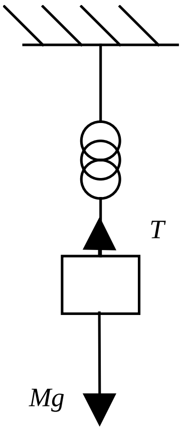
# Mass-Spring-Damper Systems
# The Theory

## The Unforced Mass-Spring System

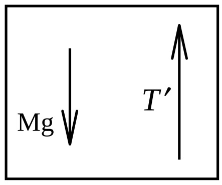



The diagram shows a mass, *M*, suspended from a spring of natural length *l* and modulus of elasticity $\lambda$. If the elastic limit of the spring is not exceeded and the mass hangs in equilibrium, the spring will extend by an amount, *e*, such that by Hooke's Law the tension in the spring, *T*, will be given by $T = \frac{\lambda e}{l}$

For system equilibrium, this will be balanced by the weight

$$\text{so } Mg = T = \frac{\lambda e}{l} \qquad (1)$$

If the spring is pulled down a further distance, *y*, (with *y* positive downwards) the restoring force will now be the new tension in the spring, $T'$, given by $T' = \frac{\lambda(e+y)}{l}$, and so the net force acting DOWNWARDS is $Mg - T'$




$$= Mg - \frac{\lambda(e+y)}{l} = Mg - \frac{\lambda e}{l} - \frac{\lambda y}{l}.$$

But, from equation (1), $Mg = \frac{\lambda e}{l}$,

so the net force downwards $= -\frac{\lambda y}{l}$ (2)

From Newton's 2nd Law, Force = mass x acceleration $= M\frac{d^2y}{dt^2}$ (3)

so, combining (2) and (3)

$$\boxed{M\frac{d^2y}{dt^2} + \frac{\lambda y}{l} = 0}$$

The above analysis has resulted in a ***second-order*** *differential equation* with dependent variable *y* (displacement) and independent variable *t* (time) and system parameters *M*, $\lambda$ and *l*. **(See box on next page for discussion on parameters and variables)**

For the mass-spring-damper's ***2nd*** order differential equation, ***TWO*** *initial conditions* are given, usually the mass's *initial displacement* from some datum and its *initial velocity*.

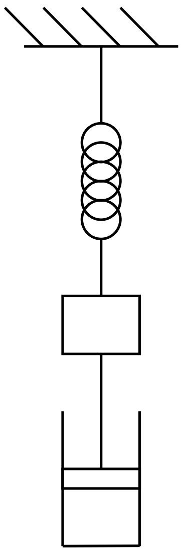
Since the system above is unforced, any motion of the mass will be due to the initial conditions ONLY. Typical initial conditions could be $y(0) = -2$ and $\dot{y}(0) = +4$. With *downward* as the positive direction, *y* measured in centimetres and *t* in seconds, these initial conditions say that at *t* = 0 the mass is instantaneously 2 cm *above* the datum and is travelling with a velocity of 4 cm/s in the *downward* direction.

***Digression on Variables and Parameters*** In this system $y(t)$ is the *output* of the system once the mass has been initially displaced and released. It is a time dependent variable.

$y(t)$ is called the ***dependent variable*** and $t$ is the ***independent variable*** since the value of displacement $y$ *depends on* (is a function of) time, $t$. Note that in any such system, the displacement $y$ will *vary* (unless it is a constant) as time, $t$, *varies*. However, in any given system $M$, $\lambda$ and $l$ will always take just the one value for all time. It ***is*** possible to change them - but if they are changed, this results in a different mass-spring-damper system and hence a completely different differential equation to solve. Quantities that remain constant like this within any system (such as $M$, $\lambda$ and $l$) are ***parameters*** of the system.

Note that the system above has no input – it is unforced. Nothing forces the system to move; any movement is a consequence only of an initial displacement or an initial velocity.

## The Unforced Mass-Spring-*Damper* System



The above system is unrealistic since it does not take into account the resistance to motion due to friction in the spring or air resistance. Once the mass is set in motion, that system will continue moving forever.

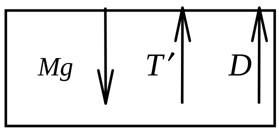
Damping can be introduced into the system physically, schematically and mathematically by incorporating all resistances into a dashpot (see diagram).

