Name ……………………………………… Course ………………………………

Print and use this sheet in conjunction with ***MathinSite***'s 'LRC Series Circuit' applet and tutorial sheets.

The LRC Series Circuit

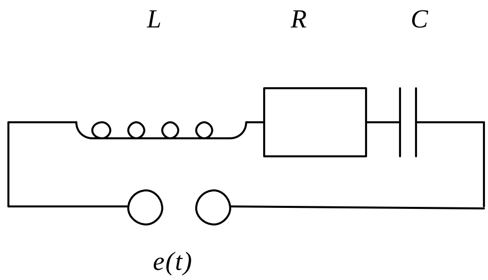
Theory Sheet 2
The Three Types of Complementary Function

Learning Outcomes, Prerequisites & Background

These are all outlined in Theory Sheet 1

The applet accompanying these work sheets can be found on the ***MathinSite*** web site at http://mathinsite.bmth.ac.uk/html/applets.html.

The LRC series circuit



The governing differential equation for this circuit in terms of current, *i*, is

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{de(t)}{dt}$$

Finding the Complementary Function (CF) of the Differential Equation

Investigation of the CF alone is possible whether using the Assumed Solution method or the Laplace Transform method (both of which were outlined in Theory Sheet 1).

A more comprehensive explanation of these methods can be found in a variety of textbooks.
One such example is "Engineering Mathematics" by Stroud, K.A. and Booth, D.J.
(Fifth edition, published by Palgrave, UK, 2001) See pages 1071 – 1117.

Method 1 The Assumed Solution method

The general differential equation above, with an assumed solution of $i = Ae^{mt}$, gives rise to the ***auxiliary equation***

$$Lm^2 + Rm + \frac{1}{C} = 0$$

which is a quadratic equation in m.

Any quadratic of the form $am^2 + bm + c = 0$ has

(a) 2 real, distinct roots, m_1 and m_2, (m_1, m_2 negative)
(b) 2 real, equal roots, m (and m) (m negative)
(c) 2 complex roots, $m_1 = a + bj$, $m_2 = a - bj$ (a negative)

depending on the sign of the *discriminant* $b^2 - 4ac$.

m_1, m_2, m and a will always be negative since the coefficients of the auxiliary equation (involving L, R and C) are always positive.

For the auxiliary equation, then, this gives rise to the following types of CFs:

	Discriminant	***Roots***	***CF***
(a)	$R^2 - \frac{4L}{C} > 0$	m_1 and m_2	$i = Ae^{m_1 t} + Be^{m_2 t}$
(b)	$R^2 - \frac{4L}{C} = 0$	m (and m)	$i = (At + B)e^{mt}$
(c)	$R^2 - \frac{4L}{C} < 0$	$m_1 = a + bj,\ m_2 = a - bj$	$i = e^{at}\left(A\cos bt + B\sin bt\right)$

Method 2 The Laplace Transform method

During the Laplace Transform method of solution, the following compound fraction in s will occur:

$$\bar{i} = \frac{\ldots}{\left(Ls^2 + Rs + \frac{1}{C}\right)(\ldots)}$$

where the "…" represent expressions in s.

This time, $Ls^2 + Rs + \frac{1}{C} = 0$ is called the ***characteristic equation*** (rather than the *auxiliary equation* in m obtained using the Assumed Solution method).

The characteristic expression, $Lm^2 + Rm + \frac{1}{C}$ will either factorise into

(a) 2 real, distinct linear factors (if the discriminant, $R^2 - \frac{4L}{C} > 0$),

(b) 2 real, equal linear factors (if $R^2 - \frac{4L}{C} = 0$), or

(c) will require the "completing the square" process (if $R^2 - \frac{4L}{C} < 0$).

In which case the following types of partial fractions and their inverse Laplace Transforms will occur (again with m_1, m_2, m and a negative):

	Discriminant	***Type of Partial Fractions***	***Inverse Laplace (CF only)***
(a)	$R^2 - \frac{4L}{C} > 0$	$\bar{i} = \frac{A}{s - m_1} + \frac{B}{s - m_2} + \ldots$	$i = Ae^{m_1 t} + Be^{m_2 t} + \ldots$
(b)	$R^2 - \frac{4L}{C} = 0$	$\bar{i} = \frac{A}{(s - m)^2} + \frac{B}{s - m} + \ldots$	$i = (At + B)e^{mt} + \ldots$
(c)	$R^2 - \frac{4L}{C} < 0$	$\bar{i} = \frac{A(s - a) + Bb}{(s - a)^2 + b^2} + \ldots$	$i = e^{at}\left(A\cos bt + B\sin bt\right) + \ldots$

Interpretation of the CF

Note that the CF is dependent only on the system itself (i.e. it is found using values of L, R and C only and not on the applied voltage).

