$$A_1 = -E_0, \tag{4-167}$$

and that

$$B_n = \frac{b_n}{2n+1}\frac{a^{n+2}}{\epsilon_0}. \tag{4-168}$$

To evaluate the b_n's, we use the fact that the potential is zero at $r = a$. Substituting the above coefficients into Eq. *4-139* and setting $r = a$, we have

$$0 = -E_0 a P_1(\cos\theta) + \frac{b_0 a}{\epsilon_0} + \frac{b_1 a}{3\epsilon_0} P_1(\cos\theta) + \frac{b_2 a}{5\epsilon_0} P_2(\cos\theta) + \cdots, \tag{4-169}$$

which must be true for all θ. Thus both the term $b_0 a/\epsilon_0$, which is independent of θ, and the coefficients of all the P_n's must be equal to zero. Thus

$$b_0 = 0, \tag{4-170}$$

and

$$-E_0 a + \frac{b_1 a}{3\epsilon_0} = 0, \tag{4-171}$$

or

$$b_1 = 3\epsilon_0 E_0. \tag{4-172}$$

All other b_n's are zero.

The potential V at any point (r, θ) is thus given by substituting into Eq. *4-139* $A_1 = -E_0$ as in Eq. *4-167*, and

$$B_1 = E_0 a^3, \tag{4-173}$$

as in Eqs. *4-168* and *4-172*. The field is the same as that found in Eq. *4-151*.

The surface charge density $\sigma(\theta')$ on the conducting sphere can be obtained from Eq. *4-164* now that the b_n's are known: we find the value previously found in Eq. *4-156*.

4.6.2. Dielectric Sphere in a Uniform Electrostatic Field. We can calculate this field by either of the formal methods discussed above if we write a general solution as in Eq. *4-139* for points outside the sphere and write another solution with different coefficients for points inside the sphere. The coefficients must be chosen such that the boundary conditions are satisfied:

$V \longrightarrow -E_0 r\cos\theta$ $\quad(r \longrightarrow \infty)$;

V is continuous across the boundary $\quad(r = a)$;

the normal component of $\mathbf{D}$ is continuous $\quad(r = a)$.

Instead of following such a formal procedure, however, we shall write down a combination of spherical harmonics which will satisfy all the boundary conditions.

Outside the sphere, we must have $-E_0 r\cos\theta$ as one of the terms in the solution to satisfy the condition at $r \longrightarrow \infty$. Furthermore, this is the only harmonic with a positive power of r which we can permit, for otherwise the condition at $r \longrightarrow \infty$ would be violated. As regards this condition, all the terms with inverse powers of r are acceptable.

Consider now the solution for points inside the dielectric sphere. No inverse powers at all are permissible here, since such terms would make the potential infinite at the center. This is clearly impossible, since the only charges in the system are those which produce the field E_0 and those induced on the surface of the sphere, if we assume a Class A dielectric, with the result that no volume distribution of induced charge exists.

Writing V_0 for the potential outside the sphere and V_i for that inside, we have

$$V_0 = -E_0 r \cos\theta + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta), \tag{4-174}$$

$$V_i = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta). \tag{4-175}$$

We also require that

$$V_0(a,\theta) = V_i(a,\theta) \tag{4-176}$$

and that

$$-\left(\frac{\partial V_0(r,\theta)}{\partial r}\right)_{r=a} = -\left(K_e \frac{\partial V_i(r,\theta)}{\partial r}\right)_{r=a}, \tag{4-177}$$

where K_e is the dielectric coefficient of the sphere. These are the second and third boundary conditions discussed above. Therefore

$$-E_0 a P_1(\cos\theta) + \frac{B_0}{a} + \frac{B_1 P_1(\cos\theta)}{a^2} + \frac{B_2 P_2(\cos\theta)}{a^3} + \cdots$$
$$= C_0 + C_1 a P_1(\cos\theta) + C_2 a^2 P_2(\cos\theta) + \cdots, \tag{4-178}$$

and

$$E_0 P_1(\cos\theta) + \frac{B_0}{a^2} + \frac{2B_1 P_1(\cos\theta)}{a^3} + \frac{3B_2 P_2(\cos\theta)}{a^4} + \cdots$$
$$= -K_e C_1 P_1(\cos\theta) - 2K_e C_2 a P_2(\cos\theta) + \cdots. \tag{4-179}$$

