single charge $Q'' = +\frac{2}{K_e + 1} Q$ at the position of Q, as in Figure 4-14b. Since these combinations of charges produce fields which satisfy the boundary conditions, we know from the uniqueness theorem that they provide the correct solution. The shape of the field is shown in Figure 4-15.

In general, for two media having dielectric constants K_{e1} and K_{e2}, with the point charge Q in the first medium, the point charges which give the correct field are the following:

$$Q' = -\frac{K_{e2} - K_{e1}}{K_{e2} + K_{e1}} Q \tag{4-69}$$

at the image position together with Q gives the field in the first medium, and

$$Q'' = +\frac{2K_{e2}}{K_{e2} + K_{e1}} Q \tag{4-70}$$

at the position of Q gives the field in the second medium.

4.5. General Solution of Laplace's Equation*

The methods which we have considered until now for the calculation of electrostatic fields are useful only in special cases. We shall discuss here a more general method which will involve solving Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{4-71}$$

To begin with, we shall confine our attention to problems in which the charge density ρ is equal to zero, thus we shall have to deal with Laplace's equation,

$$\nabla^2 V = 0. \tag{4-72}$$

Solutions of Laplace's equation are known as *harmonic functions*, and there are an infinite number of them. These functions have a number of general properties, of which we shall use the following one. If the functions $V_1, V_2, V_3, \cdots$ are solutions, then any linear combination $A_1V_1 + A_2V_2 + A_3V_3 + \cdots$ of these functions, where the A's are arbitrary constants, is also a solution. This can be demonstrated readily by substitution into the original equation.

4.5.1. Solutions in Rectangular Coordinates. It is usually possible to find solutions of Laplace's equation which will satisfy required boundary conditions by the process of *variable separation*. In Cartesian coordinates, for example, we can usually find a solution of the form

$$V = X(x)\,Y(y)\,Z(z), \tag{4-73}$$

where $X(x)$, $Y(y)$, and $Z(z)$ are functions only of the variables x, y, and z,

* Sections 4.5 to 4.6.3 may be omitted without losing continuity.

respectively. We can then fit boundary conditions by adding a series of such solutions multiplied by suitable coefficients. The uniqueness theorem assures us that the solution thus found is the proper solution.

We can find the form of the functions $X(x)$, $Y(y)$, and $Z(z)$ by substituting V of Eq. *4-73* into Laplace's equation. Then

$$YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0, \quad (4\text{-}74)$$

where we have written total instead of partial derivatives, since the X, Y, and Z functions are each a function of a single variable. On dividing through by XYZ, we find that

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \quad (4\text{-}75)$$

Now since the second and third terms are independent of x, and since the three terms must add to zero at all points, the first term must also be independent of x. It is therefore constant in value, and

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1. \quad (4\text{-}76)$$

Similarly,

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad (4\text{-}77)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = C_3. \quad (4\text{-}78)$$

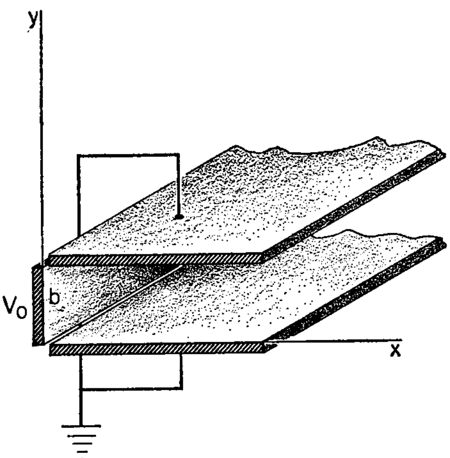

Figure 4-16. *Grounded, plane-parallel electrodes terminated by a plane electrode at potential V_0. The electrodes are assumed to be infinite in the direction perpendicular to the paper and are assumed to extend infinitely on the right.*

Then

$$C_1 + C_2 + C_3 = 0. \quad (4\text{-}79)$$

The problem now becomes one of solving the three *ordinary* differential equations, subject to the condition of Eq. *4-79* and to the boundary conditions.

4.5.2. Field Between Two Grounded Semi-infinite Parallel Electrodes Terminated by a Plane Electrode at Potential V_0. As an example, consider Figure 4-16, which shows two grounded, semi-infinite, parallel electrodes separated by a distance b. The plane at $x = 0$ is occupied by a conducting electrode maintained at a potential V_0. The problem is to find the potential V at any point between the plates.