Since m_1, m_2, m and a are always negative, each of three types of CF involves exponential *decay* and so the CF is a ***transient*** part of the overall solution, that is, it is short-lived. Short-lived is a relative term, so how short-lived is short-lived? This can be found from the ***time constant***, τ , for each exponential.

Note on time constants: Any exponential ***decay*** term of the form e^{-kt} ($k > 0$) [or e^{mt} ($m < 0$)] has a time constant defined by $\tau = 1/k$ [or $\tau = 1/m$]. τ is a useful quantity since $5\times\tau$ gives a measure of approximately how long the exponential term takes to decay away to zero. In this theory sheet and in ***MathinSite***'s LRC Series Circuit applet, $5\times\tau$ is used, although some other authors use $6\times\tau$.

Further information on time constants can be found in the theory and tutorial sheets of ***MathinSite***'s 'Exponential Function' applet.

So the current settles down to a ***steady state*** after a time $5\times\tau$ approximately.

In the case of $i = Ae^{m_1 t} + Be^{m_2 t}$, the time for the transient to decay is the longer of $5\times\frac{1}{m_1}$ and $5\times\frac{1}{m_2}$.

The solution $i = Ae^{m_1 t} + Be^{m_2 t} + \ldots$ occurs when $R^2 > \dfrac{4L}{C}$ and can be achieved using a combination of large resistance, small inductance and large capacitance. The transient exponential decay is likely to be 'slow'. This type of response, with large resistance, is called '***heavy damping***'.

The solution $i = e^{at}\left(A\cos bt + B\sin bt\right) + \ldots$ occurs when $R^2 < \dfrac{4L}{C}$ and can be achieved using a combination of small resistance, large inductance and small capacitance. The transient will exhibit exponential decay with oscillations (courtesy of the sin/cos terms). With comparatively small resistance, this type of response is called '***light damping***'.

The solution $i = (At + B)e^{mt} + \ldots$ occurs when $R^2 = \dfrac{4L}{C}$. The transient exponential decay here will be the fastest it possibly can be without resulting in oscillations (there are no sin/cos terms). This type of response, the critical boundary between heavy damping and light damping, is called '***critical damping***'.

The LRC Series Circuit

Theory Sheet 1

Learning Outcomes

After using the ***MathinSite*** LRC Series Circuit applet and its accompanying tutorial and theory sheets you should

- be aware of the composition of an LRC Series circuit
- be able to model the LRC Series Circuit mathematically
- be aware of solution methods for the LRC circuit's associated differential equation
- be aware of the different types of solution of the LRC circuit differential equation
- have developed, through experimentation with the applet, an understanding of how the current in an LRC series circuit responds to changes in system parameters and applied voltages
- be able to answer "what if ...?" questions about the LRC Series Circuit.

Prerequisites

Before using the applet and the accompanying theory and tutorial sheets, familiarity with the following mathematics would be useful (but not vital).

- The Straight Line
- The Exponential Function – including the notion of 'time constants'
- Trigonometrical Functions (in particular sines and cosines)
- Differentiation and Integration, and
- The solution of linear second-order differential equations

However, ***even without this knowledge***, just understanding how the circuit responds can help in your appreciation of the mathematics involved. Applets covering most of the above mathematical topics can also be found on the ***MathinSite*** web site at (http://mathinsite.bmth.ac.uk/html/applets.html).

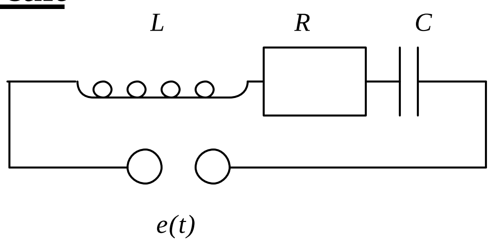
Background

Inductance, L, resistance, R, and capacitance, C, are the building blocks of basic electrical systems studied in first-year undergraduate engineering. A voltage (the input) applied to such a circuit containing these elements will result in a current flow (the output, or response) and a change in the charge on the capacitor. But what form does the resulting response take?

