On matching coefficients with Eq. *4-194*, we find that

$$B_0 = \frac{Q}{4\pi\epsilon_0}, \tag{4-197}$$

$$B_1 = 0, \tag{4-198}$$

$$B_2 = -\frac{Q}{4\pi\epsilon_0}\frac{a^2}{2}, \tag{4-199}$$

$$B_3 = 0, \cdots, \tag{4-200}$$

and, from Eq. *4-193*,

$$V(r, \theta) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{2}\frac{a^2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right]. \tag{4-201}$$

Figure 4-26 shows the equipotential lines in this case. The components of the field intensity may be found, as usual, by calculating $-\nabla V$.

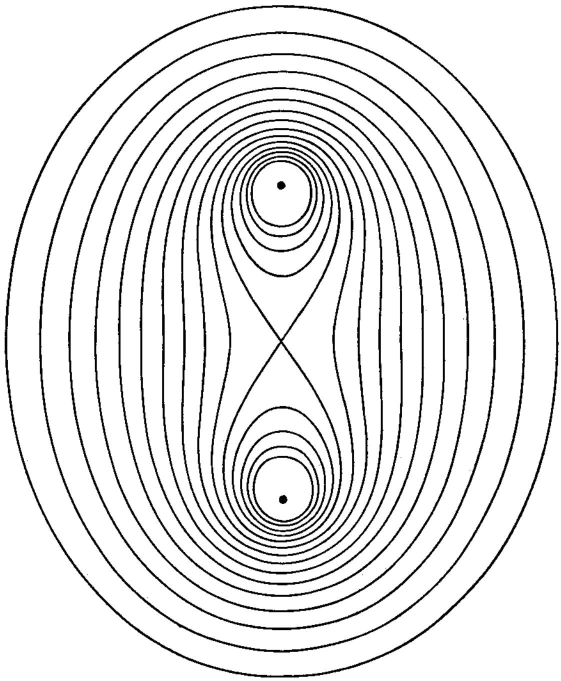

Figure 4-26

Equipotentials for a charged ring. None are shown in the vicinity of the ring, where they are too close together to be depicted graphically. At about two diameters from the ring the equipotentials are approximately circular, and the field is quite similar to that of a point charge.

4.7. Solutions of Poisson's Equation

We have as yet dealt only with solutions to Laplace's equation, since we have concerned ourselves only with cases in which the charge density ρ is zero. As we pointed out earlier, however, there are important fields in which a *space charge* exists and in which ρ is not zero. For these, we must find a solution of

Poisson's equation, and again the solution must be consistent with the boundary conditions which obtain in the particular problem. We have already shown in Section 4.2 that the solution is unique.

4.7.1. The Vacuum Diode. As an example of such a field let us find the potential distribution between the plates of a vacuum diode whose cathode and anode are plane parallel surfaces separated by a distance which is small compared to their linear extent. The anode is maintained at a positive potential V_0 relative to the cathode whose potential we shall take to be zero. The cathode is heated in order that electrons will be emitted thermionically and will be accelerated toward the anode under the action of the electric field. We shall assume that the electrons are emitted with zero velocity and that the current is not limited by the cathode temperature but can be increased at will by increasing V_0.

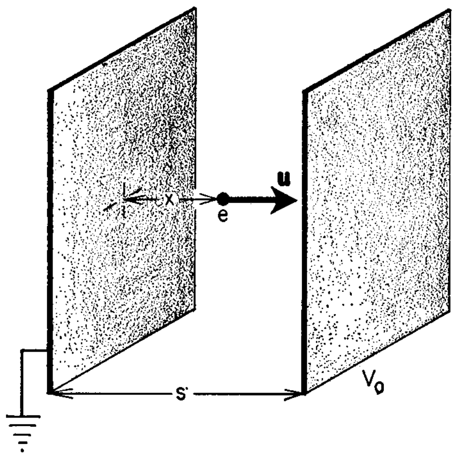

Figure 4-27. *A plane-parallel vacuum diode. The cathode is grounded, and the anode is at a potential* V_0. *An electron of charge e moves toward the plate with a velocity* **u**.

Since the electrons move in the space between the plates with finite velocity, they constitute a space charge whose density ρ is given by

$$\rho = \frac{J}{u}, \tag{4-202}$$

where J is the current density in amperes/meter^2 at a point where the electron velocity is u meters/second. The space charge density ρ is then measured in coulombs/meter^3; since the electron carries a negative charge, ρ is negative. The current density J is also negative, since we take the velocity u as positive for motion from cathode to anode, as in Figure 4-27.

Since the potential can depend only on the coordinate x in the direction perpendicular to the plates, Poisson's equation reduces to

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0}. \tag{4-203}$$

Thus the second derivative of V with respect to x is everywhere positive, since ρ is a negative quantity, and, for a given potential difference between the plates, V is everywhere lower than the corresponding free space value.

Expressing ρ in terms of the current density J and of the velocity u, we have

$$\frac{d^2V}{dx^2} = \frac{J}{\epsilon_0 u}, \tag{4-204}$$

where we now take J to be the magnitude of the current density, as read on a meter, without regard to sign.