Since the plates have infinite extent in the positive and negative z directions, the potential must be independent of z, thus the last term of Eq. *4-75*, together with the constant C_3, is zero. We must therefore solve the two ordinary differential equations

$$\frac{d^2X}{dx^2} - k^2X = 0 \tag{4-80}$$

and

$$\frac{d^2Y}{dy^2} + k^2Y = 0, \tag{4-81}$$

where we have substituted k^2 for C_1 and $-k^2$ for C_2 to eliminate square roots in the solution. The choice between C_1 and C_2 as the negative constant is immaterial; the boundary conditions will force us to the same final solution in either case.

Equation *4-81* is solved by setting

$$Y = A \sin ky + B \cos ky, \tag{4-82}$$

where A and B are arbitrary constants. This can be easily verified by substitution.

Our value of Y must satisfy the boundary conditions

$$V = 0 \qquad (y = 0, y = b), \tag{4-83}$$

$$V = V_0 \qquad (x = 0), \tag{4-84}$$

$$V \longrightarrow 0 \qquad (x \longrightarrow \infty). \tag{4-85}$$

In order to have $V = 0$ at $y = 0$ we must have $B = 0$; and in order to have $V = 0$ at $y = b$ we must have

$$kb = n\pi \qquad (n = 1, 2, \cdots), \tag{4-86}$$

thus

$$Y = A \sin \frac{n\pi y}{b} \qquad (n = 1, 2, \cdots). \tag{4-87}$$

The value $n = 0$ must be omitted, for it corresponds to a sine term which is zero, and therefore to zero field.

Turning now to the X equation, we have

$$\frac{d^2X}{dx^2} - \left(\frac{n\pi}{b}\right)^2 X = 0, \tag{4-88}$$

thus

$$X = Ge^{n\pi x/b} + He^{-n\pi x/b}, \tag{4-89}$$

where G and H are arbitrary constants. We can again verify this solution by substitution. The condition that $V \longrightarrow 0$ as $x \longrightarrow \infty$ requires that $G = 0$.

Altogether then, we have

$$V'(x, y) = C \sin \frac{n\pi y}{b} e^{-n\pi x/b}, \tag{4-90}$$

where C is another arbitrary constant.

The solution as it is will obviously satisfy the boundary conditions stated in

Eqs. *4-83* and *4-85*. It will not, however, satisfy Eq. *4-84*. We therefore take an infinite sum of such solutions and set

$$V(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{b} e^{-n\pi x/b}. \qquad (4\text{-}91)$$

To evaluate the coefficients C_n, we use the boundary condition at $x = 0$, namely,

$$V(0, y) = V_0 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{b}. \qquad (4\text{-}92)$$

The expression on the right is called a *Fourier series*. It can be shown that the functions in Eq. *4-92*, provided an infinite series of cosine terms is also included, constitute a *complete* set of functions. This means that *an arbitrary boundary condition can be satisfied with such an infinite series.*

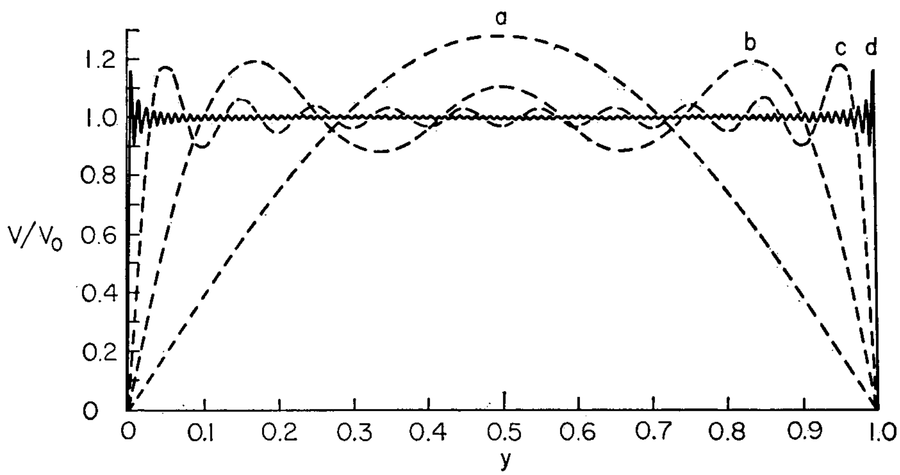

Figure 4-17. *The condition $V = V_0$ as satisfied by a Fourier series taking (a) only the first term, (b) the first 3 terms, (c) the first 10 terms, and (d) the first 100 terms. The Fourier series provides an increasingly better approximation as the number of terms is increased.*

