TABLE 4-2. Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{3}{2}\cos^2\theta - \frac{1}{2}$
3	$\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$
4	$\frac{35}{8}\cos^4\theta - \frac{15}{4}\cos^2\theta + \frac{3}{8}$

A general solution of Laplace's equation in spherical polar coordinates, assuming axial symmetry, is therefore the following:

$$V = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \tag{4-139}$$

The various terms are shown in Table 4-3.

TABLE 4-3. Solutions of Laplace's Equation in Spherical Polar Coordinates in the Case of Axial Symmetry

n	$r^n P'_n\ (\cos\theta)$	$r^{-(n+1)} P'_n \cos\theta$
0	1	r^{-1}
1	$r\cos\theta$	$r^{-2}\cos\theta$
2	$\frac{1}{2}r^2(3\cos^2\theta - 1)$	$\frac{1}{2}r^{-3}(3\cos^2\theta - 1)$
3	$\frac{1}{2}r^3(5\cos^3\theta - 3\cos\theta)$	$\frac{1}{2}r^{-4}(5\cos^3\theta - 3\cos\theta)$
4	$\frac{1}{8}r^4(35\cos^4\theta - 30\cos^2\theta + 3)$	$\frac{1}{8}r^{-5}(35\cos^4\theta - 30\cos^2\theta + 3)$

It can be shown that the functions in Eq. *4-139* are a *complete set of functions*, thus an arbitrary boundary condition with axial symmetry can be satisfied with such an infinite series. Moreover, any function of the polar angle θ can be represented as a series of Legendre polynomials, provided the function is continuous within the range of θ considered and provided the function has a finite number of maxima and minima.

It can be shown that

$$\int_{-1}^{+1} P_m(\cos\theta) P_n(\cos\theta)\, d(\cos\theta) = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{2}{2n+1} & \text{if } m = n. \end{cases} \tag{4-140}$$

This property of *orthogonality* of the Legendre polynomials is important in evaluating the coefficients of Eq. *4-139*.

4.6.1. Conducting Sphere in a Uniform Electrostatic Field. To illustrate the use of Eq. *4-139* in calculating electrostatic fields, we consider the case of an

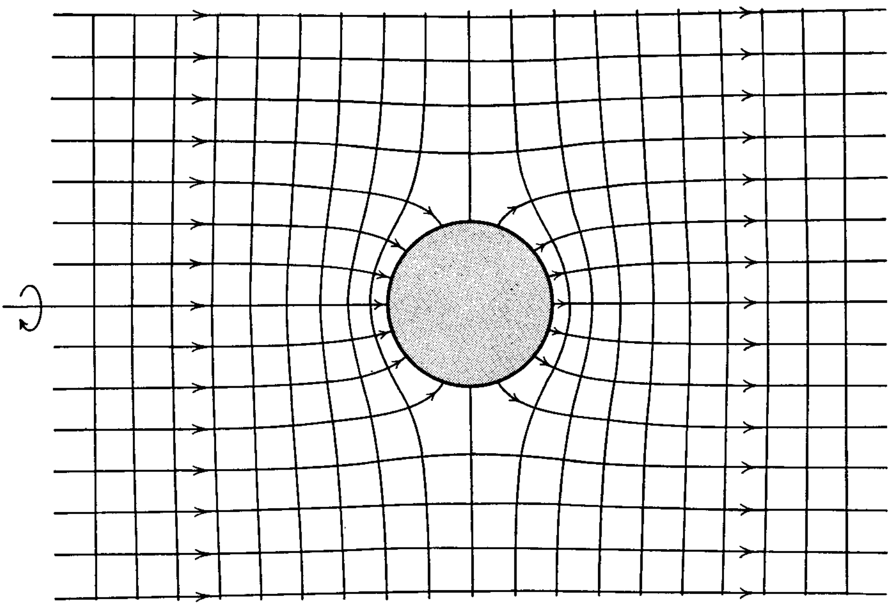

Figure 4-22. *Lines of force (indicated by arrows) and equipotentials for a* conducting sphere *in a uniform electrostatic field. The lines of force are normal at the surface of the sphere, and there is zero electric intensity inside. Observe that the field is hardly disturbed at distances larger than one* radius *from the surface of the sphere.*

insulated conducting sphere situated in a uniform electrostatic field $\mathbf{E}_0$, as in Figure 4-22.