It can be shown experimentally that in such cases ***the resistance to motion is directly proportional to the velocity*** of the mass and, naturally, opposes the motion. This is not unreasonable - the faster the mass moves, the greater the resistance is exerted upon it (compare how much more difficult it is running, rather than walking, through water).

So the damping force, $D = -R\dfrac{dy}{dt}$.  ($R > 0$)

Here, $R$ is the constant of proportionality and is called the ***damping factor***.

The inclusion of the damping modifies the equations of the previous case thus:




This time, the net downward force will be $Mg - T' - D$

$$= Mg - \frac{\lambda(e+y)}{l} - R\frac{dy}{dt} = -\frac{\lambda y}{l} - R\frac{dy}{dt}.$$

And, again using Newton's 2nd Law, this results in

$$M\frac{d^2y}{dt^2} + R\frac{dy}{dt} + \frac{\lambda y}{l} = 0$$

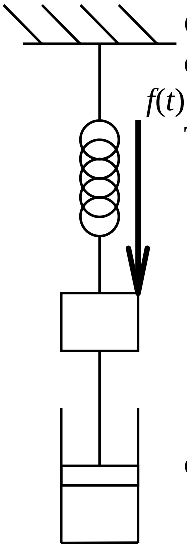
or,

$$M\frac{d^2y}{dt^2} + R\frac{dy}{dt} + ky = 0$$

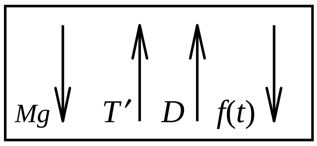
where $k = \lambda / l$

This is, once again, a second-order differential equation, but this time with parameters *M*, *R* and *k*. Parameter *k* is in terms of parameters $\lambda$ and *l*, and parameter *R* is dependent upon the viscosity of the fluid in the dashpot, for example.

## The *Forced* Mass-Spring-Damper System




Consider now the case of the mass being subjected to a force, *f*(*t*), in the direction of motion.




This time, the net downward force will be

$$Mg - T' - D + f(t)$$

$$= Mg - \frac{\lambda(e+y)}{l} - R\frac{dy}{dt} + f(t) = -\frac{\lambda y}{l} - R\frac{dy}{dt} + f(t).$$

Again using Newton's 2nd Law, this results in

$$\boxed{M\frac{d^2y}{dt^2} + R\frac{dy}{dt} + \frac{\lambda y}{l} = f(t)}$$

or,

$$\boxed{M\frac{d^2y}{dt^2} + R\frac{dy}{dt} + ky = f(t)}$$ ---(#)←used later

Note how the only difference here is that ***the input to the system, f(t),*** the forcing term, appears on the right hand side of the differential equation rather than the zero when the system was unforced (i.e. zero input).

## Forcing Terms

The forcing term, the input to the system, given by *f*(*t*) can take various forms and can be modelled readily by standard mathematical functions:

- An unforced system, modelled by using $f(t) = 0$
- A constant applied force uses $f(t) = c$ (where *c* is a constant)
- A constantly changing force (ramp input), $f(t) = mt + c$ (*m*, *c* constants)
- A quadratically changing force, $f(t) = at^2 + bt + c$ (*a*, *b* and *c* constants)
- An oscillating force (sinusoidal input), $f(t) = a\sin\omega t + b\cos\omega t$ ( here *a*, *b* & $\omega$ are constants where $\omega$ is the angular frequency of the applied oscillations)
- An exponentially changing input, $f(t) = ae^{bt}$ (*a*, *b* constants)

## Solving the Mass-Spring-Damper Second-Order Differential Equation

Obtaining the solution of second order differential equations is outside of the remit of this theory sheet. You should be learning these methods on your course - methods such as the classical "Complementary Function and Particular Integral" method, or the "Laplace Transforms" method. Here the emphasis is on using the accompanying applet and tutorial worksheet to interpret (and even anticipate) the types of solutions obtained.

## Types of Solution of Mass-Spring-Damper Systems and their Interpretation

The solution of mass-spring-damper differential equations comes as the *sum* of two parts:

- the *complementary function* (which arises solely due to the system itself), and
- the *particular integral* (which arises solely due to the applied forcing term).

