Carl von Ossietzky Universität Oldenburg – Faculty V - Institute of Physics
Module Introductory laboratory course physics – Part I

Forced mechanical oscillations

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HOOKE's law, harmonic oscillation, harmonic oscillator, eigenfrequency, damped harmonic oscillator, resonance, amplitude resonance, energy resonance, resonance curves

References:

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/2/ TIPLER, P.A.: „Physik“, Spektrum Akademischer Verlag, Heidelberg among others.
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1 Introduction

It is the object of this experiment to study the properties of a „harmonic oscillator“ in a simple mechanical model. Such harmonic oscillators will be encountered in different fields of physics again and again, for example in electrodynamics (see experiment on electromagnetic resonant circuit) and atomic physics. Therefore it is very important to understand this experiment, especially the importance of the amplitude resonance and phase curves.

2 Theory

2.1 Undamped harmonic oscillator

Let us observe a set-up according to Fig. 1, where a sphere of mass m_K is vertically suspended (x-direction) on a spring. Let us neglect the effects of friction for the moment. When the sphere is at rest, there is an equilibrium between the force of gravity, which points downwards, and the dragging resilience which points upwards; the centre of the sphere is then in the position $x = 0$. A deflection of the sphere from its equilibrium position by x causes a proportional dragging force F_R opposite to x:

(1) $$F_R \propto -x$$

The proportionality constant (*elastic* or *spring constant* or *directional quantity*) is denoted D, and Eq. (1) becomes the well-known HOOKE*'s law*:

(2) $$F_R = -D\,x$$

Following deflection and release the dragging force causes an acceleration a of the sphere. According to *Newton's second law* and in combination with Eq. (2) we therefore obtain:

(3) $$m_K a = m_K \frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} = m_K\,\ddot{x} = -D\,x \qquad (t\text{: time})$$

the three terms on the left side merely representing different ways to write the relation force = mass × acceleration. Eq. (3) is the important differential equation (equation of motion), by means of which all systems can be described which react with a dragging force on a deflection from their position of rest or equilibrium that is proportional to the degree of deflection. Such systems will be encountered very often in different fields of physics.

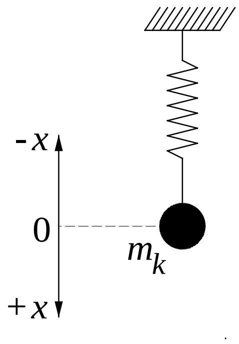


Fig. 1: Mass/spring system.

We are interested in learning which movement the sphere makes when it is deflected from its position at rest and then released, its initial velocity v at the moment of release being zero. So we look for the function $x(t)$, which fulfils the differential equation (3) under the condition $v(t = 0) = 0$. As a resolution we guess a function $x(t)$, which represents a so-called *harmonic (sinusoidal) oscillation*:

(4) $$x = x_0 \sin\left(\omega\, t + \varphi\right)$$

x_0 is the *amplitude*, $(\omega t+\varphi)$ the *phase*, φ the *initial phase* and ω the *angular eigenfrequency* of the oscillation. Inserting Eq. (4) into Eq. (3) and performing differentiation twice with respect to time t we find the value ω, for which Eq. (4) is a solution of Eq. (3):

(5) $$\omega = \sqrt{\frac{D}{m_K}} := \omega_0$$

Thus, the sphere performs oscillations with this angular eigenfrequency ω_0 when it is released. Since we assume that there is no friction, the amplitude x_0 of the oscillation remains constant. x_0 as well as the initial phase φ are free parameters which have to be chosen such that Eq. (4) is „adjusted“ to the process to be described, i.e. that Eq. (4) reflects the observed motion with the correct amplitude and initial phase.

