LECTURE 65 6-29-09

SOLVING LAPLACE'S EQUATION

$$\nabla^2 V = 0$$

DESCRIBES MANY THINGS

FOR NOW, ELECTRIC POTENTIAL (AKA VOLTAGE) IN SPACE

LAPLACE IS A SPECIAL CASE OF POISSON

$$\nabla^2 V = \rho(\vec{r})$$ POISSON EQN

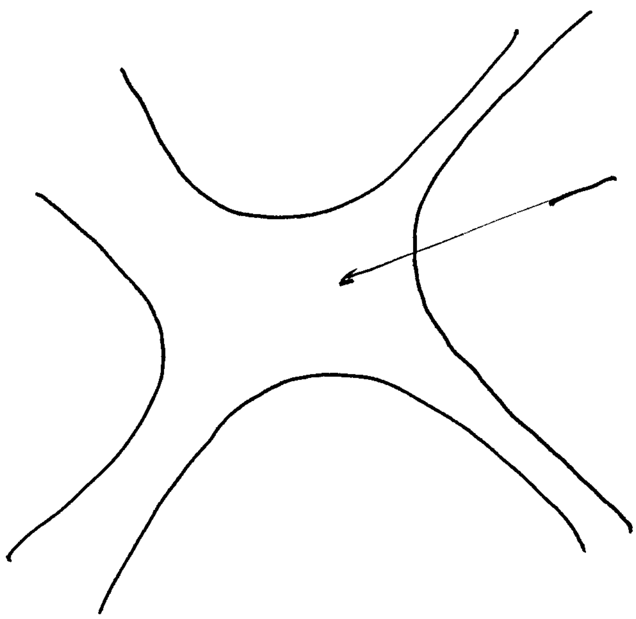
INHOMOGENEOUS RHS $\neq 0$

$$\nabla^2 V = 0$$ LAPLACE EQN

HOMOGENEOUS RHS $= 0$

START IN 2d

given V_1, V_2, V_3, V_4



FIND $V(x,y)$ INSIDE

USEFUL, SYMMETRIC

COMPLEX VARIABLES WORK FOR A VARIETY OF

INTERACTING SHAPES, BUT NOT FOR ALL SHAPES

3d WORK

OTHER ANALYTIC METHODS FOR SYMMETRIC SHAPES

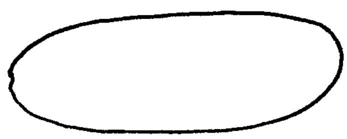
BUT COMPLICATED (REAL) SHAPES YOU MUST DO NUMERICALLY!

TODAY : COMPLEX VARIABLES

SIMPLEST NUMERICAL METHOD

IN 3d, CAN SOLVE IN SOME SPECIAL COORD SYSTEMS

ELLIPSOIDAL COORDS



ELLIPSOID

INFINITELY OBLATE => DISC

INFINITELY PROLATE => NEEDLE

IN 3d

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$V = (x, y, z)$

IN 2d

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

$V(x, y)$ constant along z

COMPLEX VARIABLE METHOD

$z = x + iy$

CONSIDER SOME SIMPLE FCNS

$F(z) = z^2$

$F(z) = \frac{1}{z^3} = z^{-3}$

$F(z) = z \log z$

EXPRESS $F(z)$ IN TERMS OF x AND y

$$z^2 = (x+iy)(x+iy)$$

$$= (x^2-y^2) + i(2xy)$$

$F(z) = u(x,y) + i\,v(x,y)$ in general case

$u(x,y) = x^2-y^2$ in this case

$v(x,y) = 2xy$

FOR ANY "ORDINARY FCN" FEYNMAN

" " ANALYTIC FCN PADDLEFORD

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$ CAUCHY-RIEMANN EQUATIONS

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\Rightarrow$ BOTH $u(x,y)$ AND $v(x,y)$ ARE SOLUTIONS TO LAPLACE EQN

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ WE GET A PAIR OF SOLNS TO LAPLACE EQN

