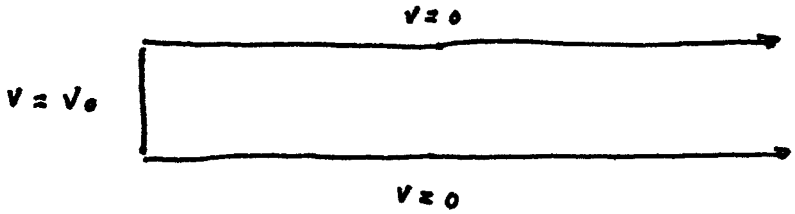
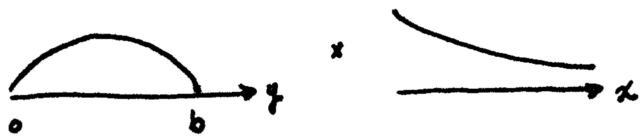
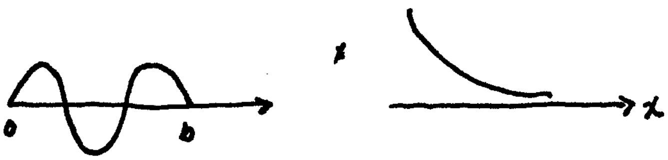
LECTURE 8 JULY 6, 2009





$\sin(k_1 y)$ $\exp(-k_1 x)$

~~sin (k~~

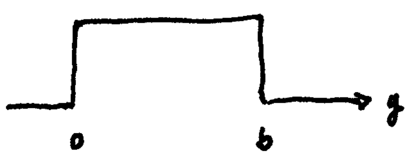


$\sin(k_3 y)$ $\exp(-k_3 x)$

+

⋮

AT $x = 0$



FINDING THE FOURIER COEFFICIENTS

ANY $f(x)$

$$f(x) = a_0 + \sum_m a_m \cos(k_m x) + b_m \sin(k_m x)$$

a_0 = AVERAGE AKA "DC PIECE"

$$a_m = \frac{1}{L}\int_0^L f(x)\cos(k_m x)\,dx$$

$$b_m = \frac{1}{L}\int_0^L f(x)\sin(k_m x)\,dx$$

$\Rightarrow$ SINE AND COSINE ARE A COMPLETE SET OF ORTHOGANAL FCNS ON AN INTERVAL

$$\int \sin(k_m x)\sin(k_m x) = 0 \quad \text{unless } k_m = k_m$$

$$\int \cos(k_m x)\ \cos(k_m x) = 0 \quad \text{unless } k_m = k_m$$

$$\int \sin(k_m x)\cos(k_m x) = 0$$

LIKE ORTHOGONAL VECTORS

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f(x) = a \sin(k_1 x) + b \sin(k_2 x) + c \sin(k_3 x)$$

WHAT IS b?

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = b$$

MULTIPLY COMPONENT BY COMPONENT, THEN SUM

$$\sin(k_1 x) \cdot f(x)$$

MULTIPLY x_i BY x_i, THEN INTEGRATE

INDEFINITE INTEGRALS

$$\int \sin(mx)\sin(nx)\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$$

$m^2 \neq n^2$

$$\int \cos(mx)\cos(nx)\,dx = \;\; '' \;\; + \;\; ''$$

$$\int \sin(mx)\cos(nx)\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)}{2(m+n)}$$

$$\int \sin^2(ax) = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$$

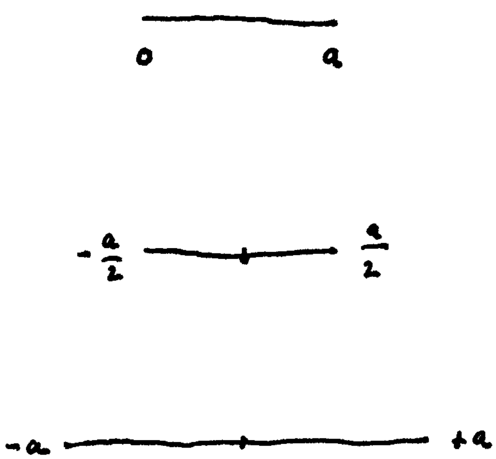
$$\int \cos^2(ax)\,dx = \frac{1}{2}x + \frac{1}{4a}\sin(2ax)$$

$$\int \sin(ax)\cos(ax)\,dx = \frac{1}{2a}\sin^2(ax)$$

UGH!

