7

The Electric Field in Various Circumstances (Continued)

7–1 Methods for finding the electrostatic field

This chapter is a continuation of our consideration of the characteristics of electric fields in various particular situations. We shall first describe some of the more elaborate methods for solving problems with conductors. It is not expected that these more advanced methods can be mastered at this time. Yet it may be of interest to have some idea about the kinds of problems that can be solved, using techniques that may be learned in more advanced courses. Then we take up two examples in which the charge distribution is neither fixed nor is carried by a conductor, but instead is determined by some other law of physics.

As we found in Chapter 6, the problem of the electrostatic field is fundamentally simple when the distribution of charges is specified; it requires only the evaluation of an integral. When there are conductors present, however, complications arise because the charge distribution on the conductors is not initially known; the charge must distribute itself on the surface of the conductor in such a way that the conductor is an equipotential. The solution of such problems is neither direct nor simple.

We have looked at an indirect method of solving such problems, in which we find the equipotentials for some specified charge distribution and replace one of them by a conducting surface. In this way we can build up a catalog of special solutions for conductors in the shapes of spheres, planes, etc. The use of images, described in Chapter 6, is an example of an indirect method. We shall describe another in this chapter.

If the problem to be solved does not belong to the class of problems for which we can construct solutions by the indirect method, we are forced to solve the problem by a more direct method. The mathematical problem of the direct method is the solution of Laplace's equation,

$$\nabla^2\phi = 0, \tag{7.1}$$

subject to the condition that ϕ is a suitable constant on certain boundaries—the surfaces of the conductors. Problems which involve the solution of a differential field equation subject to certain *boundary conditions* are called *boundary-value* problems. They have been the object of considerable mathematical study. In the case of conductors having complicated shapes, there are no general analytical methods. Even such a simple problem as that of a charged cylindrical metal can closed at both ends—a beer can—presents formidable mathematical difficulties. It can be solved only approximately, using numerical methods. The *only* general methods of solution are numerical.

There are a few problems for which Eq. (7.1) can be solved directly. For example, the problem of a charged conductor having the shape of an ellipsoid of revolution can be solved exactly in terms of known special functions. The solution for a thin disc can be obtained by letting the ellipsoid become infinitely oblate. In a similar manner, the solution for a needle can be obtained by letting the ellipsoid become infinitely prolate. However, it must be stressed that the only direct methods of general applicability are the numerical techniques.

Boundary-value problems can also be solved by measurements of a physical analog. Laplace's equation arises in many different physical situations: in steady-state heat flow, in irrotational fluid flow, in current flow in an extended medium, and in the deflection of an elastic membrane. It is frequently possible to set up a physical model which is analogous to an electrical problem which we wish to solve. By the measurement of a suitable analogous quantity on the model, the solution to the problem of interest can be determined. An example of the analog technique is the use of the electrolytic tank for the solution of two-dimensional problems in electrostatics. This works because the differential equation for the potential in a uniform conducting medium is the same as it is for a vacuum.

There are many physical situations in which the variations of the physical fields in one direction are zero, or can be neglected in comparison with the variations in the other two directions. Such problems are called two-dimensional; the field depends on two coordinates only. For example, if we place a long charged wire along the z-axis, then for points not too far from the wire the electric field depends on x and y, but not on z; the problem is two-dimensional. Since in a two-dimensional problem $\partial/\partial z = 0$, the equation for ϕ in free space is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{7.2}$$

Because the two-dimensional equation is comparatively simple, there is a wide range of conditions under which it can be solved analytically. There is, in fact, a very powerful indirect mathematical technique which depends on a theorem from the mathematics of functions of a complex variable, and which we will now describe.

7–2 Two-dimensional fields; functions of the complex variable

The complex variable $\mathfrak{z}$ is defined as

$$\mathfrak{z} = x + iy.$$

(Do not confuse $\mathfrak{z}$ with the z-coordinate, which we ignore in the following discussion because we assume there is no z-dependence of the fields.) Every point in x and y then corresponds to a complex number $\mathfrak{z}$. We can use $\mathfrak{z}$ as a single (complex) variable, and with it write the usual kinds of mathematical functions $F(\mathfrak{z})$. For example,

$$F(\mathfrak{z}) = \mathfrak{z}^2,$$

or

$$F(\mathfrak{z}) = 1/\mathfrak{z}^3,$$

or

$$F(\mathfrak{z}) = \mathfrak{z} \log \mathfrak{z},$$

and so forth.