The **particular integral** is the easier part of the solution to consider. The mass-spring-damper differential equation is of a special type; it is a ***linear*** second-order differential equation. In mathematical terms, linearity means that $y$, $dy/dt$ and $d^2y/dt^2$ only occur to the power 1 (no $y^2$ or $(d^2y/dt^2)^3$ terms, for example). In real-world terms, linearity means "What goes in, comes out"! If you apply an oscillating force to such a system, oscillations will result. A constant applied force (input) will produce a constant deflection, $y$ (output). As you can imagine, if you hold a mass-spring-damper system with a constant force, it will maintain a constant deflection from its datum position. This is the ***steady state*** part of the solution.

*How* it gets to the steady state solution is governed by the system itself (is it light and springy or perhaps heavy and slow?) and hence dependent on the values of $M$, $R$ and $k$. The way in which the mass reaches its steady-state solution, called the ***transient***, is reflected in the **complementary function**, which itself is dependent on the relative sizes of $R^2$ and $4Mk$.

A linear second order *differential* equation is related to a second order *algebraic* equation, i.e. $M\frac{d^2y}{dt^2}+R\frac{dy}{dt}+ky$ is related directly to $ax^2+bx+c$. For a second order algebraic equation the discriminant $b^2 - 4ac$ plays an important part in deciding the type of solution to the equation $ax^2+bx+c=0$. Similarly the 'discriminant' $R^2 - 4Mk$ determines the type of solution to the differential equation $M\frac{d^2y}{dt^2}+R\frac{dy}{dt}+ky = 0$, i.e. the system with the forcing term taken out – it is this equation (with $f(t) = 0$) that produces the transient response.

$\boldsymbol{R^2 - 4Mk > 0}$ (or $R^2 > 4Mk$) produces a complementary function (transient) of the form

$y = Ae^{m_1t}+Be^{m_2t}$ with $A$, $B$, $m_1$ and $m_2$ all constant with $m_1$ and $m_2$ both negative.

This produces an exponential decaying transient. How long the transient takes to die away will depend upon the time constants of the two exponential decay terms (*see next section for discussion on time constants*). This is a 'sluggish' response and corresponds to large $R$-values compared with $M$ and $k$ ($R^2 > 4Mk$) and so represents a ***heavily damped system***.

$\boldsymbol{R^2 - 4Mk < 0}$ (or $R^2 < 4Mk$) produces a complementary function (transient) of the form

$y = e^{pt}(A\sin\omega t + B\cos\omega t)$ with $A$, $B$, $p$ and $\omega$ all constant with $p$ negative.

This produces a sinusoidal transient modulated by pure exponential decay. How long the sinusoids take to die away will again depend upon the time constant of the exponential. This is a 'fast' response and corresponds to small values of $R$ compared with $M$ and $k$ (remember $R^2 < 4Mk$) and so is called a ***lightly damped system***.

$\boldsymbol{R^2 - 4Mk = 0}$ (or $R^2 = 4Mk$) produces a complementary function (transient) of the form

$y = (A+Bt)e^{mt}$ with $A$, $B$ and $m$ constant with $m < 0$.

This is a linear function ($A + Bt$), modulated by exponential decay. How long this transient takes to die away will depends on the time constant of the exponential. This is the fastest response possible without setting up oscillations in the system and corresponds to a ***critically damped system***.

## Time Constants and the Time to Decay

The transient is the way in which the system responds during the time it takes to reach its steady state. Transient means "short lived". But how short is "short lived"? This can be determined from the following table:

| $t$ | $e^{-\frac{t}{\tau}} \times 100\%$ (as a percentage) |
|---|---|
| 0 | $e^0$ x 100 =100 |
| $\tau$ | $e^{-1}$ x 100 = 36.7879 |
| $2\tau$ | $e^{-2}$ x 100 = 13.5335 |
| $3\tau$ | $e^{-3}$ x 100 = 4.9787 |
| $4\tau$ | $e^{-4}$ x 100 = 1.8316 |
| $5\tau$ | $e^{-5}$ x 100 = 0.6738 |

**$5\tau$ is an important value!**

The right hand column shows that the value of $e^{-\frac{t}{\tau}}$ varies from 100% at $t = 0$ to about 0.7% by $t = 5\tau$.