Equation (5) is only valid if the mass of the spring, m_F , is negligible compared to the mass m_K of the sphere. If this is not true, we have to consider that the spring's different elements of mass also oscillate following its deflection and release. The oscillation amplitudes of these elements of mass, however, are very different: They increase from zero at the point of suspension of the spring to a value x_0 at the end of the spring. An exact calculation (e.g. in /3/) shows that the oscillation of the single elements of mass with different amplitudes equals the oscillation of one third of the whole spring mass with the amplitude x_0. Therefore, the correct equation for the angular eigenfrequency reads:

(6) $$\omega_0 = \sqrt{\frac{D}{m_K + \frac{1}{3}m_F}} := \sqrt{\frac{D}{m}} \qquad \text{with} \quad m := m_K + \frac{1}{3}m_F$$

In the experiment to be performed the sphere is not directly fixed to the spring but by a bar S with a indicator Z (Fig. 7). In that case, m_K in Eq. (6) has to be replaced by the sum

(7) $$M = m_K + m_S + m_Z$$

m_S and m_Z being the masses of S and Z.

An example illustrates the described relationships. According to Fig. 1 we observe a sphere of the mass $m_K = 0.11$ kg suspended by bar and indicator ($m_S + m_Z = 0{,}07$ kg) on a spring with the spring constant $D = 28$ kg/s^2 and the mass $m_F = 0.02$ kg. The sphere is deflected by $x_0 = 0.05$ m downwards from its position at rest. Then we release the sphere and it performs oscillations with the amplitude x_0 and the eigenfrequency $\nu_0 = \omega_0/(2\pi) \approx 1.9$ Hz (Eq. (6)). If we start to record the motion $x(t)$ of the sphere exactly when it has achieved its maximum upward deflection, the sinus according to Eq. (4) „starts“ at an initial phase of $\varphi = 3\pi/2 = 270°$ (mind the sign of x in Fig. 1!). This situation is represented in Fig. 2.

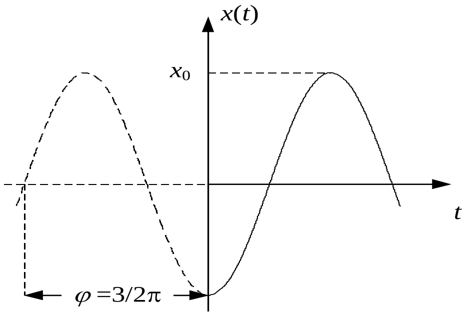


Fig. 2: Definition of the amplitude x_0 and initial phase φ.

A system according to the arrangement considered here (also called *mass/spring system*) that performs harmonic oscillations is called a *harmonic oscillator*. The harmonic oscillator is characterized by a dragging force proportional to the deflection leading to a typical equation of motion in the form of (3) with a solution in the form of (4). Equally characteristic of the harmonic oscillator is the *parabolic* behaviour of its *potential energy* E_p as a function of the position:

(8) $$E_p = \frac{1}{2} D\, x^2$$

2.2 Damped harmonic oscillator

Now we observe the more realistic case of a mass/spring system under the influence of friction. We start from the simple case that the system is subject to a force of friction F_b proportional to the velocity v. For F_b we can write:

(9) $$F_b = -bv = -b \frac{\mathrm{d}\,x}{\mathrm{d}\,t}$$

b being a *constant of friction*, which represents the magnitude of the friction.

Question 1:
- Which unit does b have? Why is there a minus sign in Eq. (9)?

In this case the equation of motion (3) takes on the form:

(10) $$m \frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} = -D\,x - b \frac{\mathrm{d}\,x}{\mathrm{d}\,t}$$

Usually, this differential equation is written in the form:

(11) $$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\frac{D}{m}x=0$$

Here again, it is interesting to know what type of motion the sphere performs after being deflected *once* from its position at rest and then released with an initial velocity of zero. Thus, we are once again searching for the function $x(t)$ which resolves the differential equation (11) under the condition $v(t = 0) = 0$. As a consequence of damping, we expect a decreasing amplitude of the oscillation and therefore try a solution with an exponentially decreasing amplitude (cf. Fig. 3):

(12) $$x = e^{-\alpha t} x_0 \sin(\omega t + \varphi)$$ (α : *damping constant*)

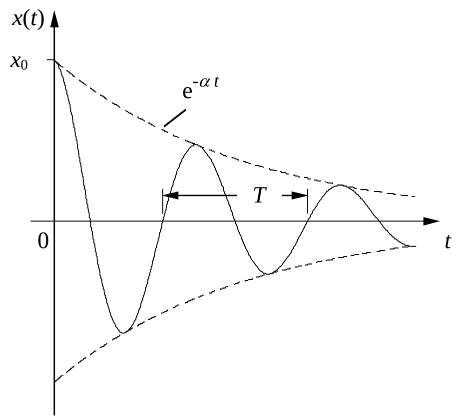


Fig. 3: Damped harmonic oscillation.

