The Harmonic Oscillator

Math 24: Ordinary Differential Equations
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Introduction

The harmonic oscillator is a common model used in physics because of the wide range of problems it can be applied to. For example atoms in a lattice (crystalline structure of a solid) can be thought of as an infinite string of masses connected together by springs, whose equation of motion is oscillatory. In fact, the solutions can be generalized to many systems undergoing oscillations, of which the mass-spring system is just one example. Since the mass-spring system is easy to visualize it will serve as the primary example as we develop a more complete general theory describing harmonic motion.

1 Theory

1.1 Hooke's Force Law

We will begin with the restoring force $F(x)$, where x is a measure of the distance from the origin of the system (taken as $x = 0$). Assuming F is analytic in the sense that it can be described by an infinite order polynomial this implies that F has continuous derivatives of all orders and can be Taylor expanded to form the series

$$F(x) = F_0 + x\left(\frac{dF}{dx}\right)_0 + \frac{1}{2!}x^2\left(\frac{d^2F}{dx^2}\right)_0 + \cdots \tag{1}$$

where $F_0 = F(0)$ and $(d^nF/dx^n)_0$ is the n^{th} derivative of F evaluated at $x = 0$.

We will assume that the perturbations from the origin of the system ($x = 0$) are small so that all second order terms and above can be neglected. Since we began under the assumption $F_0 = 0$ this leaves us with

$$\boxed{F(x) = -kx} \tag{2}$$

where $k \equiv -(dF/dx)_0$. Equation (2) is known as Hooke's Law and describes the class of objects which adhere to elastic deformations.

1.2 General Mass-Spring System (Undamped Motion)

The mass-spring system falls into this group of elastic deformations governed by Hooke's Law. From Newton's second law $F = ma$ we arrive at the equation of motion

$$-kx = m\ddot{x} \tag{3}$$

This gives us a second order linear equation of the form

$$\ddot{x} + \omega_0^2 x = 0 \tag{4}$$

where we have defined

$$\omega_0 \equiv \sqrt{\frac{k}{m}}. \tag{5}$$

This has the general solution

$$\boxed{x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)}. \tag{6}$$

1.3 Damped Motion

While the undamped system provides a measure of elegant beauty and simplicity in its solution it is to a certain extent boring. Once the initial conditions are set it will continue to oscillate forever, never deviating from its simple sinusoidal pattern. This is also unrealistic as any physical system will eventually come to rest. To create a more accurate model a damping (resistive) force must be added.

It does not make sense for this to be a constant force. If a mass-spring system is sitting at rest at its equilibrium point it will not all of a sudden begin moving under the influence of some mysterious force. It also does not make sense for the force to depend on the displacement, or position, of the mass. If the entire system were to be translated intuition says the motion will be the same, simply moved to another location. It then makes sense to model the damping force as a function of the objects velocity. This adds an additional force of the form $F_d = b\mathbf{v}$, so that our equation of motion is

$$m\ddot{x} = -kx - b\dot{x}. \tag{7}$$

Rearranging terms we are left with the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{8}$$

where

$$\beta \equiv \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}} \tag{9}$$

With a bit of foresight the *damping parameter* β has been defined. Guessing the solution $x = A\exp(rt)$ we find the roots of the auxiliary equation to be

$$r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{10}$$

so that the general solution to the equation of motion is

$$\boxed{x(t) = e^{-\beta t}\left[A_1\exp(\sqrt{\beta^2-\omega_0^2}t) + A_2\exp(-\sqrt{\beta^2-\omega_0^2}t)\right]} \tag{11}$$

Equation (11) is similar in form to (6) with the addition of a decaying exponential on the left side. It is this additional term that gives the system the damping we are looking for.

Upon closer examination we find that there are three general cases for a damped harmonic oscillator. They are:

Underdamping:	$\omega_0^2 > \beta^2$
Critical damping:	$\omega_0^2 = \beta^2$
Overdamping:	$\omega_0^2 < \beta^2$

Each case corresponds to a bifurcation of the system. Overdamped is when the auxiliary equation has two roots, as they converge to one root the system becomes critically damped, and when the roots are imaginary the system is underdamped.

Underdamped Motion

We start by defining the characteristic frequency of the underdamped system as

$$\omega_1^2 = \omega_0^2 - \beta^2. \tag{12}$$

For underdamped motion $\omega_1^2 > 0$ so that the roots in (11) are imaginary. This leaves us with

$$x(t) = e^{-\beta t}\left[A_1\exp(i\omega_1 t) + A_2\exp(-i\omega_1 t)\right] \tag{13}$$

This can be simplified to the form

$$\boxed{x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)} \tag{14}$$

which is the general solution for underdamped motion.

