Name ………………………….……….   Course ……………………………………….
Print and use these sheets with ***MathinSite***'s 'Second Order Systems' applet to have a permanent record of your work.

Second Order Linear Systems
The Mathematics

Learning Outcomes

After using the applet and the theory and tutorial worksheets you should

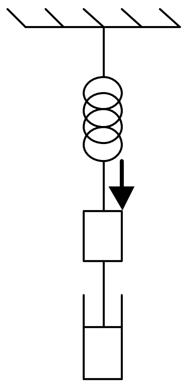
- be aware of the second-order differential equation for the Mass-Spring-Damper system.
- be aware of the second-order differential equation for the LCR series circuit.
- be able to relate second-order differential equations to second-order linear systems.
- understand how a second-order system behaves as the system poles are moved in the *s*-plane (Argand Diagram).

Introduction

Second order linear systems occur in various branches of Engineering and Science; two examples are given here and are both governed by *second-order* differential equations.

(1) The Forced Mass-Spring-Damper System

To the right is a schematic diagram of a spring suspended from a fixed point. A mass is attached to the lower end of the spring and, in turn, a piston is fixed below this mass. Finally, the piston is enclosed in a dashpot – this is the damper. An external force is applied to the mass (the bold arrow), which may be
a constant force, or one that vibrates, or exponentially decays or grows, etc.



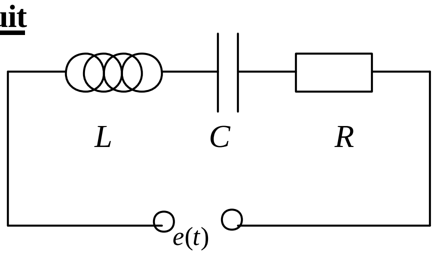
The damping may be due solely to the dashpot arrangement as shown, but may also be due to internal damping such as friction within the spring, or external damping such as aerodynamic drag. This *schematic* diagram can be used to represent, by the dashpot, whatever the cause of damping.

The displacement, y, at any time, t, of the mass is the ***output*** from the system and can be modelled by the second order differential equation (a proof of which is not given here, but can be found in the theory sheet for the "Mass-Spring-Damper System" applet):

$$M\frac{d^2y}{dt^2} + R\frac{dy}{dt} + ky = f(t) \quad \text{where}$$

- M is the mass
- R is the damping factor. The damping force here is assumed to be proportional to the velocity of the mass - a reasonable approximation. R is the constant of proportionality.
- k is the spring stiffness (it is assumed the spring is not extended beyond its elastic limit, so Hooke's Law applies)
- $f(t)$ is the time-dependent, or constant, external force driving the system – it is the ***input*** to the system (e.g. $f(t) = 20$, $f(t) = 10\sin 2t$, $f(t) = 60e^{-3t}$)

The LCR Series Circuit



Applying, amongst other things, Kirchhoff's 2nd Law and Ohm's Law to this circuit leads to

$$\boxed{L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=f(t)}$$

Again $f(t)$ is the ***input*** to the system (here, it is actually the *rate of change* of applied voltage, $e(t)$) and i is the current (***output***) in the circuit at time t. L, R and C (inductance, resistance and capacitance) are the parameters of the system.

Note the similarity between the differential equations of the two quite different systems given above. This similarity is used in analogue systems in which a mechanical system can be simulated by the equivalent electrical system.

These differential equations can be rewritten generally as

$$\boxed{a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t)}$$

When Laplace Transforms are used to solve this equation with zero initial conditions, the transformed compound algebraic fraction obtained (in terms of Laplace Transform's s) is given, in the usual notation, by

$$L\{y\}=\bar{y}=\frac{L\{f(t)\}}{as^2+bs+c}$$

In order not to overcomplicate the output displayed on the screen, the input $f(t)$ used in the accompanying software is $f(t) = P$ where P is a constant. So the system here is 'driven' by a constant forcing term.
The value of P can be changed from 0 to 200 by moving the slider on the right of the screen.
The Laplace Transform of a constant value P is, in the usual notation, P/s,
giving the full Laplace Transform of the system plus constant forcing term as

$$L\{y\}=\bar{y}=\frac{P}{s(as^2+bs+c)}$$

The as^2+bs+c part of the denominator of $L\{y\}$ indicates that this is a second-order system, which has the ***characteristic equation***,

$$as^2+bs+c=0$$

The solution of this equation determines the position of the system's two (but sometimes repeated) poles in the s-plane (Argand Diagram). Note that there are *three* different types of solution to this quadratic (i.e. second-order) equation depending on whether $b^2 - 4ac$ is less than zero, equal to zero, or greater than zero. Consider each of these cases separately:

(1) $b^2 - 4ac < 0$

This is the only one of the three cases where the poles will be complex and, in particular, complex conjugates. This is because the solutions for the case $b^2 - 4ac < 0$ are

$$s = -\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}j \qquad \text{and} \qquad s = -\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}j$$

or, for compatibility with the accompanying applet, the system poles are placed at:

$$s = k + jw \qquad \text{and} \qquad s = k - jw$$

(Note: not the same k that was used in the differential equation for the mass-spring-damper system)