This applet allows investigation of a circuit containing an inductor, a resistor and a capacitor in series. The parameters L, R and C can be varied, along with the initial current in the circuit and applied voltages (constant, ramp, sinusoidal and exponentially varying voltages can be input). The applet displays the circuit's response as graphical output together with the governing differential equation and its solution, i.e. the mathematical equation of the resulting current.

Using the LRC Series Circuit applet will help give you a feel for how changes of the circuit parameters affect circuit response – both graphically and mathematically.

The LRC series circuit



The electrical circuit above shows an inductance L, a resistance R and a capacitance C, in series. A current $i(t)$ will flow in the circuit when a voltage $e(t)$ is applied.

Using $i(t)$ and $e(t)$ indicates that current and voltage vary with time (i.e. they are time-dependent variables). Of course this doesn't preclude the possibility that the voltage, for example, could be constant (e.g. the voltage source is a battery).

The drop in voltage (i.e. potential drop) across the resistance is iR (from Ohm's Law), across L it is $L\frac{di}{dt}$ and across C it is $\frac{1}{C}\int_0^t i\,dt$.
Kirchhoff's 2nd Law says that the sum (i.e. addition) of potential drops across all of the non-supply elements in the circuit equals the applied voltage of the supply. So,

$$\boxed{L\frac{di}{dt}+Ri+\frac{1}{C}\int_0^t i\,dt=e(t)}$$

and this an *integro-differential equation*!

Solving general equations of this type can be difficult. However, for this particular equation there are two ways forward that simplify the problem.

Method 1

Bearing in mind that *the current in the circuit is given by the rate of change of charge on the capacitor*, it is possible to relate i and q using $i=\frac{dq}{dt}$ where q is the charge on the capacitor measured in coulombs.

Upon integration, this gives $q=\int_0^t i\,dt$ and upon differentiation, $\frac{di}{dt}=\frac{d^2q}{dt^2}$

So rewriting the above integro-differential equation in terms of q rather than i gives

$$\boxed{L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e(t)}$$

This is now a ***linear second-order differential equation with constant coefficients***, which can be solved by a variety of methods to find the charge, q, in terms of time, t. Methods of solution will be discussed later.

Quite often, it is the expression for current that is required and, here, differentiating this solution for q with respect to t will give the solution for the current, i, in terms of t (since $i=\frac{dq}{dt}$).

Method 2

Differentiating the above integro-differential equation with respect to t gives

$$\boxed{L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{de(t)}{dt}}$$

(noting that $\dfrac{dq}{dt}=\dfrac{d\int_0^t i\,dt}{dt}=i$).

This is again a *linear second-order differential equation with constant coefficients* which can be solved by a variety of methods to find directly the current, i, in terms of time, t. Note here that the right hand side is the ***differential*** of the applied voltage.

Integrating the solution for i with respect to t will give the charge, q, in terms of t.

Usually such differential equations require ***two initial conditions*** specifying values for the current (or charge) and the rate of change of current (or charge) in the circuit when the voltage source is applied (often at time zero). These could look like, for example,

$i = i_o$ when $t = 0$ (or as $i(0) = i_o$) and $\dfrac{di}{dt} = X$ when $t = 0$ (i_o and X are specified values).

Digression on Variables and Parameters In the above circuit $e(t)$ is the *input* to the circuit and it *varies* with time, i.e. it is time dependent. $i(t)$ is the *output* of the circuit once the input voltage has been applied and it, too, is a time dependent variable.

$i(t)$ and $e(t)$ are called ***dependent variables*** since they are dependent here on time, t, which is called the ***independent variable***. Note that in any such circuit, current and voltage will *vary* (unless they are constant) all the time. However, in a particular circuit L, R and C will always take just the one value for all time. This doesn't mean it is not possible to change any of them - but if they are changed, this results in a completely different circuit and hence a completely different differential equation to solve. Here, L, R and C are ***parameters*** of the system.

Solving Second-order Differential Equations

As with many real-world applications, the LRC series circuit is a time-dependent system. Generally, the ***solution*** of such a system's differential equation is the ***output*** of the system and is, in this case, i (or q) at time, t. It is therefore necessary to **solve the differential equation to obtain "*i* (or *q*) = some function in terms of *t*".**

The LRC Series Circuit's differential equation can be solved in a variety of ways. Two ways will be illustrated here. *If you are using the applet purely to investigate the type of response obtained when varying the parameters of an LRC Series Circuit, you do not need to understand the following*. However, all engineering undergraduates should be familiar with at least one, if not both, of the following methods.