Critically Damped Motion

For the case that $\omega_0^2 = \beta^2$ we find ourselves with a double root at $r = -\beta$. To obtain our second linearly independent solution, and thus our general form, we try a particular solution of the form $x_p = At\exp(rt)$. Plugging this into (8) we find that $r = -\beta$ so that our general solution is

$$\boxed{x(t) = (A + Bt)e^{-\beta t}} \tag{15}$$

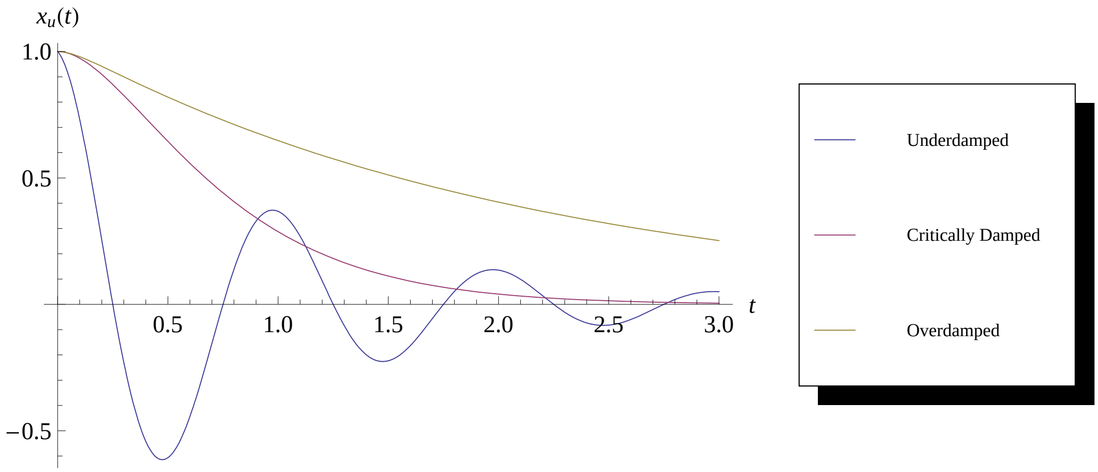

Figure 1: The different forms of damping. Notice how only underdamped crosses the equilibrium point periodically. Both critically damped and overdamped tend to zero at infinity.

Overdamped Motion

For overdamped motion we define the characteristic frequency as

$$\omega_2^2 = \beta^2 - \omega_0^2. \tag{16}$$

This means $\omega_2^2 < 0$, so that the square roots in (11) are positive. Our general solution is then

$$\boxed{x(t) = e^{-\beta t}\left[A_1 \exp(\omega_2 t) + A_2 \exp(-\omega_2 t)\right]} \tag{17}$$

Figure 1 shows the general forms of the different types of damping.

1.4 Driven Harmonic Oscillator

A common situation is for an oscillator to be driven by an external force. We will examine the case for which the external force has a sinusoidal form. The external force can then be written as $F_e = F_0 \cos\omega t$, so that the sum of the forces acting on the mass is

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos\omega t \tag{18}$$

We can rearrange this to the form

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t \tag{19}$$

where the constants are defined as

$$A \equiv \frac{F_0}{m}, \quad \beta \equiv \frac{b}{2m}, \quad \omega_0 \equiv \sqrt{\frac{k}{m}} \tag{20}$$

For the homogeneous solution we have the general solution of a damped harmonic oscillator given by (11),

$$\boxed{x_h(t) = e^{-\beta t}\left[A_1 \exp(\sqrt{\beta^2 - \omega_0^2}t) + A_2 \exp(-\sqrt{\beta^2 - \omega_0^2}t)\right]} \tag{21}$$

In order to find the particular solution we guess a solution of the form $x_p(t) = B\cos(\omega t - \delta)$ (this is equivalent to $x_p(t) = B_1\cos(\omega t) + B_2\sin(\omega t)$ but easier to solve for in this case). Substituting into (19) and expanding the sine and cosine functions we are left with

$$\begin{aligned} &\{A - B[(\omega_0^2 - \omega^2)\cos\delta + 2\omega\beta\sin\delta]\}\cos\omega t \\ &-\{B[(\omega_0^2 - \omega^2)\sin\delta - 2\omega\beta\cos\delta]\}\sin\omega t = 0 \end{aligned} \tag{22}$$