Given any particular $F(\mathfrak{z})$ we can substitute $\mathfrak{z} = x + iy$, and we have a function of x and y—with real and imaginary parts. For example,

$$\mathfrak{z}^2 = (x + iy)^2 = x^2 - y^2 + 2ixy. \tag{7.3}$$

Any function $F(\mathfrak{z})$ can be written as a sum of a pure real part and a pure imaginary part, each part a function of x and y:

$$F(\mathfrak{z}) = U(x, y) + iV(x, y), \tag{7.4}$$

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where $U(x, y)$ and $V(x, y)$ are real functions. Thus from any complex function $F(\mathfrak{z})$ two new functions $U(x, y)$ and $V(x, y)$ can be derived. For example, $F(\mathfrak{z}) = \mathfrak{z}^2$ gives us the two functions

$$U(x, y) = x^2 - y^2, \tag{7.5}$$

and

$$V(x, y) = 2xy. \tag{7.6}$$

Now we come to a miraculous mathematical theorem which is so delightful that we shall leave a proof of it for one of your courses in mathematics. (We should not reveal all the mysteries of mathematics, or that subject matter would become too dull.) It is this. For any "ordinary function" (mathematicians will define it better) the functions U and V *automatically* satisfy the relations

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \tag{7.7}$$

$$\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}. \tag{7.8}$$

It follows immediately that each of the functions U and V satisfy Laplace's equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \tag{7.9}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{7.10}$$

These equations are clearly true for the functions of (7.5) and (7.6).

Thus, starting with any ordinary function, we can arrive at two functions $U(x, y)$ and $V(x, y)$, which are both solutions of Laplace's equation in two dimensions. Each function represents a possible electrostatic potential. We can pick *any* function $F(\mathfrak{z})$ and it should represent *some* electric field problem—in fact, *two* problems, because U and V *each* represent solutions. We can write down as many solutions as we wish—by just making up functions—then we just have to find the *problem* that goes with each solution. It may sound backwards, but it's a possible approach.

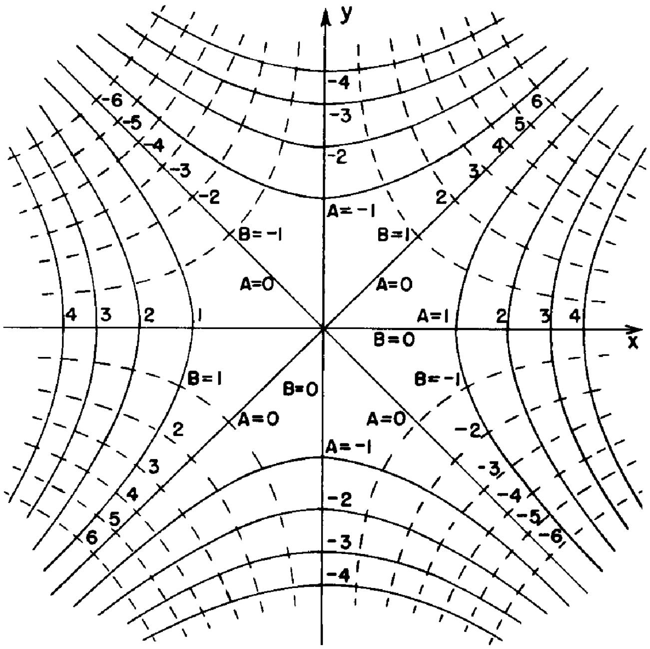


Fig. 7–1. Two sets of orthogonal curves which can represent equipotentials in a two-dimensional electrostatic field.

As an example, let's see what physics the function $F(\mathfrak{z}) = \mathfrak{z}^2$ gives us. From t we get the two potential functions of (7.5) and (7.6). To see what problem the unction U belongs to, we solve for the equipotential surfaces by setting $U = A$, constant:

$$x^2 - y^2 = A.$$

his is the equation of a rectangular hyperbola. For various values of A, we get ne hyperbolas shown in Fig. 7–1. When $A = 0$, we get the special case of diagonal raight lines through the origin.