The system output relating to complex conjugate poles of this type is of the form

$$e^{kt}\left(A\sin wt + B\cos wt\right)$$

Incorporating the constant applied input results in the full output:

$$y = e^{kt}\left(A\sin wt + B\cos wt\right) + C \qquad \text{where } A, B \,\&\, C \text{ are constants}$$

An important point to note here is that if k is negative, then the system response involves exponential decay only. In this case the system is said to be stable since, if the system is given a knock or displacement and has no forcing term, it will settle back to its equilibrium position (courtesy of the exponential *decay* multiplying the sinusoids). However, if k is positive then the system responds with exponential growth and is said to be an unstable system.

In general then, for $k < 0$ (with its corresponding pole in the left half-plane of the Argand Diagram) this system is stable. For $k > 0$ (with its corresponding pole in the right half-plane of the Argand Diagram) this system is unstable.

(2) $b^2 - 4ac > 0$

In this case the characteristic equation will have solutions:

$$s = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} \qquad \text{and} \qquad s = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$

Here both solutions are wholly real, no imaginary j term, so both poles will lie on the k axis only. Since w will be zero, there will be no oscillations. Also, the two s values are different; so the poles will be at different positions on the k axis, say k_1 and k_2.

The system output relating to two real, distinct poles, k_1 and k_2, is of the form

$$Ae^{k_1 t} + Be^{k_2 t}$$

Incorporating the constant applied input results in the full output:

$$y = Ae^{k_1 t} + Be^{k_2 t} + C \qquad \text{where } A, B \,\&\, C \text{ are constants}$$

(*both of these results are left as an exercise for the reader*)

Note that in this case, ***both*** k_1 ***and*** k_2 would have to be in the left half-plane (i.e. $k_1 < 0$ and $k_2 < 0$) for the system to be a stable system. ***Either*** pole in the right half-plane would result in a term with exponential growth, hence an unstable system.

(3) $b^2 - 4ac = 0$

In this case the characteristic equation will have solutions:

$$s = -\frac{b}{2a} \qquad \text{and} \qquad s = -\frac{b}{2a}$$

i.e. repeated solutions. Again both poles are wholly real, no imaginary j term, so both will lie on the k axis. Again, $w = 0$, so no oscillations. Now, however, the two s values are the same; so the poles will be *at the same position* on the k axis, say k_1.

The system output relating to two real, equal poles, k_1 and k_1, is of the form

$$(A+Bt)e^{k_1 t}$$

Incorporating the constant applied input results in the full output:

$$y=(A+Bt)e^{k_1 t}+C \quad \text{where } A,\ B\ \&\ C \text{ are constants}$$

Note that in this case, k_1 "and k_1" would have to be in the left half-plane (i.e. $k_1 < 0$) for the system to be a stable system.

<u>The effect of changing *k*</u>

The value of *k*, *measured on the* ***real*** *(horizontal) axis* relates to the amount of exponential decay (for k <0) or exponential growth (for k >0). k is called the instantaneous fractional growth rate. Different values of k represent different amounts of growth (or decay):

Value of *k*	**Type of Response**
$k > 0$ (large positive value)	'fast' exponential growth
$k > 0$ (small positive value; close to zero)	'slow' exponential growth
$k = 0$	neither growth nor decay
$k < 0$ (small negative value; close to zero)	'slow' exponential decay
$k < 0$ (large negative value)	'fast' exponential decay

At this point try "<u>Changing *k* only</u>" from the accompanying tutorial worksheet together with the applet

Solutions containing exponential <u>decay</u> terms only **(*stable systems*)** – as found when using the software as suggested just above - consist of two parts, namely:

- the ***transient*** (which is due to the system itself and represented in the solution by the exponential decay terms) and
- the ***steady state*** (which depends on the type of input, here the system has an applied constant force).

For constant input, as used in the applet, the steady state is necessarily a constant output. The transient is the way in which the system responds during the time it takes to reach its steady state (if it has one! – a system acted upon with an increasing force will never settle down to a steady state value). Transient means "short lived". But how short is "short lived"? This can be determined from the following table:

t	$e^{-\frac{t}{t}} \times 100\%$ (as a percentage)
0	e^0 x 100 =100
t	e^{-1} x 100 = 36.7879
$2t$	e^{-2} x 100 = 13.5335
$3t$	e^{-3} x 100 = 4.9787
$4t$	e^{-4} x 100 = 1.8316
$5t$	e^{-5} x 100 = 0.6738

$5t$ is an important value!

The right hand column shows that the value of $e^{-\frac{t}{t}}$ varies from 100% at t = 0 to about 0.7% by $t = 5t$.

t (Greek letter, "tau") is called the "***time constant***".