FALSTAD
GNYACK } APPLETS

3 CONVENTIONS



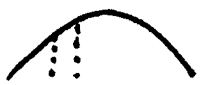
3 STANDARDS

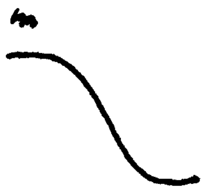
SQUARE

TRIANGLE

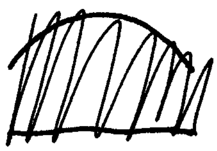
SAWTOOTH

WHAT FOURIER SERIES CAN YOU CALCULATE?

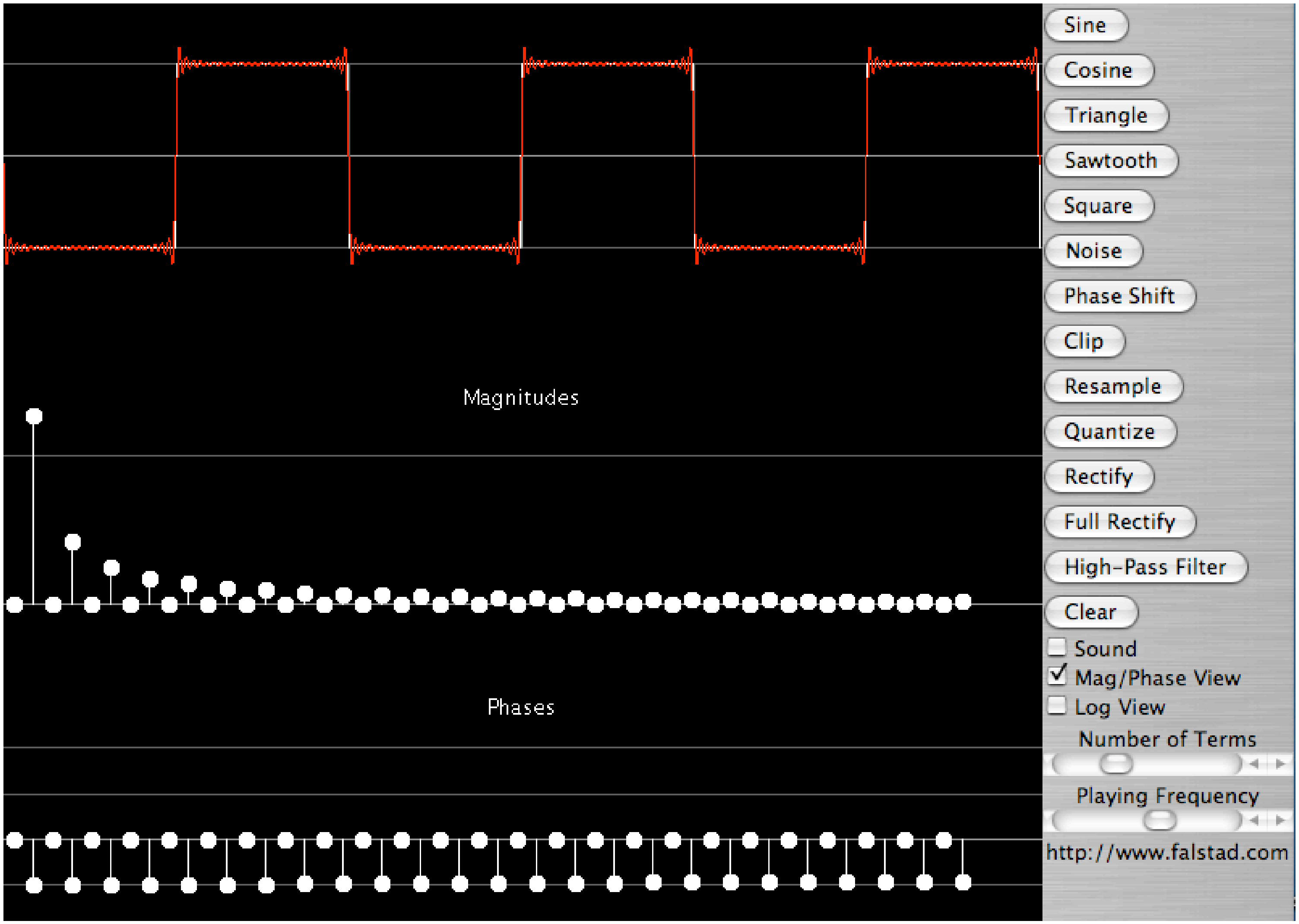




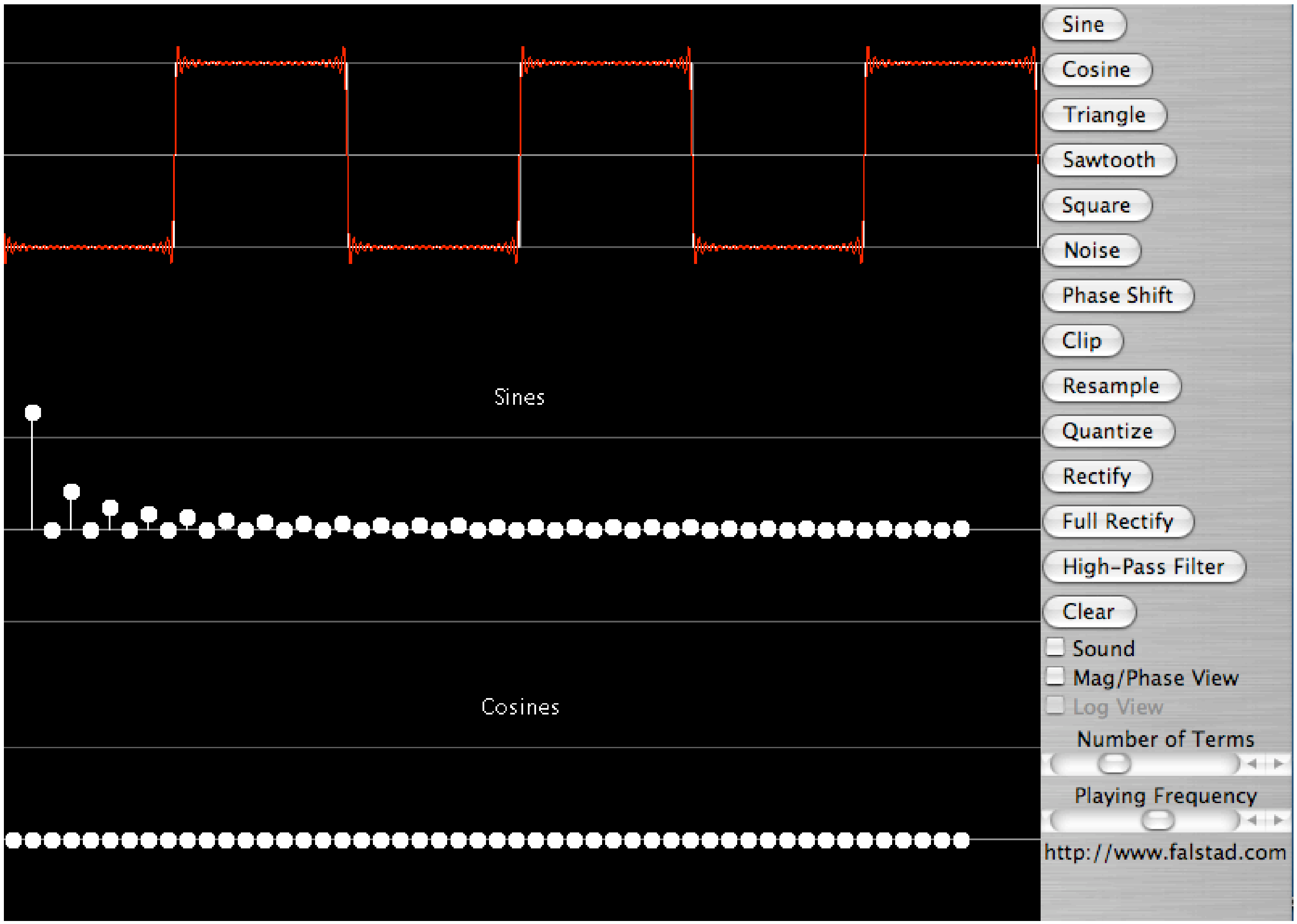
⇒ ANY PIECE-WISE CONSTANT



ANY PIECE-WISE LINEAR



Magnitudes