To see how the *current* changes in the *LRC* circuit when an applied voltage is switched on, it is necessary to solve, with appropriate initial conditions, the circuit equation

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{de(t)}{dt}$$

Suppose that for a particular circuit,

- L = 1 H and R = 3 Ω and C = 0.5 F (these are unrealistic 'real-world' values but they make the calculations easier!)
- a ramp voltage given by $e(t) = 4t$, is applied and
- initially the current is 3 amp and is changing at a rate of -2 amperes per second.

In this case, the governing differential equation is

$$\frac{d^2i}{dt^2}+3\frac{di}{dt}+2i=4$$

Remember: the ***differential*** of the applied voltage.

and the initial conditions are i = 3 and $\frac{di}{dt}=-2$ when t = 0.

Method of Solution 1. (The 'assumed solution' method)

Further examples and some background explanation of this method can be found in a variety of textbooks. One such example is "Engineering Mathematics" by Stroud, K.A. and Booth, D.J. (Fifth edition, published by Palgrave, UK, 2001) See pages 1071 – 1096.

All differential equations of this type have two parts to their solution: the ***complementary function*** (CF) and the ***particular integral*** (PI). The CF relates to the left side of the differential equation (and so relates purely to the LRC system itself) and the PI relates to the right side of the differential equation (and so relates to the applied voltage). The *overall* solution is found simply by adding the CF and the PI.

To find the CF

$$\text{put } \frac{d^2i}{dt^2}+3\frac{di}{dt}+2i=0$$

Since it is known that all differential equations of this type have exponential solutions, assume a solution of the form $i = Ae^{mt}$ and substitute for i and its first and second derivatives to obtain the ***auxiliary equation***:

$$m^2+3m+2=0$$

This has solutions m = -1 and m = -2

So the CF is

$$i_{\text{CF}}=Ae^{-t}+Be^{-2t}$$

To find the PI

Since this type of differential equation is 'linear', it is known that 'what goes in, comes out'. So if the applied term is a constant (4, in this case) then the PI is also a constant.

Assume that the PI part of the solution is $i = k$, where the value of constant k has to be determined. So substituting $i = k$ and its first and second derivatives into the full, original differential equation gives

$$0 + 3\times 0 + 2k = 4 \quad \text{i.e. } k = 2,$$
$$\text{so } i_{PI} = 2$$

The overall ***general*** solution is given by $i = i_{CF} + i_{PI}$, so $\underline{i = Ae^{-t} + Be^{-2t} + 2}$

It is now necessary to determine the unknown constants *A* and *B*, which are found using the initial conditions:

$i = Ae^{-t} + Be^{-2t} + 2 \quad \Rightarrow \quad 3 = Ae^{-0} + Be^{-2\times 0} + 2$ so $A + B = 1$

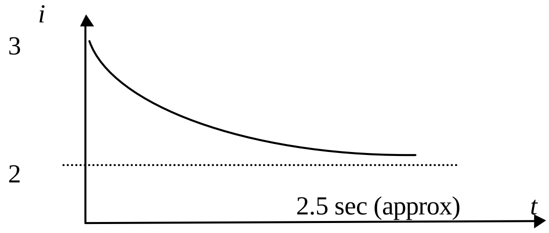
and

$\dfrac{di}{dt} = -Ae^{-t} - 2Be^{-2t} \quad \Rightarrow \quad -2 = -Ae^{-0} - 2Be^{-2\times 0}$ so $-A - 2B = -2$

These two simultaneous equations in *A* and *B* can be solved to find $B = 1$ and $A = 0$.

So, The overall ***particular*** solution is given by $\underline{i = e^{-2t} + 2}$

This has the solution curve



Note that the current settles down to a ***steady-state*** value of 2 amp (relating to the input to the system). This current has a ***time constant***, t, of ½ second so that the ***transient*** (the part of the solution relating to the system itself and its associated exponential decay) dies away in about $5\times t$ = 2½ second.

> **Note on time constants**: Any exponential ***decay*** term of the form e^{-kt} ($k > 0$) has a time constant defined by $t = 1/k$. t is a useful quantity since $5\times t$ gives a measure of approximately how long the exponential term takes to decay away to zero. In this theory sheet and in ***MathinSite***'s LRC Series Circuit applet, $5\times t$ is used, although some other authors use $6\times t$.
>
> Further information on time constants can be found in the theory and tutorial sheets of ***MathinSite***'s 'Exponential Function' applet.