Using a technique devised by Fourier, we multiply both sides of Eq. *4-92* by $\sin [(p\pi y)/b]$, where p is an integer, and integrate from $y = 0$ to $y = b$:

$$\int_0^b V_0 \sin \frac{p\pi y}{b} \, dy = \int_0^b \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{b} \sin \frac{p\pi y}{b} \, dy. \qquad (4\text{-}93)$$

On the left-hand side,

$$\int_0^b V_0 \sin \frac{p\pi y}{b} \, dy = \begin{cases} \dfrac{2bV_0}{p\pi} & \text{if } p \text{ is odd,} \\ 0 & \text{if } p \text{ is even,} \end{cases} \qquad (4\text{-}94)$$

whereas on the right-hand side,

$$\int_0^b C_n \sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}\,dy = \begin{cases} 0 & \text{if } p \neq n, \\ C_n \dfrac{b}{2} & \text{if } p = n. \end{cases} \tag{4-95}$$

Thus, for a given p, the only term of the infinite series on the right-hand side of Eq. *4-93* which differs from zero is the one for which $n = p$. Functions with such properties are said to be *orthogonal.*

Combining Eqs. *4-94* and *4-95*, we find that

$$C_n = \begin{cases} \dfrac{4V_0}{n\pi} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases} \tag{4-96}$$

We can now write down the potential V at any point (x, y):

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} \sin\frac{n\pi y}{b}\, e^{-n\pi x/b}. \tag{4-97}$$

The successive terms in the series become progressively less important because of the $(1/n)$ factor in the coefficients and because of the negative exponential factor involving n. The degree of approximation which one achieves at $x = 0$

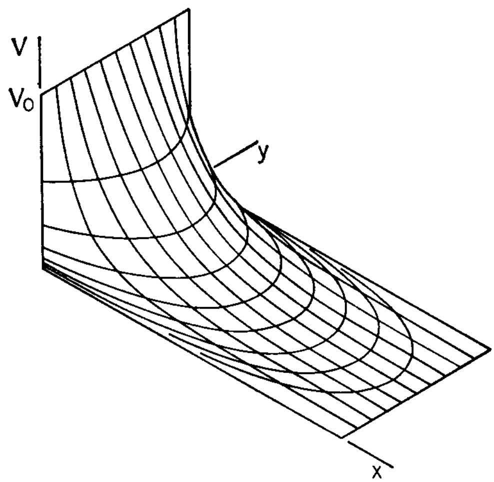

Figure 4-18

Three-dimensional plot of the potential V for the configuration of Figure 4-16. The U-shaped curves are equipotentials; the others show the intersections of the potential surface with planes parallel to the xV-plane.

with one, three, ten, and one hundred terms of the series is indicated in Figure 4-17. At $x = b$ the first term alone gives a good approximation. The equipotentials are shown in Figure 4-18.

4.5.3. Field Between Two Grounded Parallel Electrodes Terminated on Two Opposite Sides by Plates at Potentials V_1 and V_2. As a more complicated example, consider Figure 4-19, where two grounded plane parallel electrodes of width a are separated by a distance b and extend to infinity in the other direction. The plane at $x = 0$ is occupied by a conducting surface maintained at a potential V_1, and the plane at $x = a$ is occupied by a conductor maintained at a potential V_2. The problem is again to find the electrostatic potential V at any point between the plates.