Since $\cos\omega t$ and $\sin\omega t$ are linearly independent functions they must vanish to zero identically. Solving for the $\sin\omega t$ term gives us

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{23}$$

From this we arrive at

$$\sin\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{24}$$

$$\cos\delta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{25}$$

If we now look at the $\cos\omega t$ term from (22) we see that

$$\begin{aligned} B &= \frac{A}{\sqrt{(\omega_0^2 - \omega^2)\cos\delta + 2\beta\omega\sin\delta}} \\ &= \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \end{aligned} \tag{26}$$

so that our particular solution is

$$\boxed{x_p(t) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\cos(\omega t - \delta)} \tag{27}$$

where

$$\boxed{\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)} \tag{28}$$

This makes our general solution the combination of (21) and (27)

$$\boxed{x(t) = x_h(t) + x_p(t)} \tag{29}$$

2 Application

Now that the general solutions for several variations of oscillatory motion have been solved their practical applications can be shown. The following are several examples of common place phenomenon that can be described by the simple harmonic oscillator system.

2.1 Example: Beat Frequency

A "beat" is often heard when multiple frequencies are playing at the same time. The curious part is the frequency heard as the "beat" is not actually being generated by any of the external sources. For example, a 4 Hz (hertz, one oscillation per second) pitch is played at the same time as a 6 Hz pitch. In addition to those two frequencies you can also hear (if you listen closely enough) a third pitch at 2 Hz. Even though only two pitches were played, three are heard.

To explore this phenomenon we will examine a driven oscillator with no damping. Setting $b = 0$, and thus $\beta = b/2m = 0$ in (29) we have the general form

$$x_b(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t + B \sin \omega t \tag{30}$$

where $A = F_0/m$ is the adjusted amplitude of the driving force and B is amplitude of the characteristic wave of the system. Setting $F_0 = 600$, $m = 1$, and $B = 1$ we can plot this equation for $f_0 = 6$ Hz and $f_1 = 7$ Hz. With the relation $\omega = 2\pi f$ we are left with the function

$$x_b(t) = \frac{300}{(2\pi \cdot 6)^2 - (2\pi \cdot 7)^2} \cos(2\pi \cdot 7t) + \sin(2\pi \cdot 6t) \tag{31}$$

Figure 2 shows a simple plot of the above function. The third frequency heard is a result of the linear superposition of the natural frequency of the system and the frequency of the driving force. As seen in the plot, a 1 Hz sine wave matches the resulting periodicity of the third "beat" wave exactly. The Beat frequency is indeed the difference between the two original wave forms.

2.2 Example: Resonance

Resonance is what occurs when two oscillations have the same frequency. To obtain a clear picture of what happens in this situation we will examine the case $\omega = \omega_0$ for an undamped driven oscillator. Plugging this directly into (19) does not work since the second term tends to infinity for this case. We must then start with the general equation of motion for a non-damped, driven oscillator

$$\ddot{x} + \omega_0^2 x = A \cos \omega_0 t \tag{32}$$

The solution to the homogeneous equation is

$$x_h(t) = A_1 \sin \omega_0 t + A_2 \cos \omega_0 t \tag{33}$$

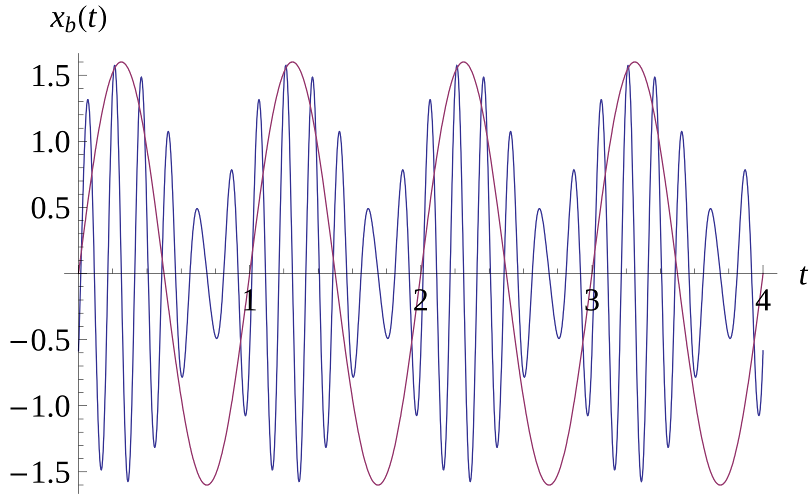

Figure 2: The graph of (31). Superimposed is a simple sine wave with frequency of 1 Hz, showing that the periodicity of the third wave form is the difference between the original 6 and 7 Hz waves.

