LECTURE 7 JULY 2, 2009

SEPARATION OF VARIABLES

IF YOUR PROBLEM HAS SYMMETRY

PUT IN PRODUCT SOLUTION

TAKE DERIVATIVES ~~AND RI~~

MULTIPLY BY $\dfrac{1}{\text{PRODUCT}}$

REGROUP THE TERMS

EACH ONE MUST BE A CONSTANT

$\Rightarrow$ SEPARATE ODE FOR EACH VARIABLE

W/ CONSTRAINT OF SUM OF THE SEPARATION CONSTANTS

OUR PROBLEM HAS CARTESIAN SYMMETRY

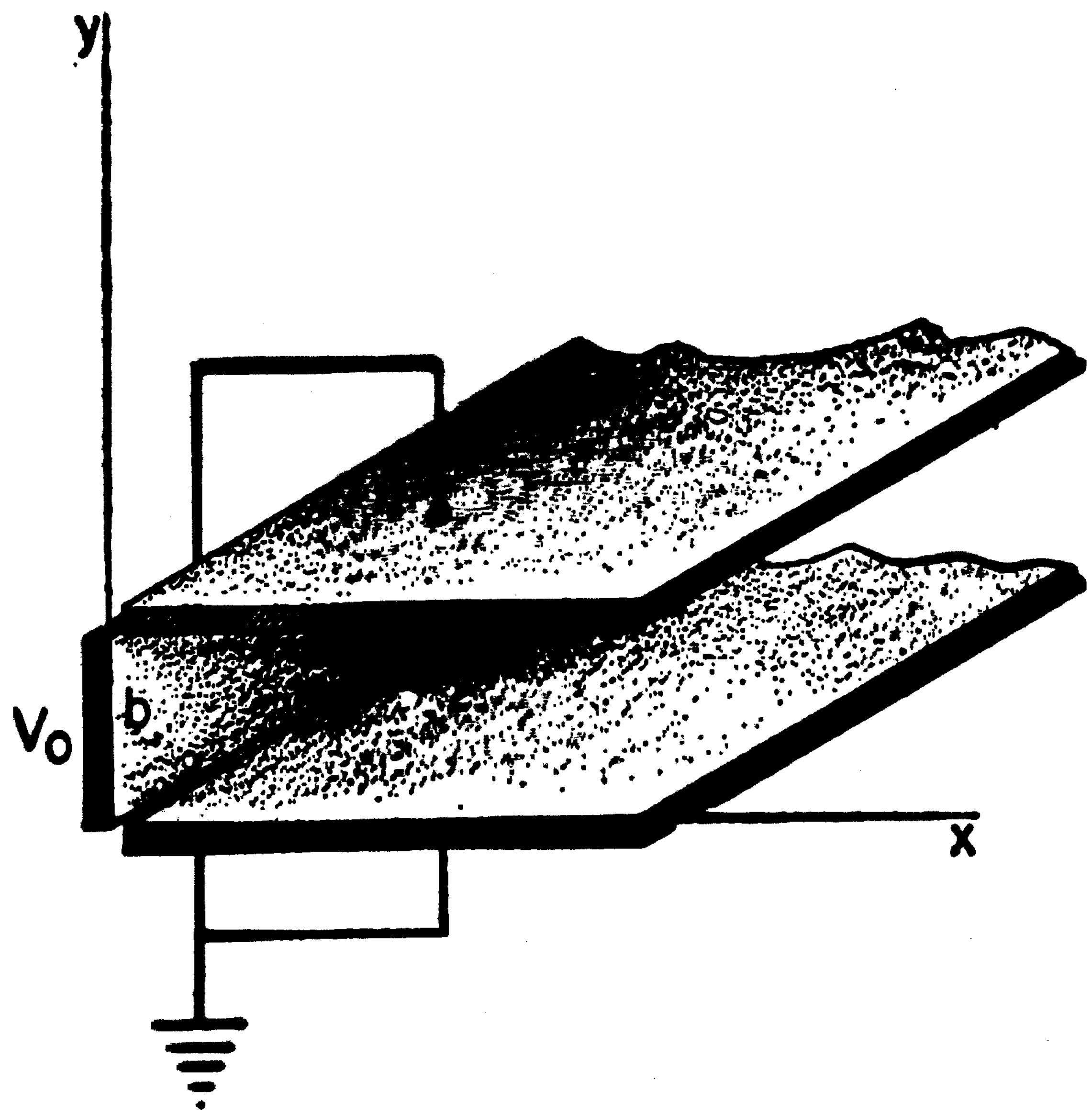
$V(x, b) = 0$

$V(x, 0) = 0$

$V(0, y) = V_0 \quad 0 \le y \le b$

SO WE SHOULD USE CARTESIAN COORD'S

BECAUSE THEN WE CAN APPLY THE BC'S


y
V_0
b
x

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = X(x) \cdot Y(y)$$

PUT IN

DO DERIVATIVES

MULTIPLY BY $\frac{1}{X\,Y}$

COLLECT TERMS ...