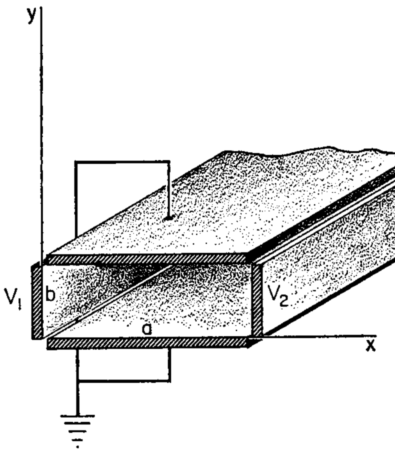

Figure 4-19. *Grounded plane-parallel electrodes terminated on two sides with plane electrodes at potentials V_1 and V_2. The electrodes are assumed to be infinite in the direction perpendicular to the paper.*

Since the plates have infinite extent in the positive and negative z directions, there is no z dependence of the potential, hence the last term of Eq. *4-75*, together with the constant C_3 in Eq. *4-79*, is again zero. Since the Y part of the solution is identical with that of the previous example, Eq. *4-87* is again valid, as are Eqs. *4-88* and *4-89*.

From this point on, the solution differs from that of the previous example, since the boundary conditions are different. Here we have

$$V = V_1 \text{ at } x = 0 \qquad (4\text{-}98)$$

and

$$V = V_2 \text{ at } x = a. \qquad (4\text{-}99)$$

The most general solution, and the one required to satisfy the boundary conditions, is

$$V(x, y) = \sum_{n=1}^{\infty} (A_n e^{-n\pi x/b} + B_n e^{n\pi x/b}) \sin \frac{n\pi y}{b}, \qquad (4\text{-}100)$$

where A_n and B_n are again constants which must be determined from the boundary conditions.

At $x = 0$,

$$V_1 = \sum_{n=1}^{\infty} (A_n + B_n) \sin \frac{n\pi y}{b}. \qquad (4\text{-}101)$$

The coefficients are evaluated by the same Fourier method used in the previous example. On multiplying by $\sin \frac{p\pi y}{b}$ and integrating from $y = 0$ to $y = b$, we have again, out of the whole infinite series, only one term corresponding to $p = n$:

$$V_1 \int_0^b \sin \frac{n\pi y}{b}\, dy = (A_n + B_n) \frac{b}{2}, \qquad (4\text{-}102)$$

thus

$$A_n + B_n = \begin{cases} \dfrac{4V_1}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \quad (4\text{-}103)$$

We can find another relationship between A_n and B_n by using the boundary condition at $x = a$: from Eq. *4-100*,

$$V_2 = \sum_{n=1}^{\infty} (A_n e^{-n\pi a/b} + B_n e^{n\pi a/b}) \sin \frac{n\pi y}{b}. \quad (4\text{-}104)$$

Multiplying by $\sin \dfrac{p\pi y}{b}$ and integrating from $y = 0$ to $y = a$, as before, we find that

$$A_n e^{-n\pi a/b} + B_n e^{n\pi a/b} = \begin{cases} \dfrac{4V_2}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \quad (4\text{-}105)$$

and from Eqs. *4-103* and *4-105*,

$$A_n = \frac{4}{n\pi}\left(\frac{V_1 - V_2 e^{-n\pi a/b}}{1 - e^{-2n\pi a/b}}\right) \quad (4\text{-}106)$$

and

$$B_n = \frac{4e^{-n\pi a/b}}{n\pi}\left(\frac{V_2 - V_1 e^{-n\pi a/b}}{1 - e^{-2n\pi a/b}}\right), \quad (4\text{-}107)$$

where $n = 1, 3, 5, \cdots$. The potential V at any point (x, y) is given by Eq. *4-100* with A_n and B_n as above.

The degree of approximation achieved with a few terms of the final solution using these coefficients in Eq. *4-100* will be left as a problem at the end of the chapter. Figure 4-20 shows the equipotentials for the case where $V_1 = V_2 = V_0$.

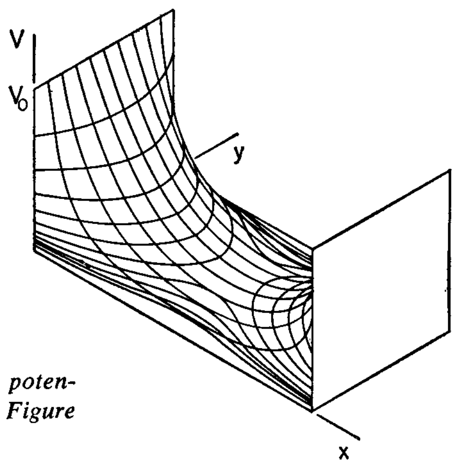

Figure 4-20

A three-dimensional plot of the potential V for the configuration of Figure 4-19 with $V_1 = V_2 = V_0$.