$\tau$ (Greek letter, "tau") is called the "***time constant***".

The implication is that by $t = 5\tau$, the contribution of $e^{-\frac{t}{\tau}}$ has died away to 'practically nothing'. For our system, the exponential terms are of the form $e^{-mt}$, so comparing with $e^{-\frac{t}{\tau}}$, gives the important result,

$$\tau = \frac{1}{m}$$

***Note that here, the time constant, $\tau$, is only appropriate for exponential decay, not growth.***

A heavily damped system that contains the complementary function (transient) $y = 7e^{-2t} - 3e^{-5t}$ in its solution, for example, has two time constants, 1/2 and 1/5. The two exponential terms will die away after about five times these values. So the $7e^{-2t}$ term (time in seconds) dies out after 2.5 seconds and the $3e^{-5t}$ term after only 1 second. If, for example, the full solution for this system had been $y = 7e^{-2t} - 3e^{-5t} + 15$ then the transient would have lasted about 2.5 seconds after which the steady state value of 15 is all that remains. Watch out for this sort of response when you use the accompanying applet.

## Transients with Exponentially Decaying Sinusoids

As seen previously, when $R^2 < 4Mk$ the complementary function (transient) had the form $y = e^{pt}\left(A\sin\omega t + B\cos\omega t\right)$ with $A$, $B$, $p$ and $\omega$ all constant and $p$ negative. This result contains oscillations - the trigonometric part - multiplied by (or modulated by, or even 'killed off' by) exponential decay (remember $p < 0$). In the section above you saw how easy it was, using the time constant, to determine how long it took for the exponential decay to kill off this part of the system response. But here, how many oscillations will occur before they disappear? This can be determined from the following.

When working with angular velocity, $\omega$, (measured in *radians* per second), two important formulae can be used:

$$\omega = 2\pi f \quad \text{and} \quad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

where $f$ is the ***frequency*** of the oscillations (measured in cycles per second or Hertz) and $T$ (in seconds) is the ***periodic time*** (time for one cycle) of the oscillations. Ordinary alternating mains electricity in the UK, for example, operates at a frequency of $f = 50$ Hz. Its angular velocity is therefore $\omega = 100\pi$ rad/sec and its periodic time is $T = 1/50$ sec or 20 ms.

Suppose the transient solution of a mass-spring-damper system is $y = 5e^{-0.2t}\sin 10t$. Here the time constant is 1/0.2, so five times the time constant will be 25 seconds – whatever the transient response, it will have disappeared by 25 seconds. The number of oscillations that will occur during this time can be found from $T = 2\pi/10 = 6.28$ seconds (2 d.p.). Each complete oscillation takes about 6.28 seconds, the exponential decay kills of the transient in about 25 seconds, and therefore there will be 25/6.28, about 4, complete oscillations. Mind you, by the time you get to the fourth oscillations, its very small amplitude will make it difficult to see. Watch out for this effect when using the applet.

**Mechanical Systems**

Many real-world systems can be modelled by the mass-spring-damper system – not just the mass-spring-damper system itself!

- If you thump a window in its frame, it will move – not by much you'd hope! The window has mass, it has a resistance to motion (not the least part of which is the fact that the window is held in a frame) and the window has natural springiness. You force the system when you thump it.
- Consider, a wine glass. This inherently has more springiness than the window, as can be heard when you 'ping' the glass with your fingernail.
- Hold down one end of a ruler at the end of a desk, pull it up at the free end and let go (initial conditions: displacement equals 2cm, say, and initial velocity zero). Here you have a good example of a lightly damped unforced system – the ruler will oscillate but the oscillations will die away. With no forcing terms, once the transient dies away, the ruler will settle down back at its datum position.
- Consider the Millennium Bridge across the Thames when first opened. It had (and still has!) mass, it had springiness (too much) and it had resistance to motion (too little) – a lightly damped system if ever there was one! Left to itself, the bridge was quite happy. However, when 'forced' by a large number of people all walking in time with a frequency close to the natural frequency of the springy bridge, it began to ***resonate*** – the oscillations set up became uncontrollably large and uncomfortable.