END UP W 2 ODES

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

NO z DEPENDENCE

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$C_x + C_y + C_z = 0 \qquad C_z = 0$

$\Rightarrow C_x + C_y = 0$

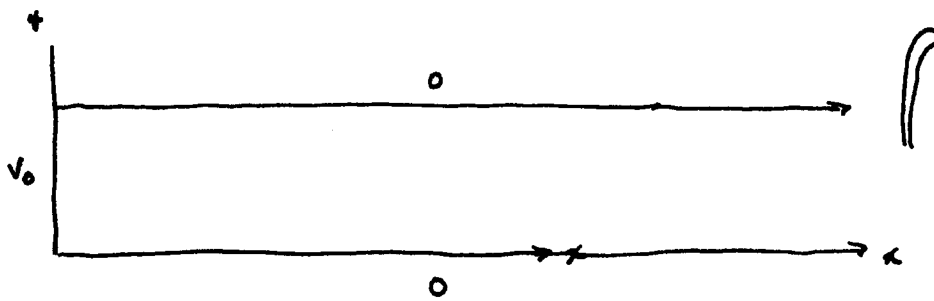
$$\frac{d^2 \bar{X}}{dx^2} + C_x \bar{X} = 0 \qquad C_x = K^2 \text{ or } C_x = -K^2$$

$$\frac{d^2 \bar{Y}}{dy^2} + C_y \bar{Y} = 0 \qquad C_y = -K^2 \text{ or } C_y = +K^2$$

$-K^2$ ~~exp~~ => sin cos

$+K^2$ other => exps

$x \to \infty$
$V \to 0$ AT ∞



=> exp along x

$\to 0$

=> sine cosine along y

$$\frac{d^2 X}{dx^2} - k^2 x = 0 \qquad X(x) = A e^{kx} + B e^{-kx}$$

$$\frac{d^2 Y}{dy^2} + k^2 Y = 0 \qquad Y(y) = C \sin(ky) + D \cos(ky)$$

GENERAL SOLNS

APPLY BC TO OBTAIN SPECIFIC SOLUTIONS

IF k^2 IS COMPLEX, EVERY THING IS IN THERE

APPLY BC's

$V(x,0) = 0$

$V(x,b) = 0$

$V = 0$ as $x \to \infty$

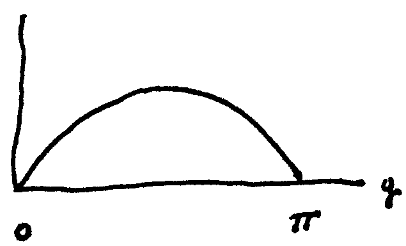
$$X(x) = A e^{-\kappa x} + \cancel{B}\, e^{+\kappa x} \to 0$$