From the conservation of energy, the velocity u is given by

$$\frac{1}{2} mu^2 = eV, \tag{4-205}$$

where m is the mass of the electron. Substituting this value of u into Eq. *4-204* gives

$$\frac{d^2V}{dx^2} = \frac{J}{\epsilon_0}\left(\frac{m}{2eV}\right)^{1/2}. \tag{4-206}$$

This equation can be integrated easily by first multiplying the left side by $2(dV/dx)\,dx$ and the right side by $2\,dV$, which are equivalent factors. Thus

$$\left(\frac{dV}{dx}\right)^2 = 4\frac{J}{\epsilon_0}\left(\frac{m}{2e}\right)^{1/2} V^{1/2} + A, \tag{4-207}$$

where A is a constant of integration which can be evaluated from the magnitude of dV/dx at the cathode, where V is zero.

In the present case, $dV/dx = 0$ at the cathode, as can be seen from the following. If we establish the potential difference V_0 between the plates when the cathode is cold, that is, with no electrons available, the electric field intensity E_c at the surface of the cathode is positive and equal to V_0/s. But if we heat the cathode, electrons are emitted, the space charge is established, and E_c diminishes. If an unlimited supply of electrons is available, the space charge increases, and E_c falls until equilibrium is reached. As long as E_c is positive, the electrons emitted are accelerated toward the anode and cannot return to the cathode. The current is then limited by the cathode emission and not by V_0, as we assumed at the beginning. On the other hand, if E_c were negative the electrons could never leave the cathode and we would have no space charge. Thus E_c cannot be either positive or negative. At equilibrium, $E_c = 0$, thus the constant of integration A in Eq. *4-207* must be zero. Then

$$\frac{dV}{dx} = 2\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{m}{2e}\right)^{1/4} V^{1/4}, \tag{4-208}$$

and

$$V^{3/4} = \frac{3}{2}\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{m}{2e}\right)^{1/4} x + B. \tag{4-209}$$

The constant of integration B is zero since $V = 0$ at $x = 0$, and so

$$V = \left(\frac{9}{4}\frac{J}{\epsilon_0}\right)^{2/3}\left(\frac{m}{2e}\right)^{1/3} s^{4/3}\left(\frac{x}{s}\right)^{4/3}. \tag{4-210}$$

When $x = s$, $V = V_0$, and so Eq. *4-210* can be written as

$$V = V_0\left(\frac{x}{s}\right)^{4/3}. \tag{4-211}$$

Expressing the field intensity E, the current density J, and the charge density ρ in similar fashion, we find that

$$E = \frac{4}{3}\frac{V_0}{s}\left(\frac{x}{s}\right)^{1/3}, \tag{4-212}$$

$$J = \frac{4\epsilon_0}{9}\left(\frac{2e}{m}\right)^{1/2}\left(\frac{V_0^{3/2}}{s^2}\right), \tag{4-213}$$

$$= 2.335 \times 10^{-6}\frac{V^{3/2}}{s^2} \quad \text{(amperes/meter}^2\text{)}. \tag{4-214}$$

Thus

$$\rho = \frac{4\epsilon_0}{9s^2}V_0\left(\frac{x}{s}\right)^{-2/3}. \tag{4-215}$$

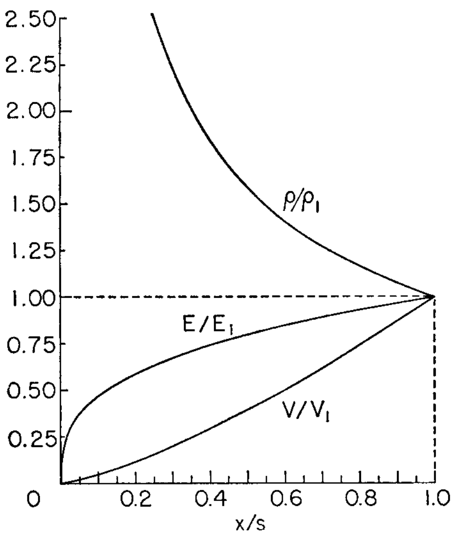

Figure 4-28. *The space charges density ρ, the electric field intensity E, and the electrostatic potential V as functions of the distance from the cathode in a plane-parallel infinite diode. The index 1 refers to the value at the anode. The distance between cathode and anode is s.*

Equation *4-214*, which is known as the *Child-Langmuir law*, is valid only for the plane parallel diode and for electrons emitted with zero velocity. Figure 4-28 shows the distribution of potential V, electric field intensity E, and charge density ρ in the plane parallel diode. One can show,* however, that, in general, no matter what the geometry of the diode may be, the current is related to the potential difference between cathode and anode by the relation

$$J = KV^{3/2}, \tag{4-216}$$

where K is a constant.

In an actual diode, electrons are emitted with finite velocities, and the equilibrium field intensity E_c at the cathode is negative. In this case, a potential minimum is established at a small distance in front of the cathode and only electrons with velocities greater than a critical value can get past the potential minimum, which is a potential energy maximum for electrons.

4.8. Summary

In this chapter we have dealt with electrostatic problems which cannot easily be solved by direct integration from Coulomb's law or by application of Gauss's law. We have sought solutions of Poisson's equation

* K. R. Spangenberg, *Fundamentals of Electron Devices* (McGraw-Hill, 1957), p. 169.