Phases
Sine
Cosine
Triangle
Sawtooth
Square
Noise
Phase Shift
Clip
Resample
Quantize
Rectify
Full Rectify
High-Pass Filter
Clear
Sound
Mag/Phase View
Log View
Number of Terms
Playing Frequency
http://www.falstad.com



Sines
Cosines
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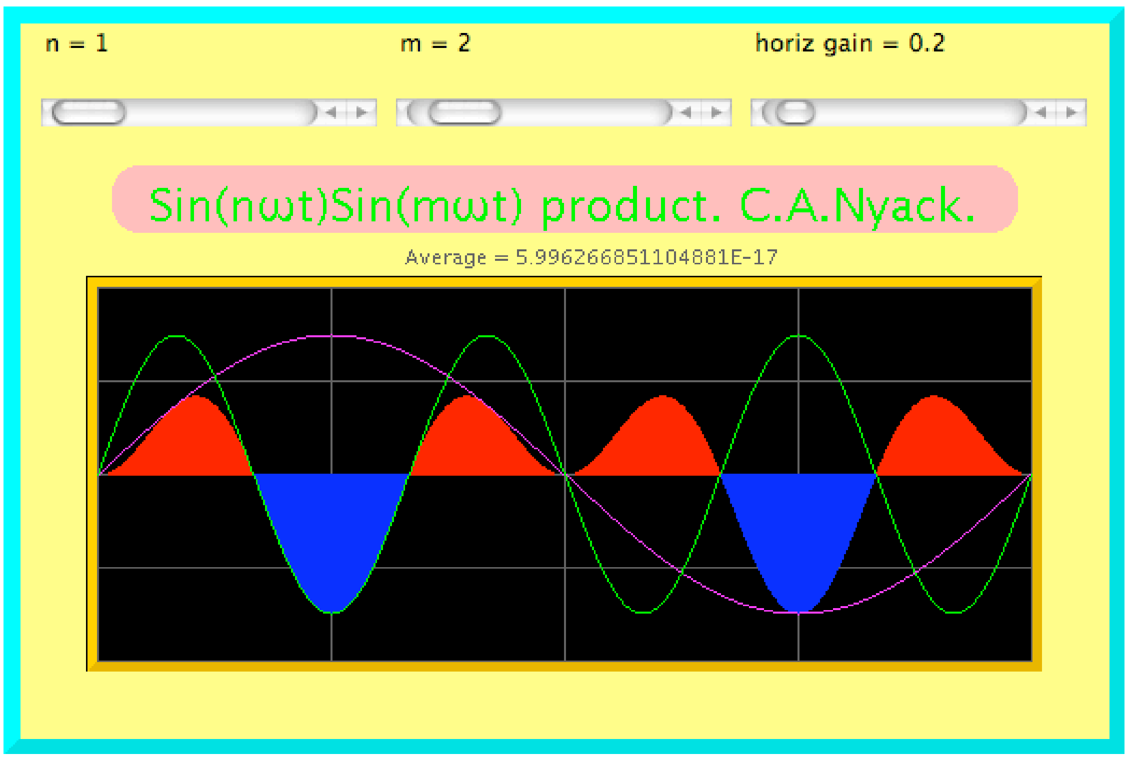
n = 1
m = 2
horiz gain = 0.2
Sin(nωt)Sin(mωt) product. C.A.Nyack.
Average = 5.996266851104881E-17

Orthogonal Polynomials

Orthogonal polynomials are classes of polynomials $\{p_n(x)\}$ defined over a range $[a, b]$ that obey an orthogonality relation

$$\int_a^b w(x)\, p_m(x)\, p_n(x)\, dx = \delta_{mn}\, c_n, \tag{1}$$

where $w(x)$ is a weighting function and δ_{mn} is the Kronecker delta. If $c_n = 1$, then the polynomials are not only orthogonal, but orthonormal.