***Resonance*** occurs when vibrations (sinusoids), whose frequency is the same as the natural frequency of the system itself, force an undamped (or very lightly damped) system. It is an effect that you will be able to set up in the accompanying applet. In mechanical systems resonance can be destructive.

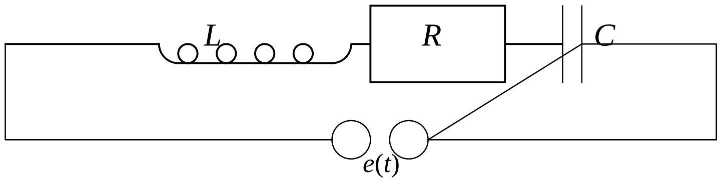
- The Tacoma Narrows Bridge disaster, for example, in which, at a certain wind speed, the vortices shed from (and hence forces acting on) the bridge were shed at the same frequency as the natural frequency of the bridge itself thus reinforcing the vibrations (now you see why they closed the Millennium Bridge!).

- The opera singer who pitches a note at the natural frequency of a wine glass can set up resonance forcing the glass to shatter.
- The passing bus that causes a badly fitting window to rattle in its frame as the engine revs (1 rev/sec = 1 Hz) pass through the natural frequency of the window.

***Beats*** are another phenomenon that can be visualised using the accompanying applet. Beats occur when you force a lightly damped system (once again) by vibrations whose frequencies are *close to* the natural frequency of the system. A good example of *hearing* beats is to tune two guitar strings to *nearly* the same note. Pluck both strings. You will hear a rather discordant sound since the notes are so close, but you will also hear what can best be described here as the “nyur, nyur”(any better offers?!) regularity of the beats.

# Appendix A: The Electrical-Mechanical Analogue

**The LCR Series Circuit** This is obviously NOT a mass-spring-damper system, yet it is of great interest to those concerned with mass-spring-damper systems, as you will see.




Here it is necessary to know that the potential drop across a capacitor is given by $\frac{q}{C}$, where $q$ is the charge on the capacitor with capacitance, $C$.

$R$ is electrical resistance, $L$ is inductance and $e(t)$ is the applied voltage. $i(t)$ is the current that results (it is the output from the system).

It is also necessary to know that $q = \int_0^t i\,dt$. The differential of both sides of this leads to $\frac{dq}{dt} = i$ (the rate of change of charge across a capacitor is equal to the current in the circuit) and hence $\frac{d^2q}{dt^2} = \frac{di}{dt}$. These will both be used shortly.

Applying Kirchhoff's 2nd Law (the sum of the potential drops across all elements in a circuit equals the applied potential)

$$L\frac{di}{dt} + Ri + \frac{q}{C} = e(t) \qquad \text{-----(A)}$$

Unfortunately, this differential equations involves TWO time-dependent output variables $i(t)$ and $q(t)$. However as seen above, they are related, so the equation can be totally written in terms of $q$ (or indeed of $i$) leading to

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)}$$

a second-order differential equation in $q$, or differentiating equation (A) with respect to time and replacing $\frac{dq}{dt}$ with $i$, gives

$$\boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{d(e(t))}{dt}} \qquad \text{----------(\#)}$$

Note the similarity between the two equations marked # (the other is on page 3.) Here they are again.

$$\boxed{M\frac{d^2y}{dt^2} + R\frac{dy}{dt} + ky = f(t)} \quad \text{and} \quad \boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{d(e(t))}{dt}}$$

These two equations are fundamentally identical and constitute an e*lectrical-mechanical analogue*.

Notice the analogy between corresponding parameters and variables. In the electrical circuit:

*L* behaves like mass, *M*

*R* behaves like mechanical resistance, *R* (surprise, surprise!)

$\frac{1}{C}$ behaves like spring stiffness, *k*

$i(t)$, the output from the electrical system, corresponds to displacement, $y(t)$, the output from the mechanical system.

$\frac{de(t)}{dt}$, the rate of change of applied voltage, behaves like the applied force, $f(t)$.

These analogies form the basis of analogue computers, aircraft simulators, etc. in which real-world mass-spring-damper type systems can be simulated with the equivalent electrical analogue circuit. In any such system, if you know the values of *M*, *R* and *k* then you can simulate that system electronically.