FOR EVERY ANALYTIC FCN

EXAMPLE u = CONSTANT

$$x^2 - y^2 = A$$

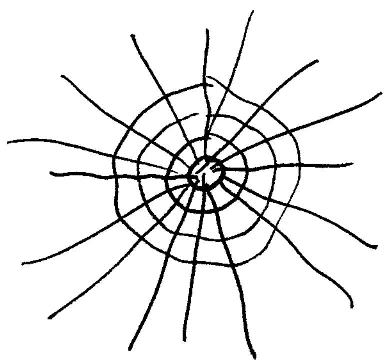
EQUATION FOR RECTANGULAR HYPERBOLAS

v = CONSTANT

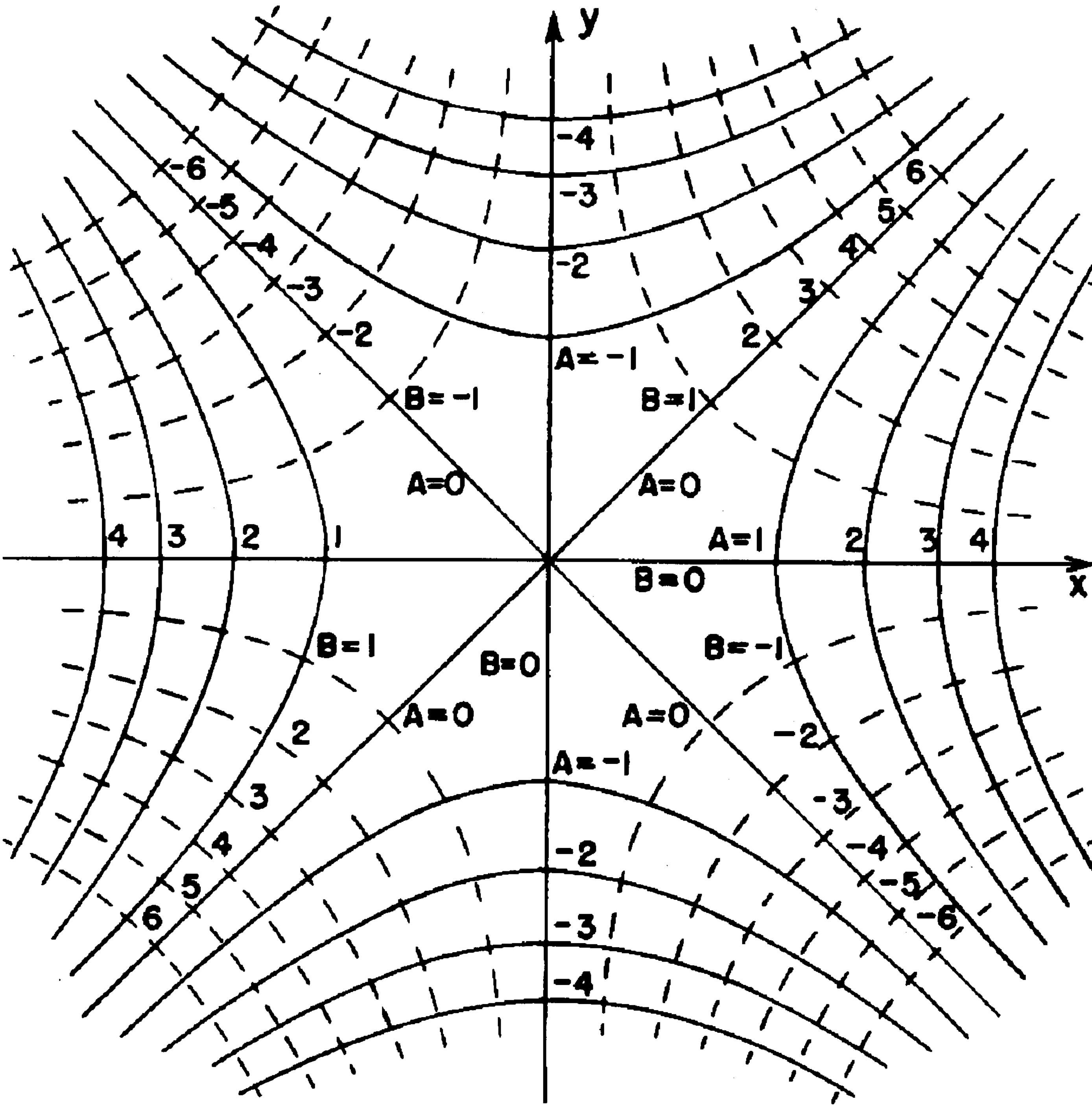
$$2xy = B$$

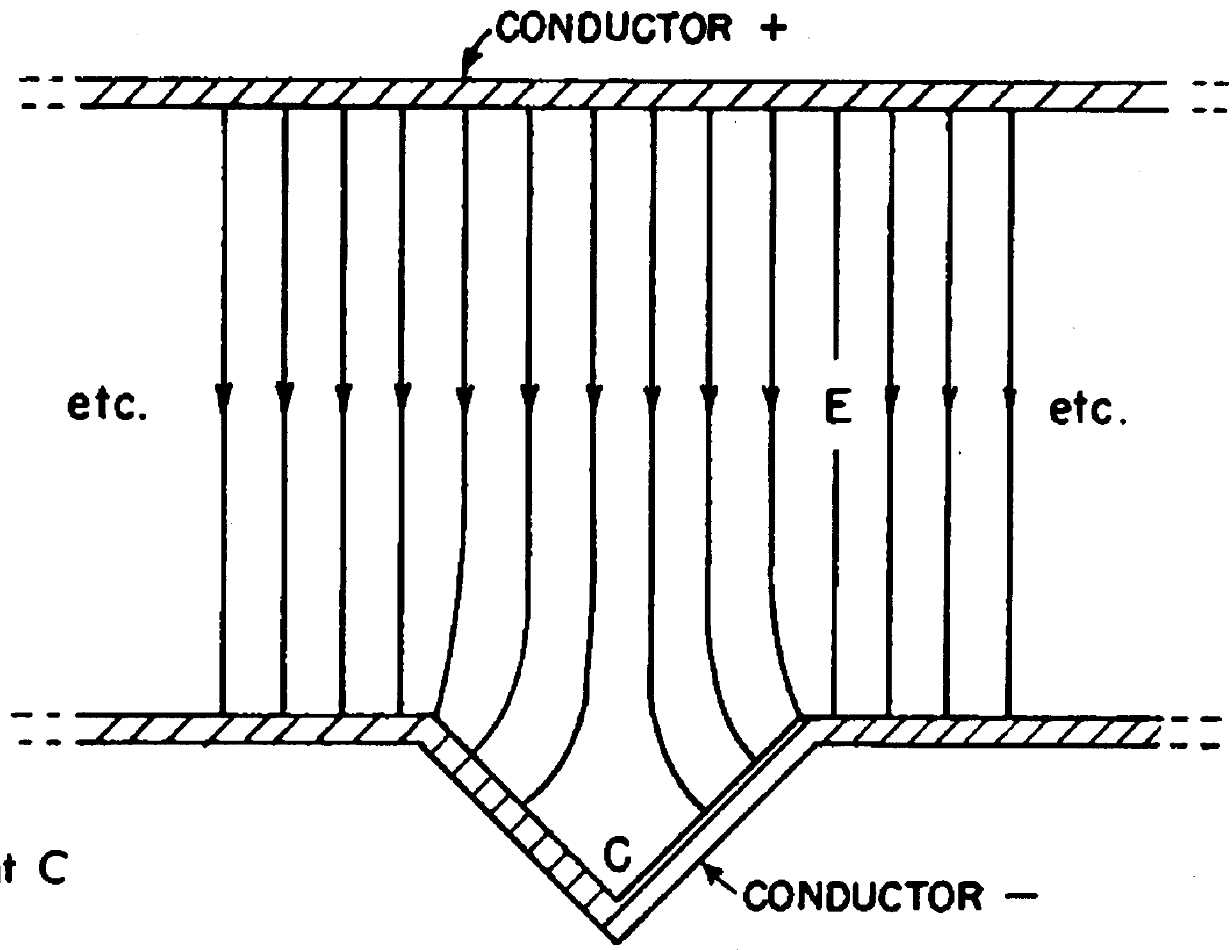
ORTHOGANAL SET OF HYPERBILAS

$\vec{E}$ field lines $\perp$ equipotentials



APPLY "A" TO ELECTRODES


y
x
-4
-3
-2
A=-1
B=-1
B=1
A=0
A=0
-6
-5
-4
-3
-2
6
5
4
3
2
4
3
2
1
A=1
2
3
4
B=0
B=1
B=0
B=-1
A=0
A=0
2
3
4
5
6
-2
-3
-4
-5
-6
A=-1
-2
-3
-4



