq. *4-112* is independent of r, the first term must also be independent of r. The first term must therefore be constant, and we may write

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = k \qquad (4\text{-}113)$$

and

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = -k, \qquad (4\text{-}114)$$

since the sum of the two constants must equal zero.

Let us examine the R equation first. Multiplying both sides by R and differentiating the bracketed factors, we obtain the relation

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0. \tag{4-115}$$

A function of the type

$$R = Ar^n + \frac{B}{r^{n+1}} \tag{4-116}$$

is a solution of Eq. *4-115*. On substituting we find that n is related to k:

$$n(n+1) = k. \tag{4-117}$$

Let us now examine Eq. *4-114* for Θ. We have

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\sin\theta\,\Theta = 0. \tag{4-118}$$

It is convenient to change variables in this equation and to let

$$\mu = \cos\theta. \tag{4-119}$$

Recall that for any function $f(\mu)$

$$\frac{df}{d\theta} = \frac{df}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{df}{d\mu} = -\sqrt{1-\mu^2}\frac{df}{d\mu}. \tag{4-120}$$

Thus Eq. *4-118* becomes

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0. \tag{4-121}$$

This is known as *Legendre's equation.* Its solutions are polynomials in $\cos\theta$ and are known as *Legendre polynomials.* They are designated by $P_n(\mu)$ or $P_n(\cos\theta)$:

$$\Theta = P_n(\mu) = P_n(\cos\theta), \tag{4-122}$$

where n is called the *degree* of the polynomial. A different polynomial exists for each value of n. We shall limit our discussion to integral values of n.

Before proceeding to find solutions of Eq. *4-121*, we may point out an interesting property of Legendre's equation. The index n must satisfy Eq. *4-117*. But

$$n' = -(n+1) \tag{4-123}$$

will equally satisfy this equation because

$$n'(n'+1) = n(n+1) = k. \tag{4-124}$$

That is, Eq. *4-121* remains unchanged when the index n' is substituted for n. Hence the solutions must be the same, and

$$P_{-(n+1)}(\cos\theta) = P_n(\cos\theta). \tag{4-125}$$

Therefore, for every solution of Laplace's equation of the form

$$V = Ar^nP_n(\cos\theta) \tag{4-126}$$

there is another solution of the form

$$V = \frac{B}{r^{n+1}} P_n(\cos\theta). \tag{4-127}$$

This also follows directly from the R function of Eq. *4-116.*

Let us now proceed to find the Legendre polynomials $P_n(\cos\theta)$ which are solutions of Legendre's equation. We know from our experience with point charges that

$$V_1 = \frac{C}{r}, \tag{4-128}$$

is a solution of Laplace's equation, C being a constant. This is readily verified by substitution in Eq. *4-108.* Since we are looking for solutions of the form indicated in Eqs. *4-126* or *4-127*, it follows from the latter equation that

$$P_0'(\cos\theta) = 1. \tag{4-129}$$

We use a prime on the symbol P because the polynomials which we shall derive here differ from the Legendre polynomials by constant factors, as we shall see below. They are nevertheless solutions of Eq. *4-121.* Substituting $\Theta = P_0'(\cos\theta) = 1$ and $n = 0$ into Eq. *4-121* does in fact solve it.

Having found the solution $P_0'(\cos\theta)$, how can we find $P_1'(\cos\theta)$ and all the other polynomials corresponding to all the possible integral values of the index n in Eq. *4-118?* We shall do this starting with Eq. *4-128*, but first we must know that any partial derivative of a solution of Laplace's equation with respect to any of the Cartesian coordinate variables is also a solution. This is easily demonstrated by substituting dV/dx in Laplace's equation and remembering that the order of differentiation in partial derivatives is immaterial.

Let us therefore find the negative partial derivative of Eq. *4-128* with respect to z:

$$-\frac{\partial}{\partial z}\left(\frac{C}{r}\right) = +\frac{C}{r^2}\frac{\partial r}{\partial z} \tag{4-130}$$

and

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{1/2} = \frac{z}{r} = \cos\theta. \tag{4-131}$$

Equation *4-130* thus gives us a new solution of Laplace's equation:

$$V_2 = C\frac{\cos\theta}{r^2}. \tag{4-132}$$

Comparing once again with Eq. *4-127* we see that

$$P_1'(\cos\theta) = \cos\theta. \tag{4-133}$$

Substitution of $P_1'(\cos\theta) = \cos\theta$ for Θ into Eq. *4-121* shows that it is really a solution when $n = 1$.