We now insert Eq. (12) into Eq. (11), perform the differentiations, and find that Eq. (12) represents a solution of Eq. (11) if the following is true for the parameters α and ω:

(13) $$\alpha = \frac{b}{2m}$$ and

(14) $$\omega = \sqrt{{\omega_0}^2 - \left(\frac{b}{2m}\right)^2}$$

We will now interpret this result. First we note that the amplitude of the oscillation decreases more rapidly the larger the *damping constant* (or *damping coefficient*) α is. In the case of invariable mass this means according to Eq. (13) that the amplitude of the oscillation decreases more rapidly the larger the constant of friction b is - which is quite plausible.

From Eq. (14) we can read how the angular eigenfrequency ω of this damped harmonic oscillation changes with the constant of friction b. We study the following different cases:

(i) $b = 0$ $\rightarrow$ $\omega = \omega_0$

In the case of vanishing friction ($b = 0$) we have the case of the undamped harmonic oscillator as discussed in Chapter 2.1; the sphere performs a *periodic oscillation* at the angular eigenfrequency ω_0.

(ii) $b/(2m)^2 = \omega_0^2$ $\rightarrow$ $\omega = 0$

This is the case of „critical damping“ in which the sphere does not perform a periodic oscillation any more. It is therefore called the *aperiodic borderline case*. The sphere only returns to its starting position exponentially (cf. remarks).

(iii) $b/(m)^2 > \omega_0^2$ $\rightarrow$ ω imaginary

In the case of „supercritical damping” there is no periodic oscillation either. This case is called *aperiodic case* or *over damped case*. Here again, the sphere only returns to its starting position, however, with additional damping, i.e., more slowly (cf. remarks).

(iv) $0 < b < 2m\omega_0$ $\rightarrow$ $\omega < \omega_0$

This most general case, the *oscillation case*, leads to a periodic oscillation at a frequency ω, which is below the angular eigenfrequency ω_0 of the undamped harmonic oscillator.

Remarks:

Under the conditions discussed above ($v(t = 0) = 0$) there is no considerable difference between the case of *critical damping* and *supercritical damping*: In both cases the sphere returns to its starting position along an exponential path; in the case of supercritical damping there is only a stronger damping. We find a different situation in the case $v(t = 0) \neq 0$. If we do not only release the sphere, but push it thus giving it a certain starting velocity, it is possible in the case of *critical damping* that the sphere oscillates beyond its position at rest once, and only then returns to its starting position along an exponential path. In the case of *supercritical damping* such an oscillation beyond that position does not occur. The sphere always returns to its position at rest along an exponential path. Detailed calculations (solution of the differential equation (11) under the conditions (2) and (3)) confirm these relationships.

2.3 Forced harmonic oscillations

In Chapters 2.1 and 2.2 we have observed how the sphere oscillates if we deflect it *once* from its position at rest and then release it. Now we will investigate which oscillations the sphere performs if the system is subject to a *periodically* changing external force F_e (Fig. 4), for which the following is true:

(15) $$F_e = F_1 \sin(\omega_1 t)$$

F_1 is the amplitude of the external force and ω_1 its angular frequency. The sign is chosen such that the forces directed downwards are counted as positive and upward forces are counted as negative.

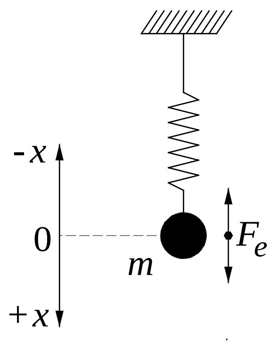


Fig. 4: Oscillation generation of a mass/spring system with an external force F_e, m being the mass according to Eqs. (6) and (7).