e point C

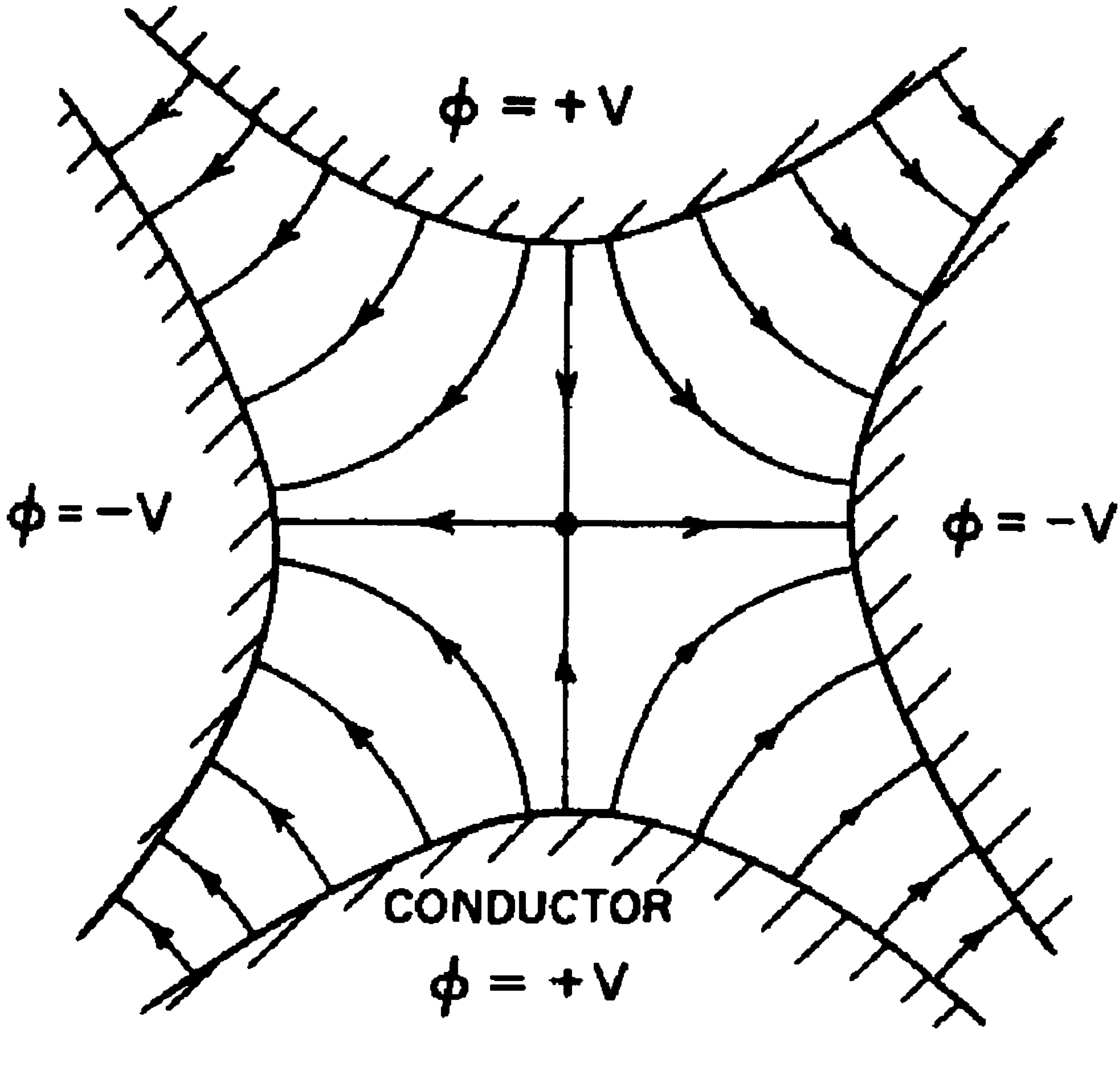


Fig. 7–3. The field in a quadrupole lens.