In order that Eqs. *4-178* and *4-179* be true for all values of θ, the coefficient of each Legendre polynomial on the left side must be equal to the coefficient of the same Legendre polynomial on the right side. Thus, from Eq. *4-178*,

$$\frac{B_0}{a} = C_0, \tag{4-180}$$

$$-E_0 a + \frac{B_1}{a^2} = C_1 a, \tag{4-181}$$

$$\frac{B_2}{a^3} = C_2 a^2, \cdots, \tag{4-182}$$

and, from Eq. *4-179*,

$$\frac{B_0}{a^2} = 0, \tag{4-183}$$

$$E_0 + \frac{2B_1}{a^3} = -K_e C_1, \tag{4-184}$$

$$\frac{3B_2}{a^4} = -2K_e C_2 a. \tag{4-185}$$

These sets of equations lead to the following values for the coefficients:

$$B_0 = C_0 = 0, \tag{4-186}$$

$$B_1 = \left(\frac{K_e - 1}{K_e + 2}\right) E_0 a^3, \tag{4-187}$$

$$C_1 = -\frac{3E_0}{K_e + 2}, \tag{4-188}$$

$$B_n = C_n = 0 \qquad (n > 1). \tag{4-189}$$

Thus

$$V_0(r, \theta) = -\left[1 - \left(\frac{K_e - 1}{K_e + 2}\right)\frac{a^3}{r^3}\right] E_0 r \cos\theta, \tag{4-190}$$

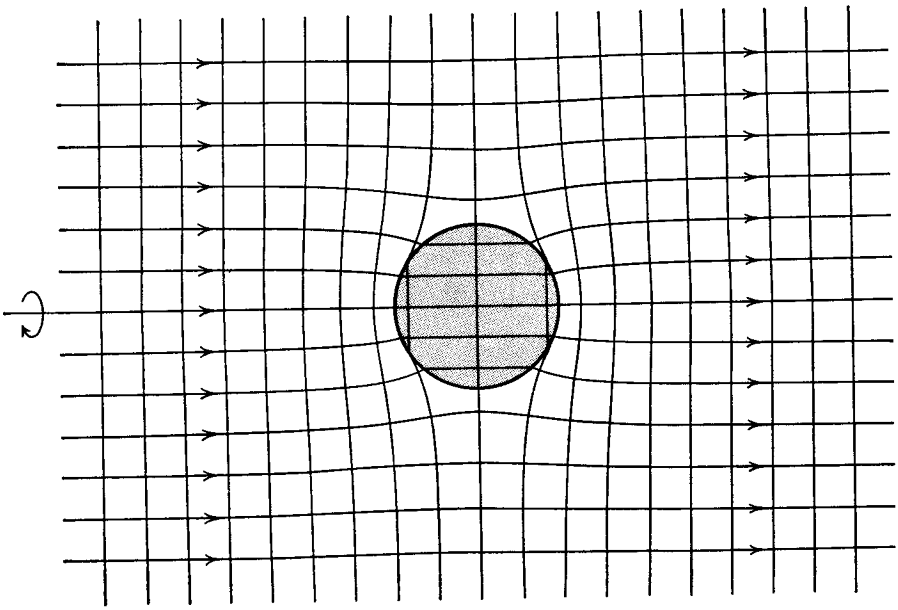

Figure 4-24. *The field near a* dielectric sphere *in a uniform electrostatic field. The lines of electric displacement (indicated by arrows) crowd into the sphere as shown, with the result that D is larger inside than outside. Since there is no free charge at the surface of the sphere, the lines of D neither originate nor terminate there, and they are continuous across the boundary. The equipotentials spread out inside, corresponding to a* lower *electric field intensity E. The electric field intensity E is discontinuous at the surface, and the density of lines of force is lower inside than outside. As in the conducting sphere, the field is hardly disturbed at distances larger than one radius from the surface. The field inside is uniform.*

and

$$V_i(r, \theta) = -\left(\frac{3}{K_e + 2}\right) E_0 r \cos\theta = -\left(\frac{3}{K_e + 2}\right) E_0 z. \qquad (4\text{-}191)$$

We may calculate the field intensity inside and outside the sphere by calculating $-\nabla V$ from Eqs. *4-190* and *4-191*. It will be observed that the field inside the sphere is uniform, is along z, and is given by

$$E_i = \left(\frac{3}{K_e + 2}\right) E_0. \qquad (4\text{-}192)$$

The lines of force and equipotentials are shown in Figure 4-24.