The external force F_e additionally acts on the spring; the equation of motion thus gets the form (cf. Eqs. (10) and (11)):

$$(16)\qquad m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -D\,x - b\,\frac{\mathrm{d}x}{\mathrm{d}t} + F_e$$

and hence

$$(17)\qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{D}{m}x = \frac{1}{m}F_1\sin\left(\omega_1 t\right)$$

It is expected that the motion of the sphere following a certain *transient time*, i.e., after the *transient oscillation*, occurs at the same frequency as does the change of the external force. (There would be no plausible explanation for another frequency.) However, a *phase shift* ϕ between the stimulating force and the deflection of the sphere could be assumed. We may expect the oscillation amplitude to remain constant upon completion of the transient oscillation since the system is provided with new external energy again and again. Based on these considerations the following ansatz is suggested for the differential equation (17):

$$(18)\qquad x = x_0\sin\left(\omega_1 t + \phi\right)$$

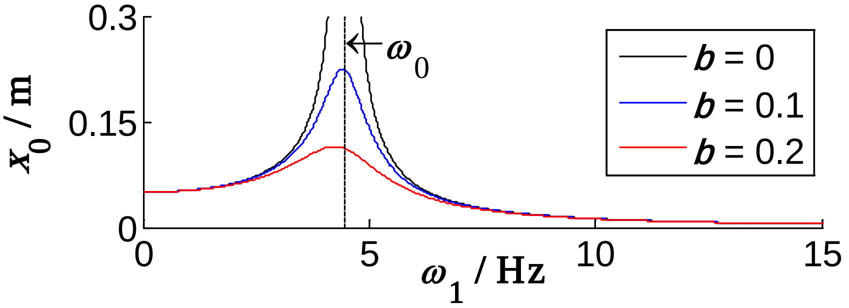
In this case ϕ is the phase shift between the deflection x and the external force F_e. For $\phi<0$ the deflection lags behind the stimulating force. By inserting Eq. (18) into Eq. (17) we find that Eq. (18) represents a solution of Eq. (17) if the following is true for the amplitude x_0 and the phase shift ϕ (cf. Appendix):

$$(19)\qquad x_0 = \frac{\dfrac{F_1}{m}}{\sqrt{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(\dfrac{\omega_1 b}{m}\right)^2}}$$

$$(20)\qquad \phi = \arctan\left(\frac{\omega_0^2 - \omega_1^2}{\dfrac{\omega_1 b}{m}}\right) - \frac{\pi}{2}$$

Contrary to the cases discussed in Chapters 2.1 and 2.2, the amplitude x_0 and the phase ϕ are no longer freely selectable parameters, rather they are definitely determined by the quantities F_1, ω_1, m, b and $\omega_0^2 = D/m$.

Eq. (19) shows that the amplitude of the sphere's motion depends on the stimulating frequency. Plotting x_0 over ω_1, we obtain the *amplitude resonance curve*. Fig. 5 (top) shows some typical amplitude resonance curves for different values of the friction constant b. In the stationary case, i.e. for $\omega_1 = 0$, we obtain the amplitude $x_0 = F_1/D$ known from HOOKE's law from Eq. (19). This is the value by which the sphere is deflected if it is affected by a constant force F_1. The position of the maximum of x_0 as a function of ω_1 is found by means of the condition $\mathrm{d}x_0/\mathrm{d}\omega_1 = 0$. From Eq. (19) follows:



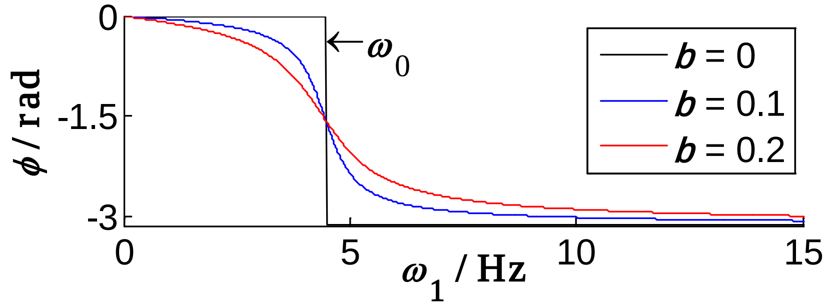


Fig. 5: Amplitude resonance curves (top) and phase curves (bottom) for a damped harmonic oscillator. (F_1 = 0.1 N, m = 0.1 kg, D = 2 kg/s^2, b in kg/s).