Such a set of equipotentials corresponds to several possible physical situations. irst, it represents the fine details of the field near the point halfway between two

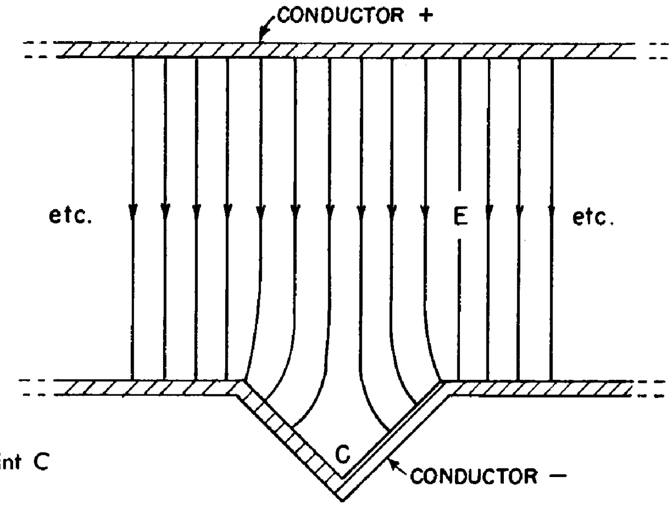


Fig. 7–2. The field near the point C is the same as that in Fig. 7–1.

equal point charges. Second, it represents the field at an inside right-angle corner of a conductor. If we have two electrodes shaped like those in Fig. 7–2, which are held at different potentials, the field near the corner marked C will look just like the field above the origin in Fig. 7–1. The solid lines are the equipotentials, and the broken lines at right angles correspond to lines of $\boldsymbol{E}$. Whereas at points or protuberances the electric field tends to be high, it tends to be *low* in dents or hollows.

The solution we have found also corresponds to that for a hyperbola-shaped electrode near a right-angle corner, or for two hyperbolas at suitable potentials. You will notice that the field of Fig. 7–1 has an interesting property. The x-component of the electric field, E_x, is given by

$$E_x = -\frac{\partial \phi}{\partial x} = -2x.$$

The electric field is proportional to the distance from the axis. This fact is used to make devices (called quadrupole lenses) that are useful for focusing particle beams (see Section 29–9). The desired field is usually obtained by using four hyperbola-shaped electrodes, as shown in Fig. 7–3. For the electric field lines in Fig. 7–3, we have simply copied from Fig. 7–1 the set of broken-line curves that represent V = constant. We have a bonus! The curves for V = constant are orthogonal to the ones for U = constant because of the equations (7.7) and (7.8). Whenever we choose a function $F(\mathfrak{z})$, we get from U and V both the equipotentials and field lines. And you will remember that we have solved either of two problems, depending on which set of curves we call the equipotentials.

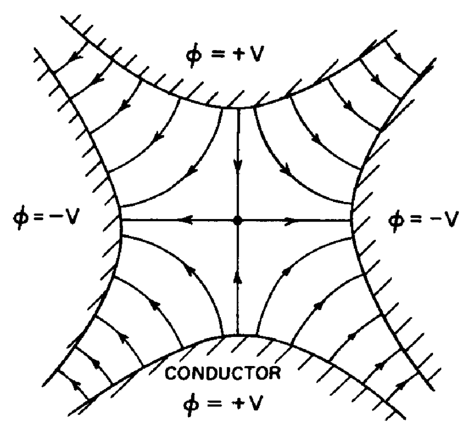


Fig. 7–3. The field in a quadrupole lens.

As a second example, consider the function

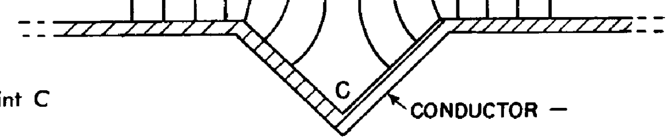
$$F(\mathfrak{z}) = \sqrt{\mathfrak{z}}. \tag{7.11}$$