Equation *4-126* shows another possible solution for a given $P_n(\cos\theta)$. In this case we have, in addition to V_2, another solution:

$$V_2' = DrP_1'(\cos\theta) = Dr\cos\theta. \tag{4-134}$$

To find $P_2'(\cos\theta)$, we differentiate V_2 with respect to z:

$$V_3 = -\frac{\partial}{\partial z}\left(C\frac{\cos\theta}{r^2}\right) = -\frac{\partial}{\partial z}\left(C\frac{z}{r^3}\right) = C\frac{(3\cos^2\theta - 1)}{r^3}. \tag{4-135}$$

Comparing this with Eq. *4-127*,

$$P_2'(\cos\theta) = (3\cos^2\theta - 1). \tag{4-136}$$

Again there is another solution:

$$V_3' = Fr^2(3\cos^2\theta - 1), \tag{4-137}$$

which corresponds to Eq. *4-126*. We shall stop here, but we could continue to find further polynomials in this way by repeated partial differentiations with respect to z.

It is convenient to multiply the above polynomials by normalizing factors to make them equal to unity at $\cos\theta = 1$. Thus Eq. *4-136* must be multiplied by the factor $\frac{1}{2}$ to make $P_2(\cos\theta) = 1$ when $\cos\theta = 1$. The *general form of the normalized Legendre polynomial* is

$$P_n(\cos\theta) = \frac{1}{2^n n!}\frac{\partial^n}{\partial(\cos\theta)^n}(\cos^2\theta - 1)^n. \tag{4-138}$$

The first five are shown in Table 4-2; those for $n = 1$, 2, and 3 are plotted as functions of θ in Figure 4-21.

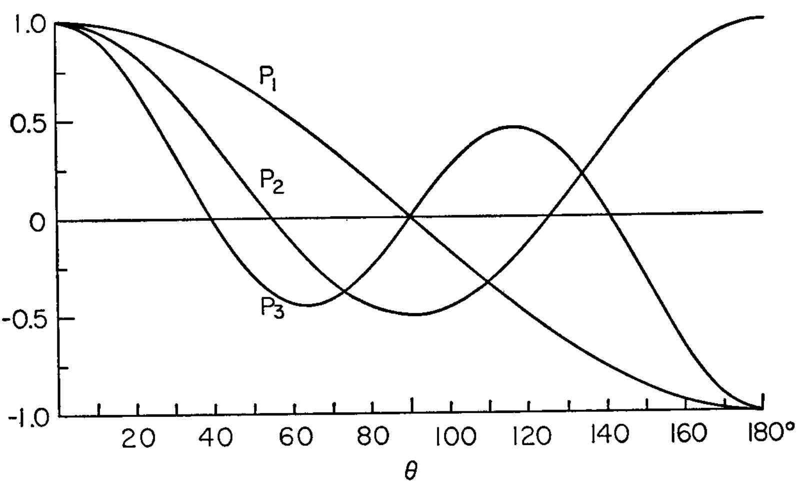

Figure 4-21. *The first three Legendre polynomials.*

TABLE 4-2. Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{3}{2}\cos^2\theta - \frac{1}{2}$
3	$\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$
4	$\frac{35}{8}\cos^4\theta - \frac{15}{4}\cos^2\theta + \frac{3}{8}$

A general solution of Laplace's equation in spherical polar coordinates, assuming axial symmetry, is therefore the following:

$$V = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \qquad (4\text{-}139)$$

The various terms are shown in Table 4-3.

TABLE 4-3. Solutions of Laplace's Equation in Spherical Polar Coordinates in the Case of Axial Symmetry

n	$r^n P'_n\ (\cos\theta)$	$r^{-(n+1)} P'_n \cos\theta$
0	1	r^{-1}
1	$r\cos\theta$	$r^{-2}\cos\theta$
2	$\frac{1}{2}r^2(3\cos^2\theta - 1)$	$\frac{1}{2}r^{-3}(3\cos^2\theta - 1)$
3	$\frac{1}{2}r^3(5\cos^3\theta - 3\cos\theta)$	$\frac{1}{2}r^{-4}(5\cos^3\theta - 3\cos\theta)$
4	$\frac{1}{8}r^4(35\cos^4\theta - 30\cos^2\theta + 3)$	$\frac{1}{8}r^{-5}(35\cos^4\theta - 30\cos^2\theta + 3)$

It can be shown that the functions in Eq. *4-139* are a *complete set of functions*, thus an arbitrary boundary condition with axial symmetry can be satisfied with such an infinite series. Moreover, any function of the polar angle θ can be represented as a series of Legendre polynomials, provided the function is continuous within the range of θ considered and provided the function has a finite number of maxima and minima.

It can be shown that

$$\int_{-1}^{+1} P_m(\cos\theta) P_n(\cos\theta)\, d(\cos\theta) = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{2}{2n+1} & \text{if } m = n. \end{cases} \qquad (4\text{-}140)$$

This property of *orthogonality* of the Legendre polynomials is important in evaluating the coefficients of Eq. *4-139*.

4.6.1. Conducting Sphere in a Uniform Electrostatic Field. To illustrate the use of Eq. *4-139* in calculating electrostatic fields, we consider the case of an