The implication is that by $5t$, the contribution of $e^{-\frac{t}{t}}$ has died away to 'practically nothing'. For our system, the exponential terms are of the form e^{-kt}, so comparing with $e^{-\frac{t}{t}}$, gives the important result, $\boxed{t = \frac{1}{k}}$

<u>Note that here, the time constant, t, is only appropriate for exponential decay, not growth.</u>

At this point try "<u>The effect of changing *k* on the time constant, t</u>" from the accompanying tutorial worksheet together with the applet

The effect of changing w

The value of w, *measured on the **imaginary** (vertical) axis* relates to the ***angular velocity*** of any oscillations that may occur in the system. If both poles lie on the horizontal axis then $w = 0$ and there will be no oscillatory motion. For oscillations to occur in the system, the system poles will appear off the horizontal axis (as a complex conjugate pair).

When working with angular velocity (measured in *radians* per second), two important formulae are

$$\boxed{w = 2pf} \text{ and } \boxed{T = \frac{1}{f} = \frac{2p}{w}}$$

where f is the ***frequency*** of the oscillations (measured in cycles per second or Hertz) and T (in seconds) is the ***periodic time*** (time for one cycle) of the oscillations. Ordinary alternating mains electricity in the UK operates at a frequency of 50 Hz. Its angular velocity is therefore $100p$ radians per sec and its periodic time is 1 / 50 sec or 20 ms.

In systems with exponentially decaying sinusoidal terms, knowing T, $5t$ and that the input into the system is a constant, it is possible to sketch a reasonable graph of the system response. But how many oscillations do you sketch before the exponential decay kills off the transient response? Consider the system with poles at $-1 \pm 5j$. This has values k = - 1 and w = 5. The k = - 1 introduces a term involving e^{-1t} into the solution, giving t = 1/1 = 1 (seconds) and hence $5t$ = 5 (seconds) and w = 5 results in sinusoidal terms with periodic time, $T = \frac{2p}{5} \approx 1.26$ (seconds). Since the exponential damps the response "completely" by 5 seconds, then the system will perform approximately 5/1.26 $\approx$ 4 oscillations before reaching steady state. Use the applet, placing the poles at $-1 \pm 5j$, to verify this is the case. Note that actually seeing the fourth oscillation is rather difficult since, by then, the damping has all but done its job!

Now try "<u>The effect of changing w only</u>" from the accompanying tutorial worksheet together with the applet

Note that if the complex conjugate poles lie on the vertical axis only, then k = 0. With $w \neq 0$, then the resulting system output will be oscillatory with neither exponential growth nor decay. Oscillations that neither grow nor decay are pure oscillations with constant amplitude. In this case the system is said to execute Simple Harmonic Motion (SHM).
This you can easily see from the applet (although you may have to ramp up the value of P to its maximum value to see this properly).

For those of you using this sheet in isolation from the tutorial worksheet and/or the applet, the following "Summary" has been extracted from the accompanying worksheet in order to complete this theory sheet – but note the extra "Postscript" below.

Summary

In a second-order system,

- poles always appear either (a) as complex conjugate pairs or (b) both on the real axis
- if *any* pole lies to the right of the vertical axis (i.e. in the right half-plane), then the system will necessarily contain at least one exponential growth term and is said to be an *unstable* system.
- if *both* (or *all*, in the case of higher order systems) poles lie in the left half-plane, then the system will necessarily contain exponential decay terms only (with no exponential growth terms) and is said to be a *stable* system.
- if the poles appear as a complex conjugate pair on the vertical axis (but not both at the origin), there will be neither exponential growth nor decay and the system responds with only pure oscillations (simple harmonic motion, SHM). This system is said to be *critically stable*.
- if both poles are at the origin, there are neither exponential nor sinusoidal terms; the analytical solution (hence output) for such a system is a pure quadratic increase ($P > 0$).
- poles vertically further away from the horizontal axis will relate to higher frequency oscillations than poles closer to the horizontal axis. Poles *on* the horizontal axis have zero frequency i.e. do not contribute any oscillatory effect to the system output.
- moving the poles horizontally away from the vertical axis results in more rapid exponential growth (or decay, depending whether the pole is in the right or left half-plane, respectively). Conversely moving the poles horizontally towards the vertical axis relates to slower exponential growth (or decay). Poles on the vertical axis relate to neither growth nor decay.

Postscript

Good systems' designers don't want to produce systems that are unstable so they must always ensure that system poles only ever occur in the left half-plane. Fortunately, in the case of an LRC series circuit or a mass-spring-damper system the parameters L, R, C and M, R and k will all be positive. This ensures that the $-b/2a$ term in the solution of the characteristic, quadratic equation is always negative, so resulting in exponential decay *always*. In fact, any "real-world" second order system will have positive parameters so in such a case then, it is impossible to build an unstable system. For example, think about giving the mass-spring-damper a kick (but keeping the spring within its elastic limit). Are any resulting oscillations likely to increase in amplitude indefinitely? Hardly! So when are you likely to encounter unstable second-order-systems? Well, if you take a second order system and use a "badly designed" feedback loop …

But this has to be for another applet / work sheet / theory sheet combination!