At any point, either inside or outside the sphere, the electric field intensity is that due to the induced charges plus $\mathbf{E}_0$. We assume that the charges which produce $\mathbf{E}_0$ are so far away that they are unaffected by the presence of the sphere. The induced charges arrange themselves on the conducting sphere such that the total field is zero *inside*. Outside the sphere, the total field is of course not zero; we shall calculate it by solving Laplace's equation.

The field is best described in terms of spherical polar coordinates with the origin at the center of the sphere and the polar axis along $\mathbf{E}_0$. Our boundary conditions are then

$$V = 0 \qquad (r = a), \qquad \textit{(4-141)}$$

$$V = -E_0 z = -E_0 r \cos\theta \qquad (r = \infty). \qquad \textit{(4-142)}$$

At $r = a$, from Eqs. *4-139* and *4-141*,

$$0 = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta). \qquad \textit{(4-143)}$$

The method of evaluating the coefficients A_n and B_n is similar to that which we used for evaluating the C_n's of Eq. *4-92*. We multiply both sides of the equation by $P_m(\cos\theta)$ and integrate from $\cos\theta = -1$ to $+1$:

$$0 = \sum_{n=0}^{\infty} \int_{-1}^{+1} A_n a^n P_n(\cos\theta) P_m(\cos\theta)\, d(\cos\theta)$$
$$+ \sum_{n=0}^{\infty} \int_{-1}^{+1} B_n a^{-(n+1)} P_n(\cos\theta) P_m(\cos\theta)\, d(\cos\theta). \qquad (4\text{-}144)$$

According to Eq. *4-140*, the only nonvanishing terms are those for which $n = m$, thus

$$0 = A_n a^n \int_{-1}^{+1} P_n^2(\cos\theta)\, d(\cos\theta) + B_n a^{-(n+1)} \int_{-1}^{+1} P_n^2(\cos\theta)\, d(\cos\theta), \qquad (4\text{-}145)$$

$$= A_n a^n \left(\frac{2}{2n+1}\right) + B_n a^{-(n+1)} \left(\frac{2}{2n+1}\right). \qquad (4\text{-}146)$$

Thus

$$B_n = -A_n a^{2n+1}. \qquad (4\text{-}147)$$

As $r \longrightarrow \infty$ the potential V is given by Eq. *4-142*, all the terms involving inverse powers of r go to zero, and

$$-E_0 r P_1(\cos\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta). \qquad (4\text{-}148)$$

Inspection of Eq. *4-148* shows that the only term which is not zero on the right hand side is that for which $n = 1$. We can show this in a formal manner by multiplying both sides by $P_m(\cos\theta)$ and integrating from $\cos\theta = -1$ to $+1$. By either method we find that

$$A_1 = -E_0 \qquad (4\text{-}149)$$

and all the other A_n's are zero. Then all the B's are zero except B_1:

$$B_1 = -A_1 a^3 = E_0 a^3. \qquad (4\text{-}150)$$

Finally, the potential at any point (r, θ) is

$$V(r, \theta) = -E_0 r \cos\theta + E_0 \frac{a^3 \cos\theta}{r^2}, \qquad (4\text{-}151)$$

$$= -E_0 \left(1 - \frac{a^3}{r^3}\right) r \cos\theta. \qquad (4\text{-}152)$$

The field intensity, illustrated in Figure 4-22, is then readily found from V:

$$E_r = -\frac{\partial V}{\partial r} = E_0 \left(1 + \frac{2a^3}{r^3}\right) \cos\theta, \qquad (4\text{-}153)$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_0 \left(1 - \frac{a^3}{r^3}\right) \sin\theta. \qquad (4\text{-}154)$$