If we write

$$\mathfrak{z} = x + iy = \rho e^{i\theta},$$

where

$$\rho = \sqrt{x^2 + y^2}$$



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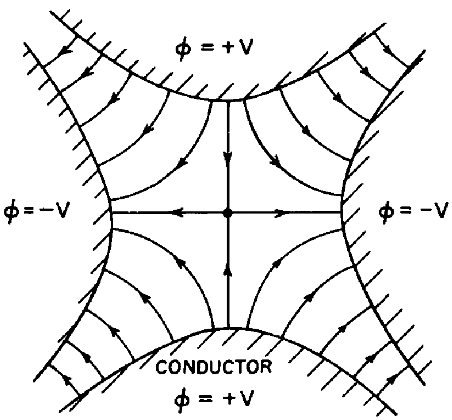

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$$\rho = \sqrt{x^2 + y^2}$$

and

$$\tan\theta = y/x,$$

then

$$F(\mathfrak{z}) = \rho^{1/2} e^{i\theta/2}$$

$$= \rho^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$

from which

$$F(\mathfrak{z}) = \left[\frac{(x^2 + y^2)^{1/2} + x}{2}\right]^{1/2} + i\left[\frac{(x^2 + y^2)^{1/2} - x}{2}\right]^{1/2}. \tag{7.12}$$

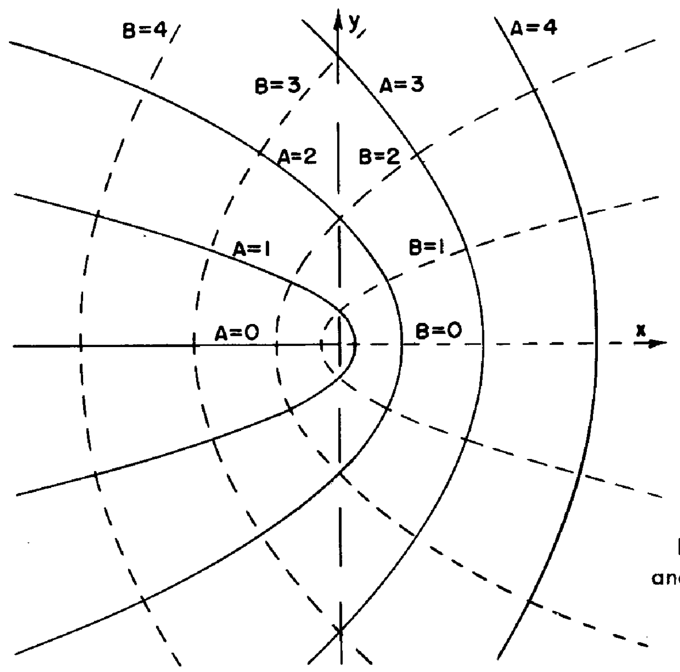


Fig. 7–4. Curves of constant $U(x, y)$ and $V(x, y)$ from Eq. (7.12).

The curves for $U(x, y) = A$ and $V(x, y) = B$, using U and V from Eq. (7.12), are plotted in Fig. 7–4. Again, there are many possible situations that could be described by these fields. One of the most interesting is the field near the edge of a thin plate. If the line $B = 0$—to the right of the y-axis—represents a thin charged plate, the field lines near it are given by the curves for various values of A. The physical situation is shown in Fig. 7–5.

Further examples are

$$F(\mathfrak{z}) = z^{3/2}, \tag{7.13}$$

which yields the field *outside* a rectangular corner

$$F(\mathfrak{z}) = \log \mathfrak{z}, \tag{7.14}$$

which yields the field for a line charge, and

$$F(\mathfrak{z}) = 1/\mathfrak{z}, \tag{7.15}$$

which gives the field for the two-dimensional analog of an electric dipole, i.e., two parallel line charges with opposite polarities, very close together.

We will not pursue this subject further in this course, but should emphasize that although the complex variable technique is often powerful, it is limited to two-dimensional problems; and also, it is an indirect method.

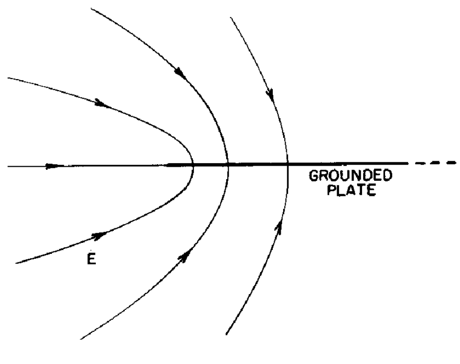


Fig. 7–5. The electric field near the edge of a thin grounded plate.