In order to find the particular solution we guess the form

$$x_p(t) = (B_1 + C_1 t)\sin\omega_0 t + (B_2 + C_2 t)\cos\omega_0 t \tag{34}$$

Plugging into (32) we find that $C_1 = A/(2\omega_0)$, $C_2 = 0$, and the rest are free variables. It follows that B_1 and B_2 are free since they are just the homogeneous solutions we already know satisfy the equation. This leaves us with the general solution for a resonant system of

$$\boxed{x_r(t) = \left(B_1 + \frac{At}{2\omega_0}\right)\sin\omega_0 t + B_2\cos\omega_0 t} \tag{35}$$

As an example take the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. This results in $B_1 = B_2 = 0$, and sets the resonant solution

$$x_r(t) = \frac{At}{2\omega_0}\sin\omega_0 t \tag{36}$$

A plot of this equation can be seen in figure 3 where $F_0 = 5$ and $m = 1$ make $A = 5$, and ω_0 has been set to 1. The oscillations quickly increase in amplitude and will continue to do so in a linear fashion due to the $At/(2\omega_0)$ term. It follows that the peaks will gain in amplitude by

$$\text{Amplitude}(t) = \frac{A}{2\omega_0}t \tag{37}$$

and this is indeed the behavior we see.

It is this *mechanical resonance* of a system that was partly responsible for the infamous collapse of the Tacoma Bridge. As air swept over the bridge, the change in pressure caused the

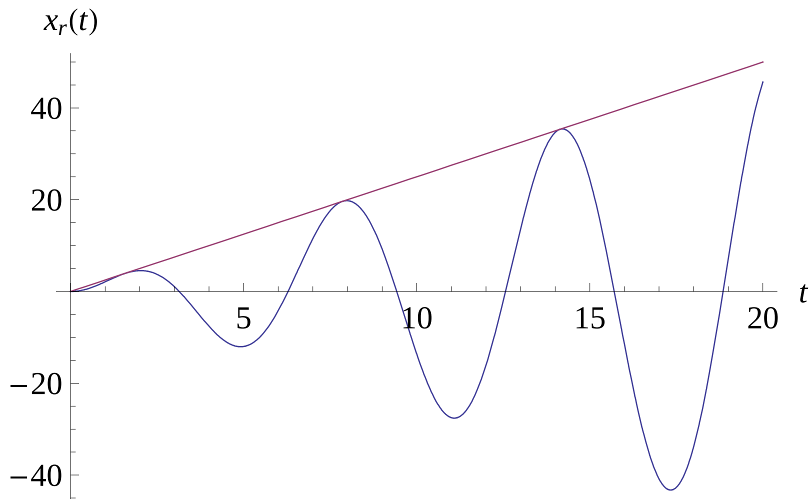

Figure 3: The graph of (32). Although the system starts at the equilibrium position at rest it quickly starts oscillating in an unbounded fashion. The line $x = (5/2)t$ has been plotted to show the linear increase of the amplitude.

bridge to sway back and forth. While this in itself presented no threat, the wind was blowing at just the right speed so that the swaying was in resonance with the natural frequency of the bridge. The amplitude of the sways increased with each cycle until the bridge eventually collapsed.

2.3 Example: Limiting Cases of a Damped, Driven System

The behavior observed from an oscillating systems doesn't just depend on the type of solution choosen (e.g. underdamped, critically damped, or overdamped). By varying the constants a wide range of responses occur in any of the systems. Take the underdamped system from (14)

$$x_u(t) = A \exp(\frac{-b}{2m}t) \cos\left(\left[\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right] t\right) \tag{38}$$

where the constants have been expanded and δ has been set to 0. There are three parameters b, k, and m that we can vary. Since the case of $x \to 0$ is often a physically significant one we will choose it to examine.

Case 1: $\mathbf{b \to 0}$

First we will take the limit of (38) as $b \to 0$ (see figure 4). This is the case were the friction is being decreased, so one would expect the decay in amplitude to also decrease, reaching a limiting case of a regular sinusoidal wave with no decay. Examining (38) as $b \to 0$ the exponential term goes to 1, and the argument of the cosine is simply ωt. Examining the plot this is indeed the behavior we see.