SECOND EXAMPLE

$$F(z) = \sqrt{z}$$

$$z = x + iy = re^{i\varphi}$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = y/x$$

$$F(z) = \sqrt{r}\; e^{i\varphi/2}$$

$$= \sqrt{r}\left(\cos(\varphi/2) + i\sin(\varphi/2)\right)$$

$$= \left[\frac{\sqrt{x^2+y^2} + x}{2}\right]^{1/2} + i\left[\frac{\sqrt{x^2+y^2} - x}{2}\right]^{1/2}$$

$$u(x,y) = A$$

$$v(x,y) = B$$

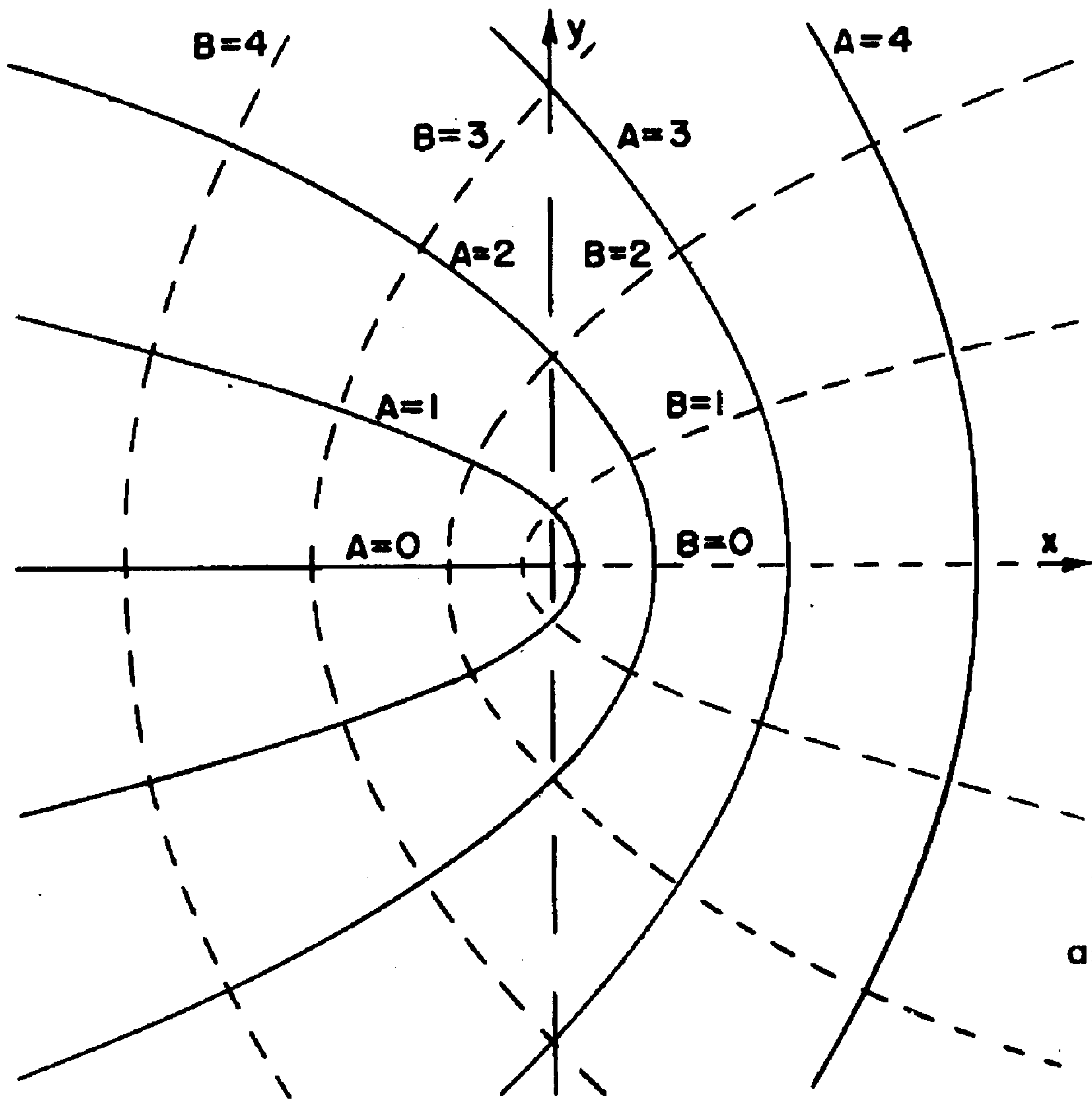


Fig. 7–4.
and $V(x, y)$ f

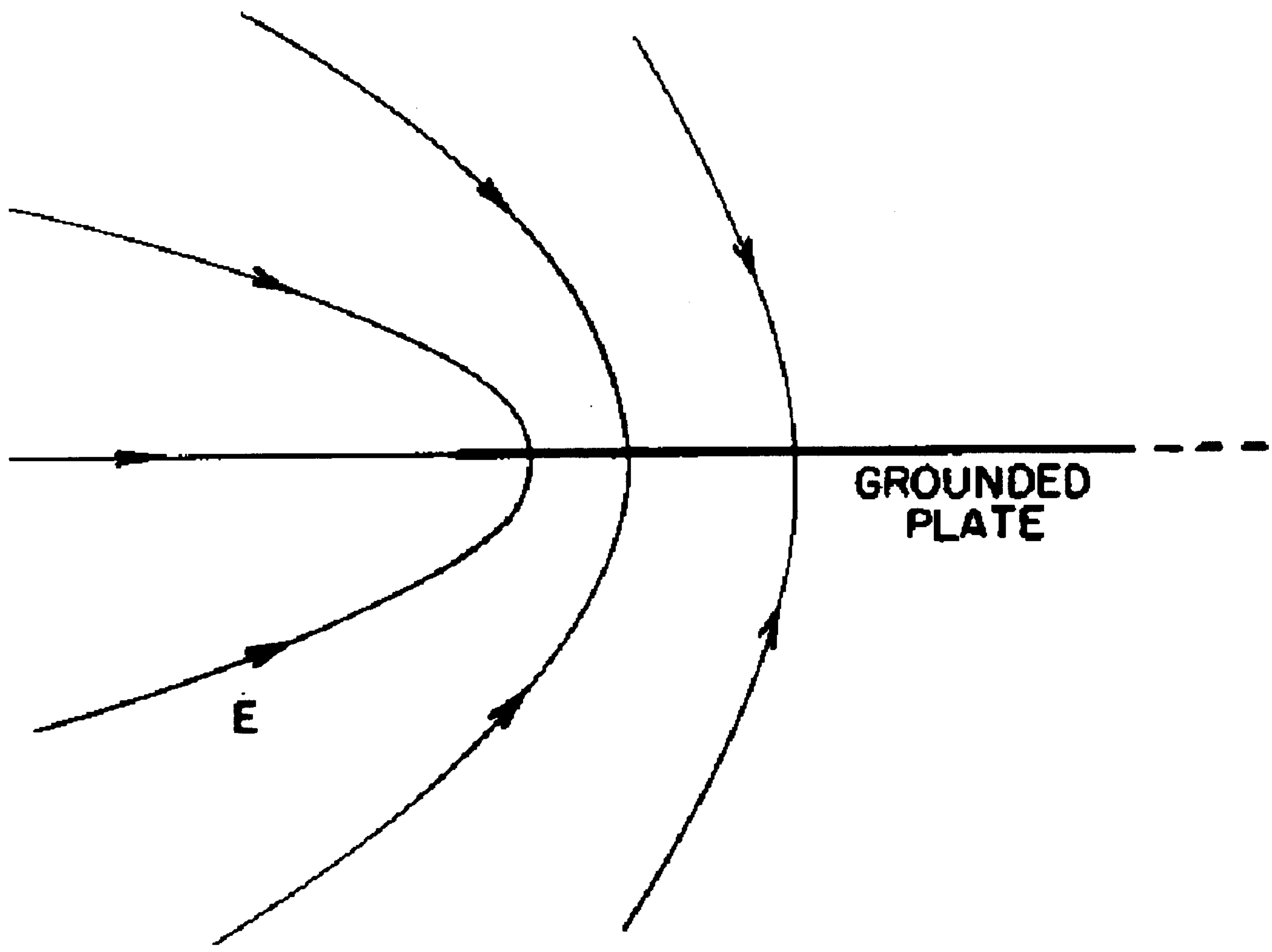


Fig. 7–5. The electric field near the edge of a thin grounded plate.