7–3 Plasma oscillations

We consider now some physical situations in which the field is determined neither by fixed charges nor by charges on conducting surfaces, but by a combination of two physical phenomena. In other words, the field will be governed simultaneously by two sets of equations: (1) the equations from electrostatics relating electric fields to charge distribution, and (2) an equation from another part of physics that determines the positions or motions of the charges in the presence of the field.

The first example that we will discuss is a dynamic one in which the motion of the charges is governed by Newton's laws. A simple example of such a situation occurs in a plasma, which is an ionized gas consisting of ions and free electrons distributed over a region in space. The ionosphere—an upper layer of the atmosphere—is an example of such a plasma. The ultraviolet rays from the sun knock energy to charge Q and capacitance C. Suppose that C depends in some manner on a linear coordinate x which measures the displacement of one "plate" of a capacitor, which might be a conductor of any shape, with respect to the other. Let F be the magnitude of the force that must be applied to each plate to overcome their attraction and keep x constant. Now imagine the distance x is increased by an increment Δx with Q remaining constant and one plate fixed. The external force F on the other plate does work $F\,\Delta x$ and, if energy is to be conserved, this must appear as an increase in the stored energy $Q^2/2C$. That increase at constant Q is

$$\Delta U = \frac{dU}{dx}\Delta x = \frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right)\Delta x \tag{26}$$

Equating this to the work $F\,\Delta x$ we find

$$F = \frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right) \tag{27}$$

OTHER VIEWS OF THE BOUNDARY-VALUE PROBLEM

3.8 It would be wrong to leave the impression that there are no general methods for dealing with the Laplacian boundary-value problem. Although we cannot pursue this question much further, we shall mention some useful and interesting approaches which you are likely to meet in future study of physics or applied mathematics.

First, an elegant method of analysis, called conformal mapping, is based on the theory of functions of a complex variable. Unfortunately it applies only to two-dimensional systems. These are systems in which φ depends only on x and y, for example, all conducting boundaries being cylinders (in the general sense) with elements running parallel to z. Laplace's equation then reduces to

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{28}$$

with boundary values specified on some lines or curves in the xy plane. Many systems of practical interest are like this or sufficiently like this to make the method useful, quite apart from its intrinsic mathematical interest. For instance, the exact solution for the potential around two long parallel strips is easily obtained by the method of conformal mapping. The field lines and equipotentials are shown in a cross-section plane in Fig. 3.17. This provides us with the edge field for any parallel-plate capacitor in which the edge is long compared with the gap. The field shown in Fig. 3.12*b* was copied from such a solution. You will be able to apply this method after you have studied in more advanced mathematics functions of a complex variable.