Orthogonal polynomials have very useful properties in the solution of mathematical and physical problems. Just as Fourier series provide a convenient method of expanding a periodic function in a series of linearly independent terms, orthogonal polynomials provide a natural way to solve, expand, and interpret solutions to many types of important differential equations. Orthogonal polynomials are especially easy to generate using Gram-Schmidt orthonormalization.

A table of common orthogonal polynomials is given below, where $w(x)$ is the weighting function and

$$c_n \equiv \int_a^b w(x)\, [p_n(x)]^2\, dx \tag{2}$$

Hermite polynomial	$(-\infty, \infty)$	e^{-x^2}	$\sqrt{\pi}\ 2^n\ n!$
Jacobi polynomial	$(-1, 1)$	$(1-x)^\alpha\ (1+x)^\beta$	h_n
Laguerre polynomial	$[0, \infty)$	e^{-x}	1
generalized Laguerre polynomial	$[0, \infty)$	$x^k\ e^{-x}$	$\frac{(n+k)!}{n!}$
Legendre polynomial	$[-1, 1]$	1	$\frac{2}{2n+1}$

Coordinate System	Variables	Solution Functions
Cartesian	$X(x)Y(y)Z(z)$	exponential functions, circular functions, hyperbolic functions
circular cylindrical	$R(r)\Theta(\theta)Z(z)$	Bessel functions, exponential functions, circular functions
conical		ellipsoidal harmonics, power
ellipsoidal	$\Lambda(\lambda)M(\mu)N(\nu)$	ellipsoidal harmonics
elliptic cylindrical	$U(u)V(v)Z(z)$	Mathieu function, circular functions
oblate spheroidal	$\Lambda(\lambda)M(\mu)N(\nu)$	Legendre polynomial, circular functions
parabolic		Bessel functions, circular functions
parabolic cylindrical		parabolic cylinder functions, Bessel functions, circular functions
paraboloidal	$U(u)V(v)\Theta(\theta)$	circular functions
prolate spheroidal	$\Lambda(\lambda)M(\mu)N(\nu)$	Legendre polynomial, circular functions
spherical	$R(r)\Theta(\theta)\Phi(\phi)$	Legendre polynomial, power, circular functions

Laplace's equation can be solved by separation of variables in all 11 coordinate systems that the Helmholtz differential equation can. The form these solutions take is summarized in the table above. In addition to these 11 coordinate systems, separation can be achieved in two additional coordinate systems by introducing a multiplicative factor. In these coordinate systems, the separated form is

Fourier Decomposition
http://www.falstad.com/fourier/

Fourier Orthogonality
http://cnyack.homestead.com/files/sinusoid/Ortho1.htm

Square Wave
http://mathworld.wolfram.com/FourierSeriesSquareWave.html
http://en.wikipedia.org/wiki/Square_wave

Triangle Wave
http://mathworld.wolfram.com/FourierSeriesTriangleWave.html
http://en.wikipedia.org/wiki/Triangle_wave

Sawtooth Wave
http://mathworld.wolfram.com/FourierSeriesSawtoothWave.html
http://en.wikipedia.org/wiki/Sawtooth_wave

Semi-Circle Wave
http://mathworld.wolfram.com/FourierSeries.html

Orthogonal Polynomials
http://mathworld.wolfram.com/OrthogonalPolynomials.html

Laplace's Equation in 13 Coordinate systems
http://mathworld.wolfram.com/LaplacesEquation.html

http://cnyack.homestead.com/files/afourse/fsdef.htm

http://www.falstad.com

http://cnyack.homestead.com/files/idxpages.htm