$V(0,0) = 0$

$V(0,b) = 0$

$$Y(y) = C \sin(\kappa y) + \cancel{D \cos(\kappa y)} \to 0$$

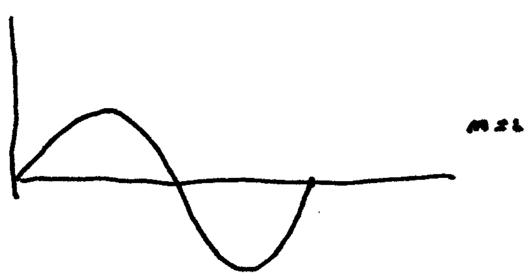
$$Y(b) = 0 = C \sin(\kappa b)$$



$\kappa = \frac{\pi}{b}$

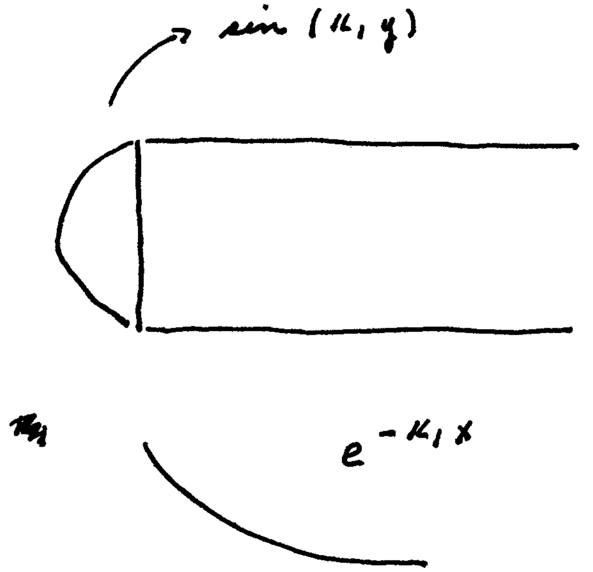
$\kappa y = \frac{\pi}{b} \cdot b = \pi$