Second, we mention a numerical method for finding approxi-

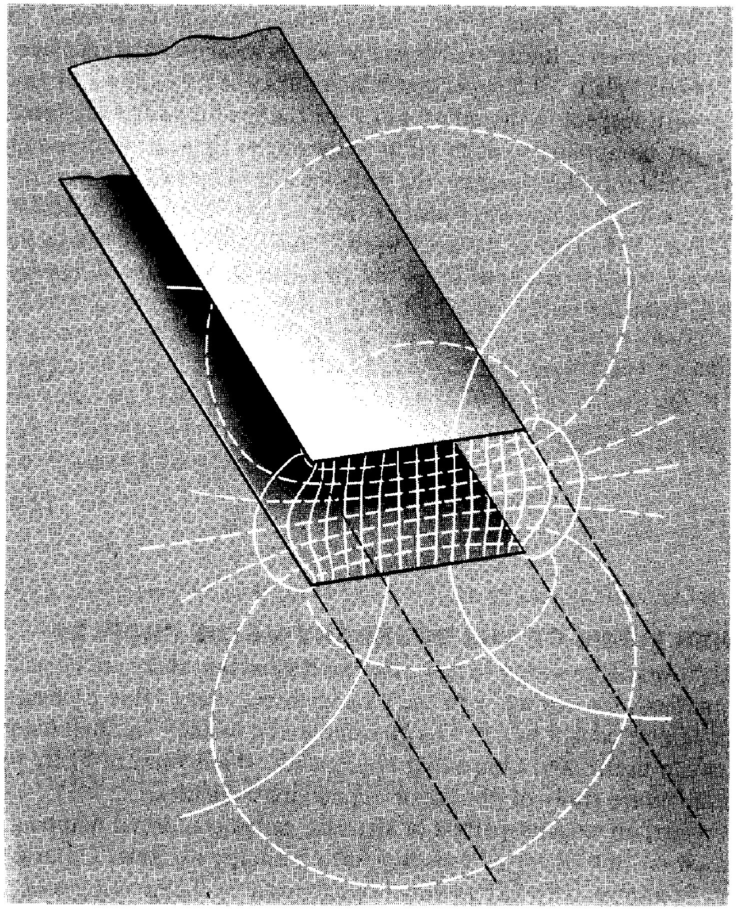


FIGURE 3.17
Field lines and equipotentials for two infinitely long conducting strips.

mate solutions of the electrostatic potential with given boundary values. Surprisingly simple and almost universally applicable, this method is based on that special property of harmonic functions with which we are already familiar: The value of the function at a point is equal to its average over the neighborhood of the point. In this method the potential function φ is represented by values at an array of discrete points only, including discrete points on the boundaries. The values at nonboundary points are then adjusted until each value is equal to the average of the neighboring values. In principle one could do this by solving a large number of simultaneous linear equations—as many as there are interior points. But an approximate solution can be obtained by the following procedure, called a *relaxation method.* Start with the boundary points of the array, or grid, set at the values prescribed. Assign starting values arbitrarily to the interior points. Now visit, in some order, all the interior points. At each point reset its value to the average of the values at the four (for a square grid) adjacent grid points. Repeat again and again, until all the changes made in the course of one sweep over the network of interior points are acceptably small. If you want to see how this method works, Problems 3.30 and 3.31 will provide an introduction. Whether convergence of the relaxation process can be ensured, or even hastened, and whether a relaxation method or direct solution of the simultaneous equations is the better strategy for a given problem are questions in applied mathematics that we cannot go into here. It is the high-speed computer, of course, that makes both methods feasible.

PROBLEMS

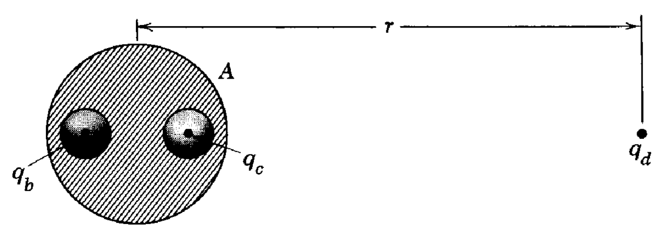
3.1 A spherical conductor A contains two spherical cavities. The total charge on the conductor itself is zero. However, there is a point charge q_b at the center of one cavity and q_c at the center of the other. A considerable distance r away is another charge q_d. What force acts on each of the four objects, A, q_b, q_c, q_d? Which answers, if any, are only approximate, and depend on r being relatively large?

3.2 What is wrong with the idea of a gravity screen, something that will "block" gravity the way a metal sheet seems to "block" the electric field. Think about the difference between the gravitational source and electrical sources. Note that the walls of the box in Fig. 3.6 do not block the field of the outside sources but merely allow the surface charges to set up a compensating field. Why can't something of this sort be contrived for gravity? What would you need to accomplish it?

✓**3.3** In the field of the point charge over the plane (Fig. 3.9), if you follow a field line that starts out from the point charge in a horizontal direction, that is, parallel to the plane, where does it meet the surface of the conductor? (You'll need Gauss's law and a simple integration.)

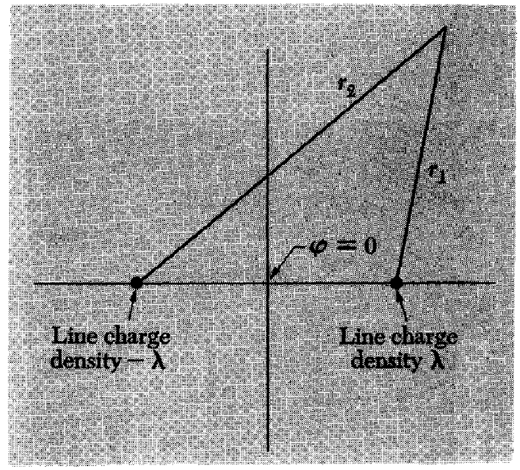
3.4 A positive point charge Q is fixed 10 cm above a horizontal conducting plane. An equal negative charge $-Q$ is to be located somewhere along the perpendicular dropped from Q to the plane. Where can $-Q$ be placed so that the total force on it will be zero?