The LRC Series Circuit differential equation can also be solved using **Laplace Transforms**. If you are studying on an engineering course, it is highly likely that you are, or soon will be, familiar with this method. As an introduction (or reminder) of the method, let's solve the same equation, but this time using Laplace Transforms (LTs).

Method of Solution 2. The Laplace Transform Method

$$\frac{d^2i}{dt^2}+3\frac{di}{dt}+2i=4$$

Same equation and initial conditions as before, but this time the Laplace Transform of the equation will be used - found from tables of LTs from formula, or text, books (e.g. Stroud & Booth).

and the initial conditions are i = 3 and $\frac{di}{dt}=-2$ when t = 0.

REMINDER: *from the definition of the Laplace Transform (LT) and tables of transforms,*

- *the LT of i(t) is* $\bar{i}$,
- *the LT of* $\frac{di}{dt}$ *is* $s\bar{i}-i(0)$, *where i(0) is the value of i when t = 0, and*
- *the LT of* $\frac{d^2i}{dt^2}$ *is* $s^2\bar{i}-s\,i(0)-i'(0)$, *where, again,* $i(0)$ *is the value of i when t = 0, and i'(0) is the value of the* ***differential*** *of i* at $t=0$
- *the LT of* 4 *is* $\frac{4}{s}$.

Take the Laplace transform of the differential equation to give

$$\left(s^2\bar{i}-3s-(-2)\right)+3\left(s\bar{i}-3\right)+2\bar{i}=\frac{4}{s}, \quad \text{or}$$

$$(s^2+3s+2)\bar{i}-3s+2-9=\frac{4}{s}, \quad \text{so}$$

$$(s^2+3s+2)\bar{i}=3s+7+\frac{4}{s}=\frac{3s^2+7s+4}{s}$$

$$\bar{i}=\frac{3s^2+7s+4}{s(s+1)(s+2)}=\frac{2}{s}+\frac{1}{(s+2)}$$

by "Partial Fractions".

Now take inverse Laplace Transform to find $i(t)$.

(*These can be found from LT tables reading from the* ***right****-hand column across to the* ***left****-hand column.*)

$$\underline{i=2+e^{-2t}} \qquad \text{as before.}$$

If you have not seen this method before, don't worry. It's here just to show you that there is another way of solving such differential equations. The LT method of solution is a most powerful and more sophisticated method, which totally bypasses the need to 'assume solutions'. Furthermore, the initial conditions are "taken care of" at the outset - there is no need to determine separately the constants A and B. Hopefully, you can see that the LT method has "possibilities"!

The LT method contained unexplained variables, (s), and needed the use of partial fractions. However, Laplace Transforms are extensively used in electronic, electrical, mechanical and control systems, so it is clear to see that knowledge of how they are manipulated becomes imperative to any engineer.

Appendix

Analysis of different parts of the solution

An LRC Series Circuit is a 'linear system', that is, it behaves according to the rule "what goes in, comes out". So a sinusoidal input (right hand side of the differential equation), for example, results in a sinusoidal output. However, before the circuit settles down to a ***steady state*** sinusoidal output, it usually exhibits an exponential response (the ***transient*** – or 'short lived' - part of the response).

The LRC differential equation's solution comes in two parts:

$$i = f(t) + g(t)$$

The **first part**, $f(t)$ is called the **Complementary Function** (CF) and, being dependent only on R, C and L, results from the system (circuit) itself. It invariably involves exponential decay for non-trivial cases.

Since $f(t)$ exponentially decays away, this part of the solution is called the ***transient***.

The **second part**, $g(t)$ is called the **Particular Integral** (PI) and results from, and takes the same form as, the right-hand side of the differential equation (which is related to the input to the system). So, if the right hand side is a sinusoid, the PI will also be a sinusoid (possibly a mix of sines **and** cosines); if it is of the form ae^{pt} then the PI is Be^{pt} (or $(Bt+C)e^{pt}$ under certain conditions).

$g(t)$ is the ***steady state*** part of the response.

… so,

"the PI (steady state output) is the same form (trig, exponential, algebraic, etc) as the right hand side of the d.e."

and this is characteristic of ***linear*** systems such as the LRC Series Circuit.

The overall solution (current or charge) is the sum of the CF and PI.

*Note that various mixes of values for the initial conditions, system parameters and applied voltage parameters **can** result in any part of the full solution being zero (as happened in the above example).*