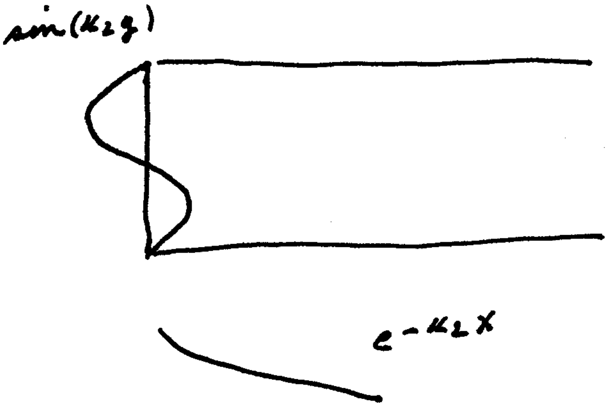
$\kappa = n \frac{\pi}{b}$

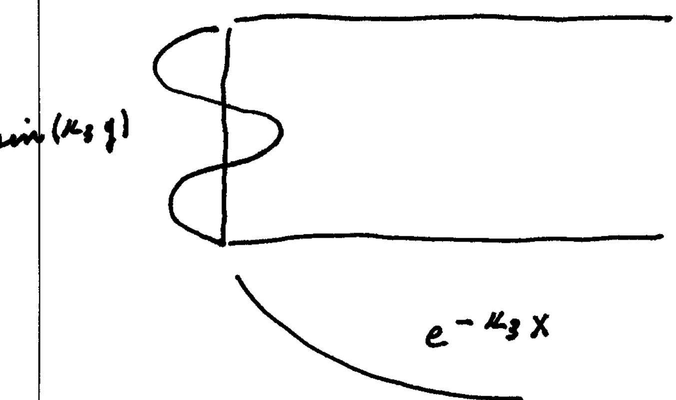


$$Y(y) = C \sin\Big(\underbrace{\frac{n\pi}{b}}_{K_n}\, y\Big)$$

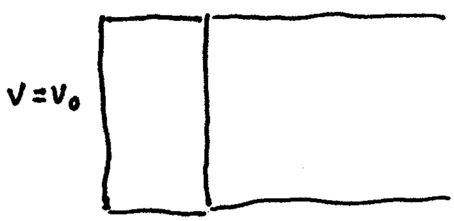
$$= C \sin(K_n y)$$

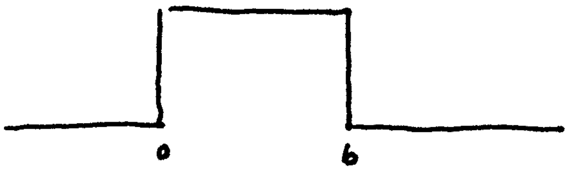






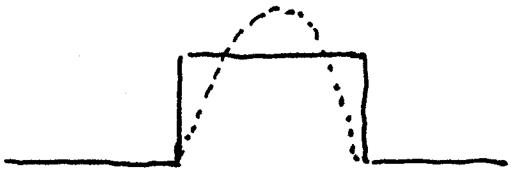
NEED





=

$$\text{sum } c_m \sin(k_m y) \cdot e^{-k_m x}$$



k_1

k_3

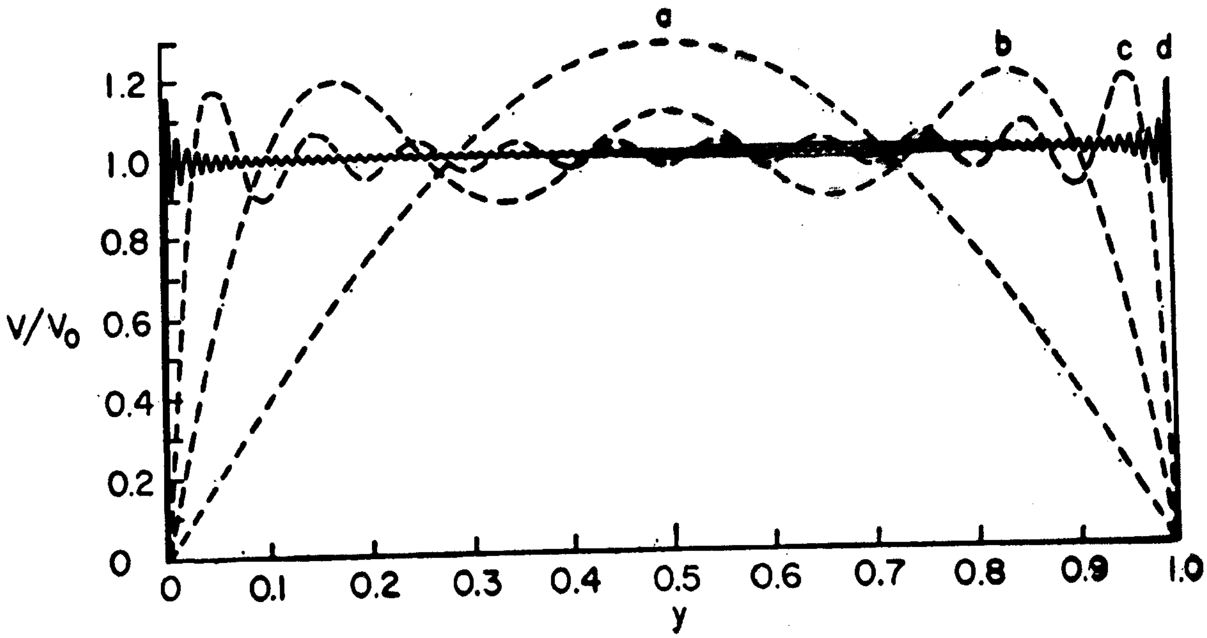
k_5

⋮

ONLY m = ODD

m = EVEN = 0

HOW CAN WE FIND c_m's ?


a
b
c
d
1.2
1.0
0.8
V/V0 0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
y

$$\Psi(y) = \sum_n c_n \sin(k_n y)$$

CALLED FOURIER SERIES

THM: GIVE ANY $g(y)$ ON AN INTERVAL

$$g(y) = \sum_n c_n \sin(k_n y) + D_n \cos(k_n y)$$

CAN EXPRESS ANY (NICE) FCN AS A UNIQUE FOURIER SERIES

$$\sum_n c_n \sin(k_n y) = g(y)$$

$$= V_0$$

sines and cosines are an orthogonal set of functions

$$V_0 = \sum_m c_m \sin(k_m y)$$

MULTIPLY BY $\sin\left(\frac{p\pi}{b} y\right)$ and integrate from 0 to b

$$\int_0^b \sin\left(\frac{p\pi}{b} y\right) V_0 \, dy = \int_0^b \sin\left(\frac{p\pi}{b} y\right)\left[\sum c_m \sin\left(\frac{m\pi}{b} y\right)\right] dy$$