Ans. $y = 3.06$ cm.



PROBLEM 3.1

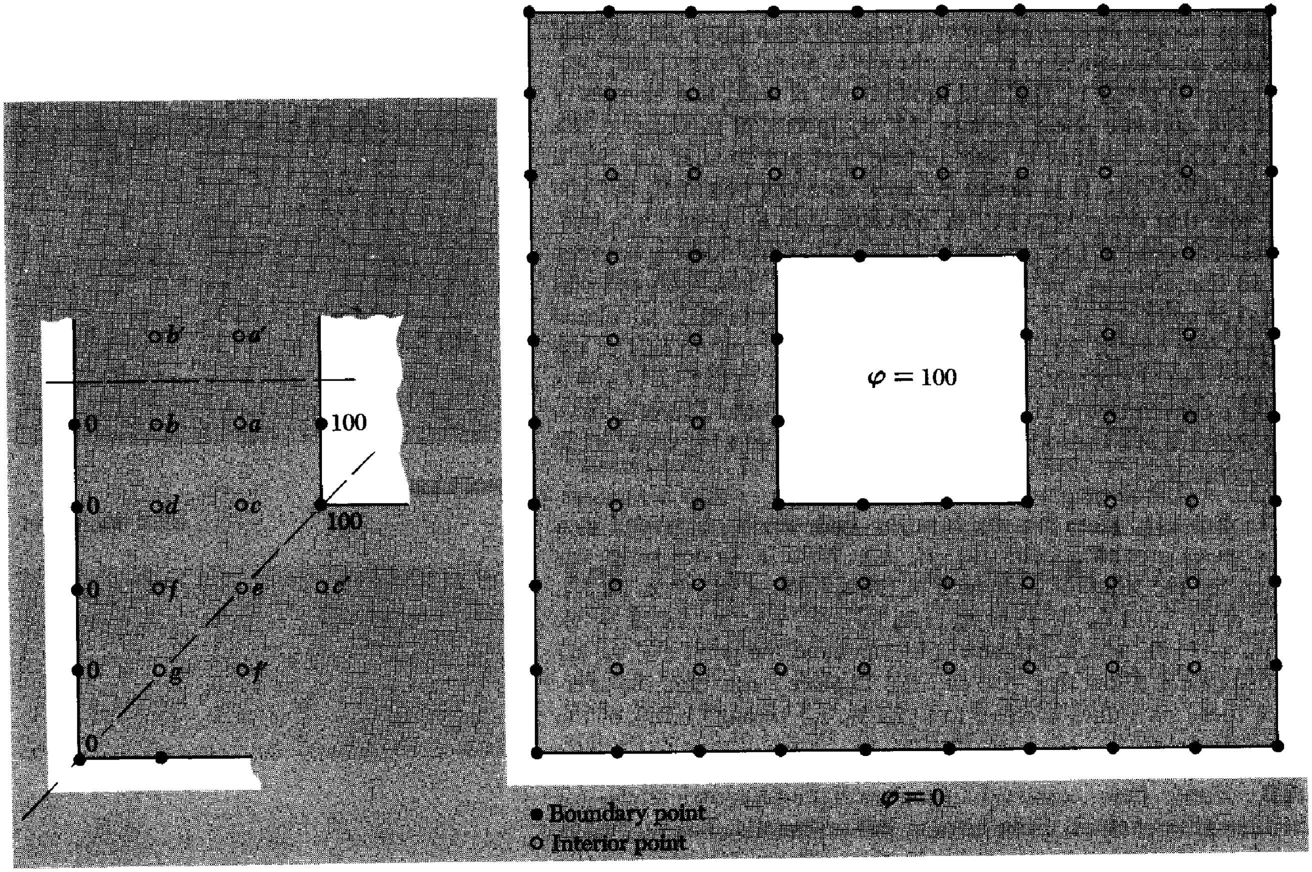
infinite length and at different potentials. These two-dimensional problems happen to be much more tractable than three-dimensional problems, mathematically. In fact, the key to all problems of the "two-pipe" class is given by the field around two parallel line charges of equal and opposite linear density. All equipotential surfaces in this field are circular cylinders! And all field lines are circular too. See if you can prove this. It is easiest to work with the potential, but you must note that one cannot set the potential zero at infinity in a two-dimensional system. Let zero potential be at the line midway between the two line charges, that is, at the origin in the cross-sectional diagram. The potential at any point is the sum of the potentials calculated for each line charge separately. This should lead you quickly to the discovery that the potential is simply proportional to ln (r_2/r_1) and is therefore constant on a curve traced by a point whose distances from two points are in a constant ratio. Make a sketch showing some of the equipotentials.