(21) $$\omega_1 = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} \quad \rightarrow \quad x_0 = \text{Max.}$$

Except for the case $b = 0$, the maximum of the amplitude resonance curve is thus *not* found at the angular eigenfrequencies ω_0, but at lower angular frequencies $\omega_1 < \omega_0$.

The lower part of Fig. 5 shows the so called phase curves which determine the development of the phase shift ϕ as a function of the angular frequency ω_1. From Eq. (20) it follows that the phase angle ϕ is always *negative*, i.e., the deflection of the sphere always *lags behind* the stimulating force except for the case $\omega_1 = 0$.

We will now discuss some special cases:

(i) In the case $\omega_1 << \omega_0$ the amplitude $x_0 \approx F_1/D$ is independent of b for „not too large" b. The phase shift ϕ goes to 0 in this case: $\phi \approx 0°$. Thus the sphere directly follows the stimulating force.

(ii) In the *resonance case* $\omega_1 \approx \omega_0$, the amplitude equals $x_0 \approx F_1/(\omega_0 b)$, i.e., it is dependent on b. The smaller b is, the larger x_0 becomes; for $b \rightarrow 0$, $x_0 \rightarrow \infty$. In this case the sphere's deflection lags behind the generating force by 90° ($\phi \approx -\pi/2$).

(iii) In the case $\omega_1 >> \omega_0$ we find $x_0 \approx F_1/(m\omega_1^2)$, i.e., the amplitude drops by $1/\omega_1^2$. The phase shift is $\phi = -\pi$ in this case, i.e., the sphere's deflection lags behind the generating force by 180°.

From the amplitude resonance curves the *damping behaviour* of a *mass-spring-system* can be read, i.e. of a vibration isolating table, which is frequently used in optical precision metrology. The eigenfrequencies of such tables are in the range of about 1 Hz. If an external disturbance has a very low frequency ($\omega_1 \rightarrow 0$), its amplitude is transferred onto the table without damping. In the range of the eigenfrequency ($\omega_1 \approx \omega_0$) it is (unintentional) amplified and in the range of high frequencies ($\omega_1 >> \omega_0$) it is damped strongly.

The damping behaviour of such a system can be influenced by changing the mass *m*. Fig. 7 shows that a larger *m* reduces the eigenfrequency with the other parameters remaining unchanged and that the damping for frequencies above the eigenfrequency can be increased significantly.

(22) $$v = \frac{dx}{dt} = \omega_1 x_0 \cos\left(\omega_1 t + \phi\right) := v_0 \cos\left(\omega_1 t + \phi\right)$$

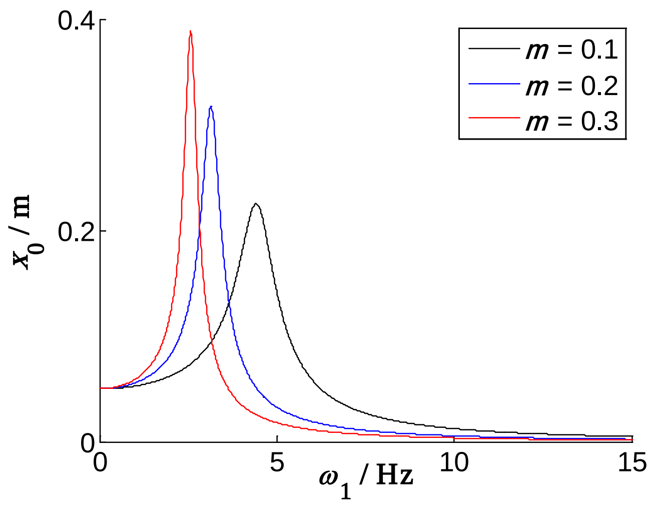


Fig. 6: Amplitude resonance curves for different masses *m* (in kg) with other parameters remaining unchanged (F = 0.1 N, D = 2 kg/s^2, b = 0.1 kg/s)

Finally we will examine at which frequency the maximal *energy transfer* occurs from the generating system to the oscillating system. As we know that the maximal kinetic energy is equivalent to the maximum velocity, we first calculate the temporal course of the velocity *v* of the sphere using Eq. (18):

With Eq. (19) we obtain for the maximal velocity v_0:

(23) $$v_0 = \omega_1 x_0 = \frac{\frac{\omega_1 F_1}{m}}{\sqrt{\left(\omega_1^2 - \omega_0^2\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2}}$$

and hence:

(24) $$v_0 = \frac{F_1}{\sqrt{\left(m\omega_1 - \frac{D}{\omega_1}\right)^2 + b^2}}$$

v_0 becomes maximal when the denominator of Eq. (24) becomes minimal, i.e., if the following is true (for b ≠ 0):

(25) $$\frac{D}{\omega_1} - m\omega_1 = 0 \quad \rightarrow \quad v_0 = \text{Max.}$$

Hence it follows:

(26) $$\omega_1 = \sqrt{\frac{D}{m}} = \omega_0 \quad \rightarrow \quad v_0 = \text{Max.}$$

Thus the velocity and also the kinetic energy become maximal if the system is stimulated with its angular eigenfrequency ω_0. Therefore, this case is called *energy resonance*, a case in which the generating system can transfer the maximal energy to the oscillating system.

Question 2:

- What is the typical course of energy resonance curves ($\sim v_0^2(\omega_1)$)? Sketch a diagram with the principal course of $v_0^2(\omega_1)$ for the cases $b \approx 0$, $b = b_1$ and $b = b_2$ (according to Fig. 5).

Summing up we note the important result that a maximum transfer of energy and a maximum oscillation amplitude are achieved at different generating frequencies in the case $b \neq 0$.

3 Experimental procedure

Equipment:

Spring ($D = (22.7 \pm 0.5)$ kg/s^2, $m_F = (0.0575 \pm 10^{-4})$ kg), sphere with suspension bar and indicator ($M = (0.1727 \pm 3 \times 10^{-4})$ kg), generation system on stand with motor and light barrier, power supply for motor and light barrier of the generation system, electronic speed controller for motor, halogen bulb with condenser and power supply, lens (focal length $f \approx 200$ mm, $\varnothing \approx 80$ mm), photo diode with pinhole diaphragm ($\varnothing = 0.5$ mm) on stand, power supply for photo diode, observation screen on stand, CCD camera with TV monitor and line selector, stand material for mounting lens, bulb, and CCD camera, 2 glasses with different glycerine/water mixtures ($b = (0.72 \pm 0.03)$ kg/s for the more viscous mixture at $T = 20$°C), desk for lens, lamp and CCD camera, digital oscilloscope, electronic counter, metal measuring tape,.

3.1 Description of experimental set-up

The experiments are performed in a set-up as sketched in Fig. 7. By optically imaging the sphere's oscillation on a) a TV monitor and b) an observation screen it is possible to measure amplitude resonance curves and phase curves without contact. This set-up is described in the following:

A sphere of mass m_K is suspended on a spring by means of a bar S. The sphere is plunged into a glass GL filled with a glycerine/water mixture to damp its oscillation. We assume that the frictional force is proportional to the velocity of the sphere. There is an indicator Z fixed on the bar S, which is illuminated with a halogen bulb from the right side. The indicator (with this illumination, its shadow) is imaged on a screen B by means of the lens L (focal length $f \approx 200$ mm) such that a magnified shadow Z' of the indicator is produced on the screen. The image distance b' is chosen to be approx. 2 m, which yields an object distance g of approx. 0.2 m according to the laws of geometric optics (imaging equation):

In order to obtain a good image, we place the lamp at a distance of 30 cm from the indicator and place the condenser K such that the *opening of the lens* L is illuminated uniformly and in a straight line in the direction of the optical axis. The indicator and the centre of the lamp and lens should be at the same level when the plunged sphere ($x = 0$, central position of the bar S_1) is at its zero position.

Now we look at the suspension of the spring. It is connected to a bar P of the length l via a joint G_1 with a bar S which runs in a guide F. The bar P is fixed on a rotary disk D via a ball bearing joint G_2. The disk can be turned at an angular frequency ω_1 via a motor. Thus, the suspension point of the spring (i.e. the position of the joint G_1) is set in a periodic motion and a time-dependent external force F_e is produced on the spring.