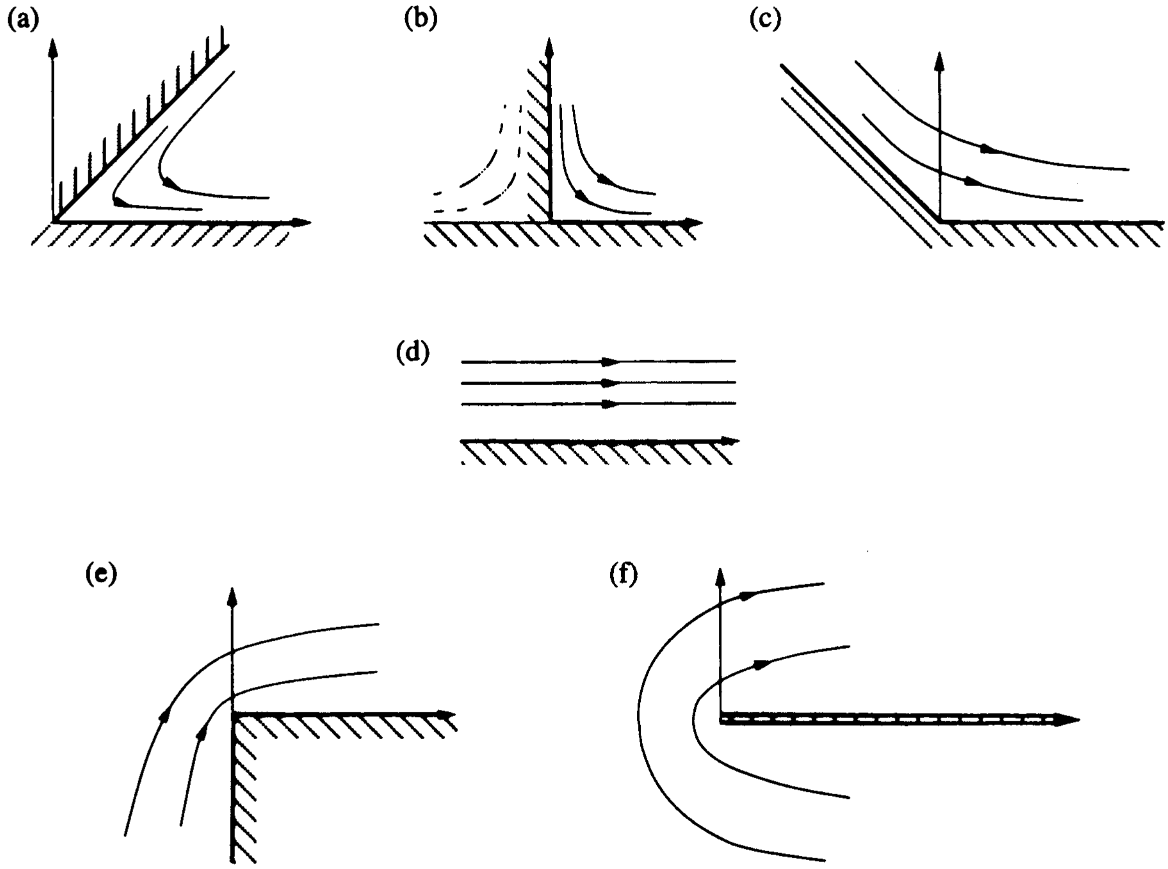


Fig. 6.24 Flows for which the velocity potential has the form $f(z) = Cz^{m+1}$. (a) $m > 1$; (b) $m = 1$; (c) $0 < m < 1$; (d) $m = 0$; (e) $-1/2 < m < 0$; (f) $m = -1/2$.

Many Applications of Laplace's Equation

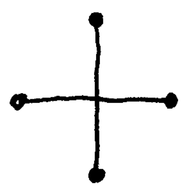
electrostatics	equipotentials	E field lines
gravitostatics	equipotentials	G field lines
magnetostatics	equipotentials	B field lines
current flow	E equipotentails	current flow lines
fluid flow	P equipotentails	fluid flow lines
heat flow	T isothermals	heat flow lines
diffusion (mass)	C isoconcentration	mass flow lines
elasticity	Strain	Stress