The surface density of induced charge on the sphere is found easily, since we know that at the surface of the conductor,

$$(E_r)_{r=a} = \frac{\sigma}{\epsilon_0}. \qquad (4\text{-}155)$$

Then

$$\sigma = 3\epsilon_0 E_0 \cos\theta. \qquad (4\text{-}156)$$

Returning now to Eq. *4-151*, we observe that the first term is the potential corresponding to the uniform field intensity $\mathbf{E}_0$. The second term has the form of the potential due to a dipole (Section 2.8). In fact, if we replace the sphere by a dipole of moment

$$p = 4\pi\epsilon_0 E_0 a^3 \qquad (4\text{-}157)$$

located at the center, the field outside the surface previously occupied by the sphere will remain unchanged. We shall examine the image aspect of this field in the problems at the end of this chapter.

We could also have determined the field quickly from Eq. *4-139* by a less formal method. We must have the term $-E_0 r\cos\theta$ to fit the condition at infinity. No other function with positive powers of r can be included. This one term, however, is inadequate to fit the condition at $r = a$, where V must be independent of θ. We must therefore add another function which also includes the $\cos\theta$ factor in order that the coefficient of $\cos\theta$ can be zero at $r = a$. Then

$$V = -E_0 r\cos\theta + \frac{B\cos\theta}{r^2}. \qquad (4\text{-}158)$$

We finally choose a value for B which will make $V = 0$ at $r = a$. Our solution satisfies both Laplace's equation and the boundary conditions; it must therefore be the correct solution, according to the uniqueness theorem.

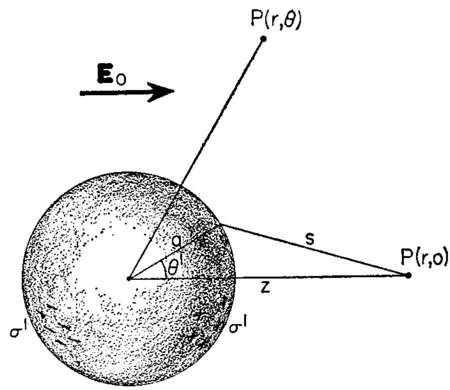

Figure 4-23. *The electrostatic potential V at $P(r, \theta)$ can be calculated from the applied electric field intensity $\mathbf{E}_0$ and the induced surface charge density σ'. We first calculate V on the axis of symmetry at $P(r, \theta)$ by a simple integration. This value is then used as a boundary condition to determine the coefficients of the Legendre polynomials in the series for V at any point $P(r, \theta)$.*

There is still another method of calculating this same field which will add to our understanding of the physical phenomenon and which will illustrate further the use of Legendre polynomials. Consider Figure 4-23. As indicated previously, the potential at any point (r, θ) arises from two charge distributions: (1) that which produces the electric field intensity $\mathbf{E}_0$ and which resides on electrodes situated far away, and (2) that which is induced on the surface of the sphere. This latter distribution is unknown, and we denote it by $\sigma(\theta')$. We use a prime on θ to distinguish it from the polar angle for a point (r, θ) outside the sphere. At the general point (r, θ) the total potential from these two sources must be of the form shown in Eq. *4-139*.

Now it is possible to compute from Coulomb's law the potential at a point P on the axis $\theta = 0$ at a distance $r = z$ from the center of the sphere:

$$V = -E_0 z + \frac{1}{4\pi\epsilon_0}\int_0^\pi \frac{\sigma(\theta')2\pi a^2 \sin\theta'\,d\theta'}{s}, \tag{4-159}$$

where a is the radius of the sphere and s is the distance from a point on the sphere to P as in Figure 4-23.

$$s = \sqrt{z^2 + a^2 - 2az\cos\theta'} \tag{4-160}$$

$$= z\sqrt{1 + \frac{a^2}{z^2} - \frac{2a}{z}\cos\theta'}. \tag{4-161}$$

Expanding $1/s$ and grouping terms involving the same power of (a/z), we obtain

$$\frac{1}{s} = \frac{1}{z}\left[1 + \frac{a}{z}\cos\theta' + \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\frac{a^2}{z^2} + \left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta'\right)\frac{a^3}{z^3} + \cdots\right], \tag{4-162}$$

$$= \frac{1}{z} + \frac{a}{z^2}P_1(\cos\theta') + \frac{a^2}{z^3}P_2(\cos\theta') + \cdots. \tag{4-163}$$