4 Appendix

We want to demonstrate that the resonance amplitude x_0 and the phase shift ϕ can be calculated with a few simple calculation steps, if we change over to complex representation. In complex representation Eq. (17) reads:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{D}{m}x = \frac{1}{m}F_1\,\mathrm{e}^{i\omega_1 t} \tag{34}$$

In analogy to Eq. (18) we choose as a complex approach:

$$x = x_0\, e^{i(\omega_1 t + \phi)} = x_0\,\mathrm{e}^{i\omega_1 t}\,\mathrm{e}^{i\phi} \tag{35}$$

Following differentiation and division by $e^{i\omega_1 t}$ insertion of Eq. (35) into Eq. (34) yields:

$$-\omega_1^2\, x_0\,\mathrm{e}^{i\phi} + i\,\omega_1 \frac{b}{m} x_0\,\mathrm{e}^{i\phi} + \frac{D}{m} x_0\,\mathrm{e}^{i\phi} = \frac{F_1}{m} \tag{36}$$

Hence it follows with the definition of the angular eigenfrequency ω_0 according to Eq. (6):

$$x_0\,\mathrm{e}^{i\phi} = \frac{\frac{F_1}{m}}{\omega_0^2 - \omega_1^2 + i\omega_1 \frac{b}{m}} := z \tag{37}$$

As already demonstrated in the experiment on the measurement of capacities, Eq. (37) is *one* representation form of a complex number z, whose absolute value (modulus) $|z| = x_0$ is given by $\sqrt{zz^*}$, with z^* being the conjugate complex quantity of z. Hence it follows:

$$x_0 = \sqrt{zz^*} = \sqrt{\left(\frac{\frac{F_1}{m}}{\omega_0^2 - \omega_1^2 + i\omega_1 \frac{b}{m}}\right)\left(\frac{\frac{F_1}{m}}{\omega_0^2 - \omega_1^2 - i\omega_1 \frac{b}{m}}\right)} \tag{38}$$

from which we obtain Eq. (19) by simple multiplication.

For calculating the phase angle we again use (cf. experiment on the measurement of capacities) the *second* representation of complex numbers, namely $z = \alpha + i\beta$, α being the real part and β, the imaginary part of z. As is generally known, the phase angle ϕ can be calculated from these quantities as

$$\phi = \arctan\left(\frac{\beta}{\alpha}\right)\begin{Bmatrix} +\pi & \text{for } \alpha < 0 \text{ and } \beta \geq 0 \\ -\pi & \text{for } \alpha < 0 \text{ and } \beta < 0 \end{Bmatrix} \tag{39}$$

In order to convert Eq. (37) into the form $\alpha + i\beta$, we extend the fraction in Eq. (37) with the conjugated complex denominator:

(40) $$x_0\,e^{i\phi} = \frac{\frac{F_1}{m}\left(\omega_0^2-\omega_1^2-i\omega_1\frac{b}{m}\right)}{\left(\omega_0^2-\omega_1^2+i\omega_1\frac{b}{m}\right)\left(\omega_0^2-\omega_1^2-i\omega_1\frac{b}{m}\right)} = \frac{\frac{F_1}{m}\left(\omega_0^2-\omega_1^2\right)-i\frac{F_1}{m}\omega_1\frac{b}{m}}{\left(\omega_0^2-\omega_1^2\right)^2+\left(\frac{\omega_1 b}{m}\right)^2}$$

from which we can read off the quantities α and β:

(41) $$\alpha = \frac{\frac{F_1}{m}\left(\omega_0^2-\omega_1^2\right)}{\left(\omega_0^2-\omega_1^2\right)^2+\left(\frac{\omega_1 b}{m}\right)^2} \qquad \text{und} \qquad \beta = -\frac{\frac{F_1}{m}\omega_1\frac{b}{m}}{\left(\omega_0^2-\omega_1^2\right)^2+\left(\frac{\omega_1 b}{m}\right)^2}$$

which yields by insertion into Eq. (39):

(42) $$\phi = \arctan\left(-\frac{\frac{\omega_1 b}{m}}{\omega_0^2-\omega_1^2}\right)\left\{-\pi \text{ für } \omega_1 > \omega_0\right\}$$

With

(43) $$\arctan(-y) = \arctan\left(\frac{1}{y}\right)-\frac{\pi}{2}$$

it finally yields Eq. (20).