PROBLEM 3.28

3.29 Let $\varphi(x, y, z)$ be any function that can be expanded in a power series around a point (x_0, y_0, z_0). Write a Taylor series expansion for the value of φ at each of the six points $(x_0 + \delta, y_0, z_0)$, $(x_0 - \delta, y_0, z_0)$, $(x_0, y_0 + \delta, z_0)$, $(x_0, y_0 - \delta, z_0)$, $(x_0, y_0, z_0 + \delta)$, $(x_0, y_0, z_0 - \delta)$, which symmetrically surround the point (x_0, y_0, z_0) at a distance δ. Show that, if φ satisfies Laplace's equation, the average of these six values is equal to $\varphi(x_0, y_0, z_0)$ through terms of the third order in δ.

✓ **3.30** Here's how to solve Laplace's equation approximately, for given boundary values, using nothing but arithmetic. The method is the relaxation method mentioned in Section 3.8, and it is based on the result of Problem 3.29. For simplicity we take a two-dimensional example. In the figure there are two square equipotential boundaries, one inside the other. This might be a cross section through a capacitor made of two sizes of square metal tubing. The problem is to find, for an array of discrete points, numbers which will be a good approximation to the values at those points of the exact two-dimensional potential function $\phi(x, y)$. For this exercise, we'll make the array rather coarse, to keep the labor within bounds. Let us assign, arbitrarily, potential 100 to the inner boundary and zero to the outer. All points on these boundaries retain those values. You could start with any values at the interior points, but time will be saved by a little judicious guesswork. We know the correct values must lie between 0 and 100, and we expect that points closer to the inner boundary will have higher values than those closer to the outer boundary. Some reasonable starting values are suggested in the figure. Obviously, you should take advantage of the symmetry of the configuration: Only seven different interior values need to be computed. Now you simply go over these seven interior lattice points in some systematic manner,



PROBLEM 3.30
Replace value at an interior point by ¼ × sum of its four neighbors: $c \rightarrow \frac{1}{4}(100 + a + d + e)$; keep $a' = a$, $b' = b$, $c' = c$, and $f' = f$. Suggested starting values:

$a = 50$	$e = 50$
$b = 25$	$f = 25$
$c = 50$	$g = 25$
$d = 25$	

replacing the value at each interior point by the average of its four neighbors. Repeat until all changes resulting from a sweep over the array are acceptably small. For this exercise, let us agree that it will be time to quit when no change larger in absolute magnitude than one unit occurs in the course of the sweep. The relaxation of the values toward an eventually unchanging distribution is closely related to the physical phenomenon of *diffusion*. If you start with much too high a value at one point, it will "spread" to its nearest neighbors, then to its next nearest neighbors, and so on, until the bump is smoothed out. Enter your final values on the array, and sketch the approximate course two equipotentials, for $\phi = 25$ and $\phi = 50$, would have in the actual continuous $\phi(x, y)$.

3.31 The relaxation method is clearly well adapted to machine computation. Write a program that will deal with the concentric square boundary problem on a finer mesh—say, a grid with four times as many points and half the spacing. It might be a good idea to utilize a coarse-mesh solution in assigning starting values for the relaxation on the finer mesh.

3.32 A capacitor consists of two concentric spherical shells. Call the inner shell, of radius a, conductor 1, and the outer shell, of radius b, conductor 2. For this two-conductor system, find C_{11}, C_{22}, and C_{12}.

Ans. $C_{11} = ab/(b - a)$; $C_{22} = b^2/(b - a)$; $C_{21} = -ab/(b - a)$.