4.6.3. Uniformly Charged Ring. As a final example of a field involving spherical harmonics, let us consider a thin ring of radius a, carrying a charge Q as in Figure 4-25. We wish to find the electrostatic potential V at a point $P(r, \theta)$ such that $r \geqq a$. The uniform charge on the ring assures azimuthal symmetry for the potential. Equation *4-139* again applies, but we can rule out all terms with positive powers of r since we require that $V \longrightarrow 0$ as $r \longrightarrow \infty$. The potential at P must therefore be of the form

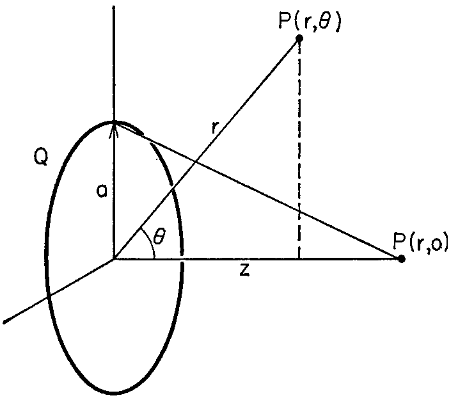

Figure 4-25. *A ring of radius a carrying a total charge Q.*

$$V(r, \theta) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \qquad (4\text{-}193)$$

We shall proceed as we did in the latter part of Section 4.6.1. On the axis, where $\theta = 0$ and $r = z$, we have $P_n(\cos\theta) = 1$ and

$$V(z, 0) = \frac{B_0}{z} + \frac{B_1}{z^2} + \frac{B_2}{z^3} + \cdots. \qquad (4\text{-}194)$$

We can, however, calculate the potential on the axis directly from Coulomb's law, and if we expand the resultant expression in inverse powers of z we may match coefficients term by term with Eq. *4-194* to determine the B_n's. The axis thus provides us with the equivalent of a boundary condition.

Following this procedure, we have

$$V(z, 0) = \frac{Q}{4\pi\epsilon_0(a^2 + z^2)^{1/2}} = \frac{Q}{4\pi\epsilon_0 z}\left(1 + \frac{a^2}{z^2}\right)^{-1/2}, \qquad (4\text{-}195)$$

$$= \frac{Q}{4\pi\epsilon_0 z}\left(1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\frac{a^4}{z^4} - \frac{5}{16}\frac{a^6}{z^6} + \cdots\right). \qquad (4\text{-}196)$$

On matching coefficients with Eq. *4-194*, we find that

$$B_0 = \frac{Q}{4\pi\epsilon_0}, \tag{4-197}$$

$$B_1 = 0, \tag{4-198}$$

$$B_2 = -\frac{Q}{4\pi\epsilon_0}\frac{a^2}{2}, \tag{4-199}$$

$$B_3 = 0, \cdots, \tag{4-200}$$

and, from Eq. *4-193*,

$$V(r, \theta) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{2}\frac{a^2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right]. \tag{4-201}$$

Figure 4-26 shows the equipotential lines in this case. The components of the field intensity may be found, as usual, by calculating $-\nabla V$.

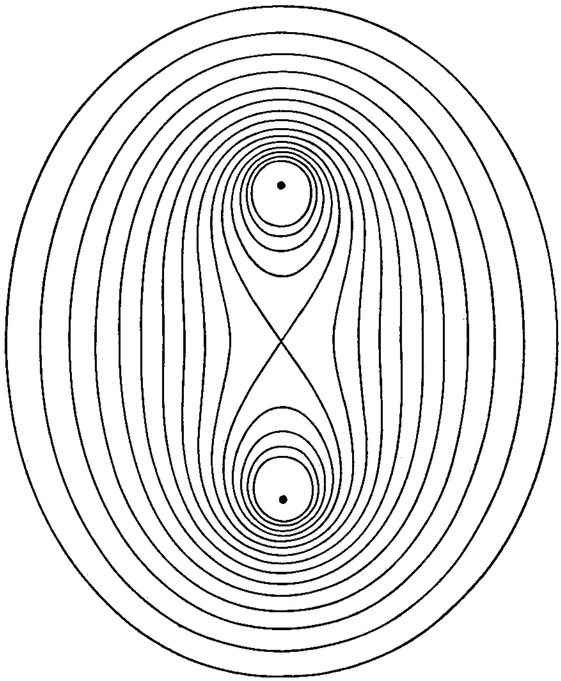

Figure 4-26

Equipotentials for a charged ring. None are shown in the vicinity of the ring, where they are too close together to be depicted graphically. At about two diameters from the ring the equipotentials are approximately circular, and the field is quite similar to that of a point charge.

4.7. Solutions of Poisson's Equation

We have as yet dealt only with solutions to Laplace's equation, since we have concerned ourselves only with cases in which the charge density ρ is zero. As we pointed out earlier, however, there are important fields in which a *space charge* exists and in which ρ is not zero. For these, we must find a solution of