NUMERICAL METHODS

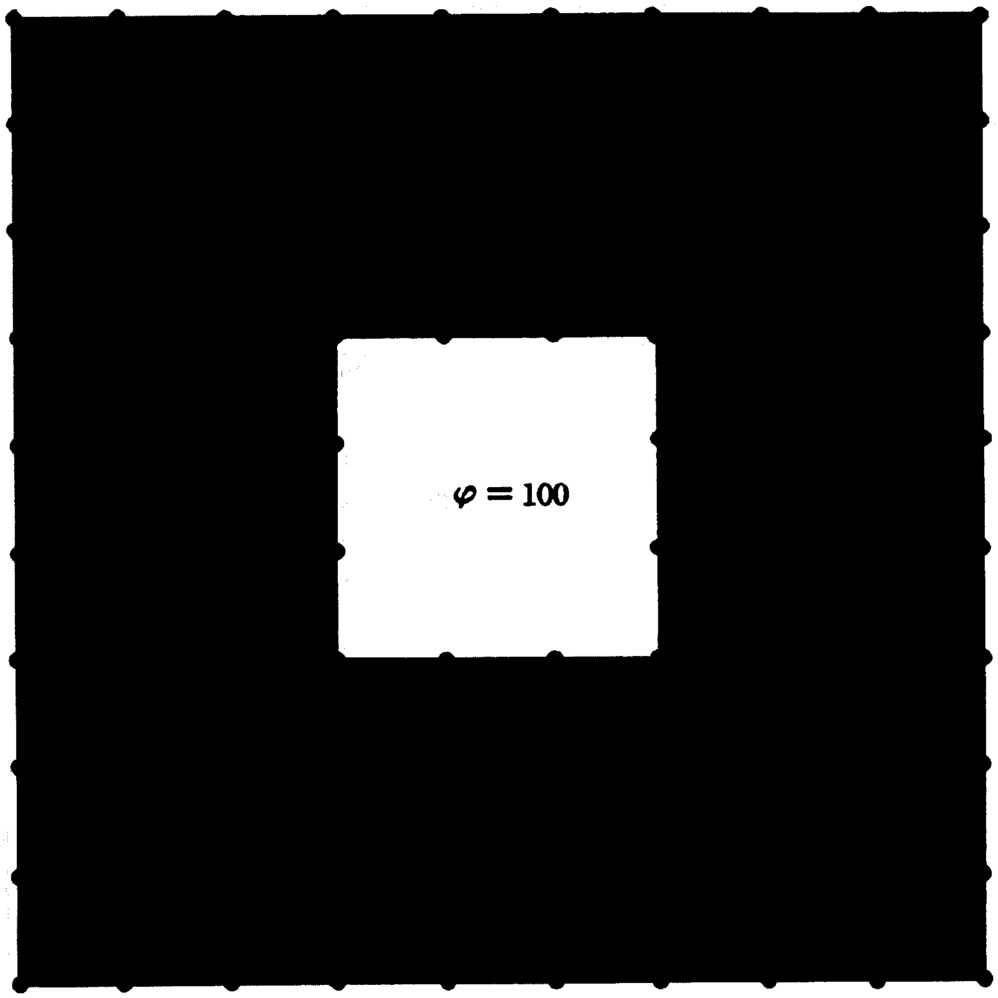
SHOW 2d CASE

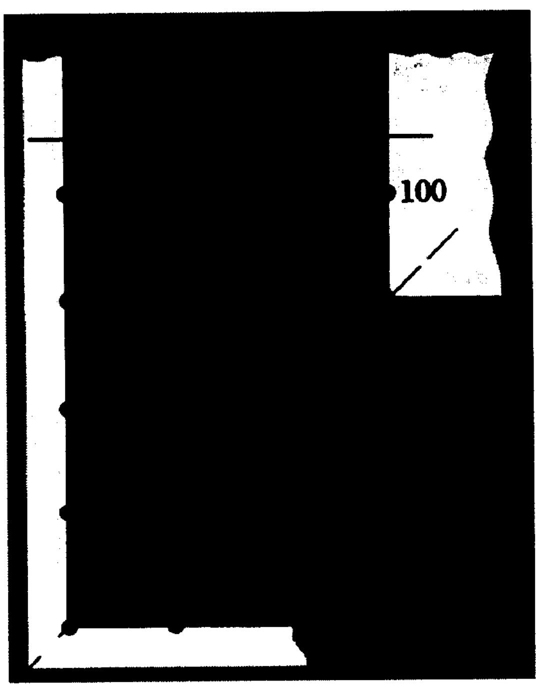
EASY TO GENERALIZE TO 3d

VALUE AT A POINT = AVERAGE OF ITS NEIGHBORS



RELAXATION METHOD


φ = 100


100

MORE EXAMPLES

$z^{3/2}$ field outside rectangular corner

$\log z$ field due to a point / line charge

z^{-1} 2d dipole

SOLUTIONS LOOKING FOR PROBLEMS