sines are ORTHOGONAL

LHS

$\frac{2b}{\pi p} V_0$ ~~if~~ p ODD

0 p EVEN

RHS

0 IF $p \neq m$

$c_m \frac{b}{2}$ IF $p = m$

$$\Rightarrow \quad c_m = \frac{4 V_0}{m\pi} \qquad m \text{ ODD}$$

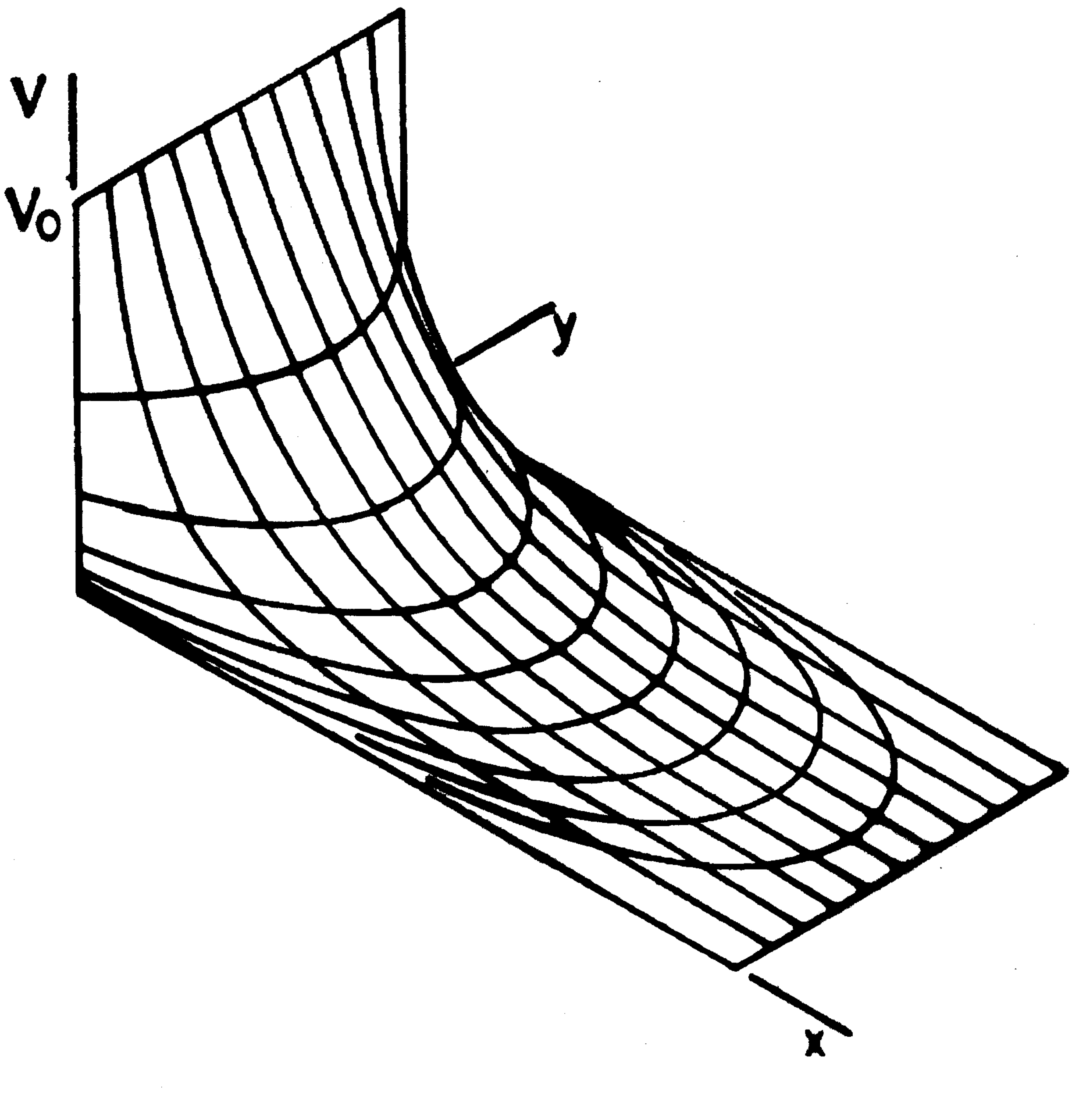
$$= 0 \qquad m \text{ EVEN}$$

PUT ALL THE PIECES TOGETHER

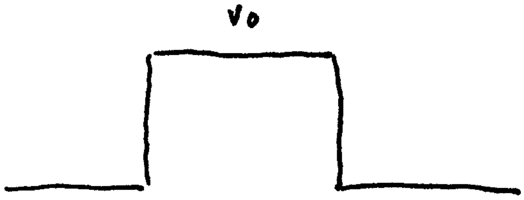
$$V(x,y) = \frac{4}{\pi} V_0 \sum_{\substack{n \\ \text{ODD}}} \left(\frac{1}{n}\right) \sin(k_n y)\, e^{-k_n x}$$

$$k_n = \frac{n\pi}{b}$$

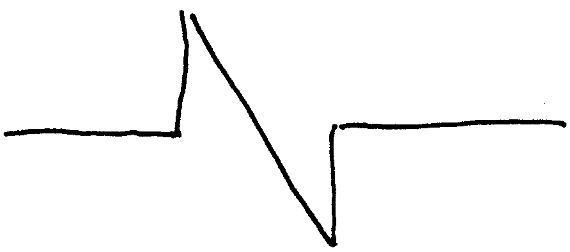
WHAT DOES THIS LOOK LIKE?


V
V0
y
x

THIS FORMALLY SOLVES THIS PROBLEM FOR ANY $V(q)$



specific fourier series



$V(q)$ EVEN $\Rightarrow$ ONLY SIN'S

$V(q)$ ODD $\Rightarrow$ ONLY COSINE'S

GENERAL $\Rightarrow$ BOTH