We have already seen that any function of the polar angle θ can be expanded as a series of Legendre polynomials. Thus

$$\sigma(\theta') = b_0 + b_1P_1(\cos\theta') + b_2P_2(\cos\theta') + \cdots, \tag{4-164}$$

where $b_0, b_1, \cdots$ are constants. Equation *4-159* then becomes

$$V = -E_0 z + \frac{a^2}{2\epsilon_0}\int_{-1}^{+1}[b_0 + b_1P_1(\cos\theta') + b_2P_2(\cos\theta') + \cdots] \times\left[\frac{1}{z} + \frac{a}{z^2}P_1(\cos\theta') + \frac{a^2}{z^3}P_2(\cos\theta') + \cdots\right]d(\cos\theta'). \tag{4-165}$$

The orthogonality property of the Legendre polynomials makes this an easy integral to evaluate, thus

$$V = -E_0 z + \frac{a^2}{2\epsilon_0}\left(\frac{2b_0}{z} + \frac{2b_1}{3}\frac{a}{z^2} + \frac{2b_2}{5}\frac{a^2}{z^3} + \cdots\right). \tag{4-166}$$

Thus, when $\theta = 0$ and $r = z$, the general solution for V shown in Eq. *4-139* must reduce to the above form, and we can match coefficients term by term to find V at *any* point (r, θ) outside the sphere. On doing this we find that all the A_n's are zero, except for

$$A_1 = -E_0, \tag{4-167}$$

and that

$$B_n = \frac{b_n}{2n+1}\frac{a^{n+2}}{\epsilon_0}. \tag{4-168}$$

To evaluate the b_n's, we use the fact that the potential is zero at $r = a$. Substituting the above coefficients into Eq. *4-139* and setting $r = a$, we have

$$0 = -E_0 a P_1(\cos\theta) + \frac{b_0 a}{\epsilon_0} + \frac{b_1 a}{3\epsilon_0}P_1(\cos\theta) + \frac{b_2 a}{5\epsilon_0}P_2(\cos\theta) + \cdots, \tag{4-169}$$

which must be true for all θ. Thus both the term $b_0 a/\epsilon_0$, which is independent of θ, and the coefficients of all the P_n's must be equal to zero. Thus

$$b_0 = 0, \tag{4-170}$$

and

$$-E_0 a + \frac{b_1 a}{3\epsilon_0} = 0, \tag{4-171}$$

or

$$b_1 = 3\epsilon_0 E_0. \tag{4-172}$$

All other b_n's are zero.

The potential V at any point (r, θ) is thus given by substituting into Eq. *4-139* $A_1 = -E_0$ as in Eq. *4-167*, and

$$B_1 = E_0 a^3, \tag{4-173}$$

as in Eqs. *4-168* and *4-172*. The field is the same as that found in Eq. *4-151*.

The surface charge density $\sigma(\theta')$ on the conducting sphere can be obtained from Eq. *4-164* now that the b_n's are known: we find the value previously found in Eq. *4-156*.

4.6.2. Dielectric Sphere in a Uniform Electrostatic Field. We can calculate this field by either of the formal methods discussed above if we write a general solution as in Eq. *4-139* for points outside the sphere and write another solution with different coefficients for points inside the sphere. The coefficients must be chosen such that the boundary conditions are satisfied:

$V \longrightarrow -E_0 r\cos\theta$	$(r \longrightarrow \infty)$;
V is continuous across the boundary	$(r = a)$;
the normal component of $\mathbf{D}$ is continuous	$(r = a)$.

Instead of following such a formal procedure, however, we shall write down a combination of spherical harmonics which will satisfy all the boundary conditions.

Outside the sphere, we must have $-E_0 r\cos\theta$ as one of the terms in the solution to satisfy the condition at $r \longrightarrow \infty$. Furthermore, this is the only harmonic with a positive power of r which we can permit, for otherwise the condition