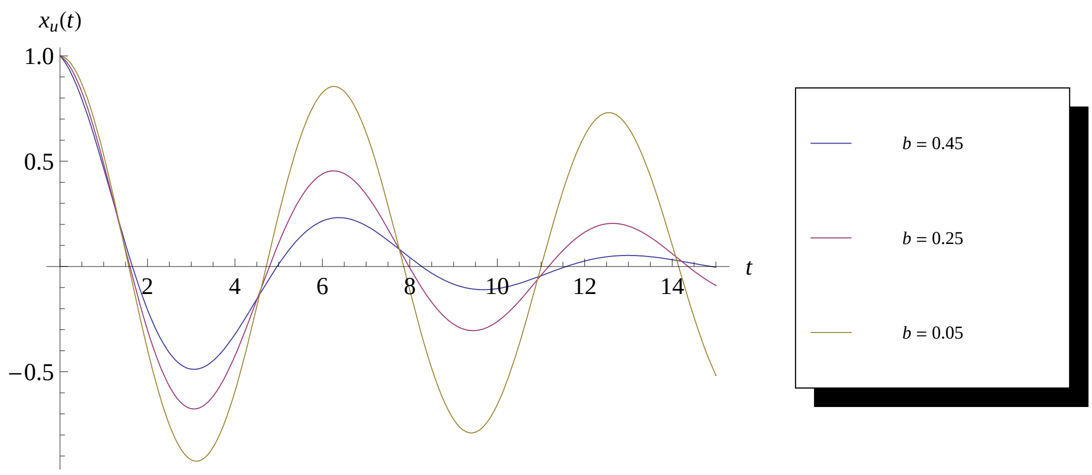

Figure 4: The limit of the underdamped oscillator as $b \rightarrow 0$.

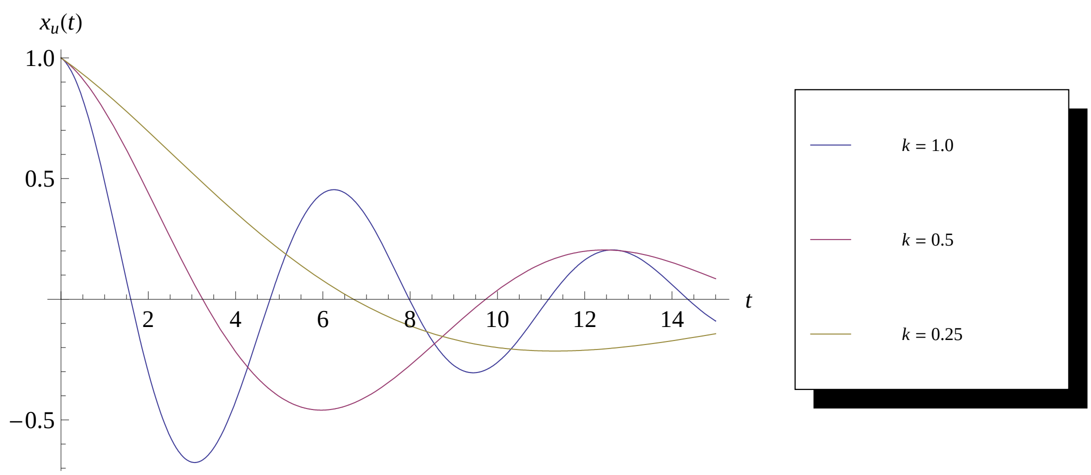

Figure 5: The limit of the underdamped oscillator as $k \rightarrow 0$.

Case 2: $\mathbf{k \rightarrow 0}$

Next we will take the limit as $k \rightarrow 0$. In this case the spring coefficient, which is directly related to the restoring force, is approaching 0. The exponential is unaffected so there will still be a normal decay, but the time it takes to cross the origin will be increased since there is not as much force working to restore it to equilibrium. Equation (38) agrees as decreasing the k term acts to decrease the frequency ω of the wave. By the relation $T = 2\pi/\omega$, where T is the period, we see this decrease in ω in turn increases T and thus the time between equilibrium crossings. Figure 5 confirms this behavior.

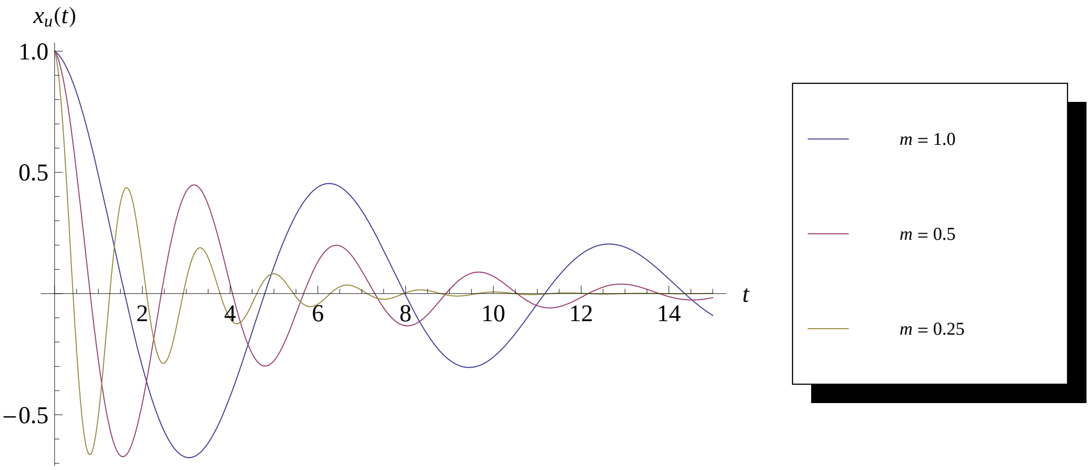

Figure 6: The limit of the underdamped oscillator as $m \to 0$.

Case 3: $\mathbf{m \to 0}$

Finally we have the case $m \to 0$. This case is similar to the previous case where we had $k \to 0$, but shrinking the period instead of lengthening it. A decrease in mass means an easier time for the spring in restoring it to the equilibrium point. Again examining (38) the frequency of the cosine function will be increased by a decrease in m, and will eventually tend to infinity. The cosine term is dominated by $1/m^2$ as for small m it will have an expansion of the form

$$\cos \frac{1}{m^2} t = 1 - \frac{1}{m^2} t + \frac{1}{2!} \frac{1}{m^4} t^2 + \cdots \tag{39}$$

Since the exponential is only $1/m$ to leading order, at small m the change in the function due to the cosine will dominate. This is shown in figure 6.

2.4 Example: Alternate Applications

While the mass-spring system has been the fall back when giving examples in this paper there are many other systems which exhibit the same time dependent, oscillatory behavior. One example is an RLC circuit (Resistor Inductor Capacitor circuit). By Kirchoff's loop rule the sum of the voltages around the entirety of the circuit must be zero, so adding up the voltage drops in figure 7 we see

$$V_{source} = V_{resistor} + V_{inductor} + V_{capacitor} \tag{40}$$

The voltage drops across each component are well known, and are given by the equations

$$V_{inductor} = L \frac{dI}{dt} = L\ddot{q} \tag{41}$$

$$V_{resistor} = RI = R\dot{q} \tag{42}$$

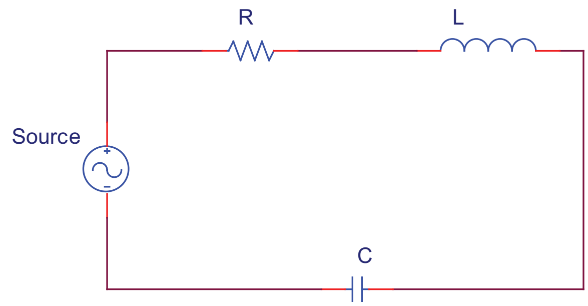

Figure 7: A Simple RLC Circuit.

$$V_{capacitor} = \frac{q}{C} \tag{43}$$

where q is the charge, $dq/dt = I$ is the current, L is the inductance, R is the resistance, and C is the capacitance of the circuit. If we assume the source voltage is an oscillating voltage source of the form $V_{source} = V_0 \sin \omega t$ we have a differential equation describing charge

$$\boxed{L\ddot{q} + R\dot{q} + \frac{q}{C} = V_0 \sin \omega t} \tag{44}$$

This is the same form as the general equation of motion for a damped, driven harmonic oscillator.

References

[1] Stephen T. Thornton and Jerry B. Marion, *Classical Dynamics of Particles and Systems, 5th Edition*, California: Thomson Books/Cole 2004