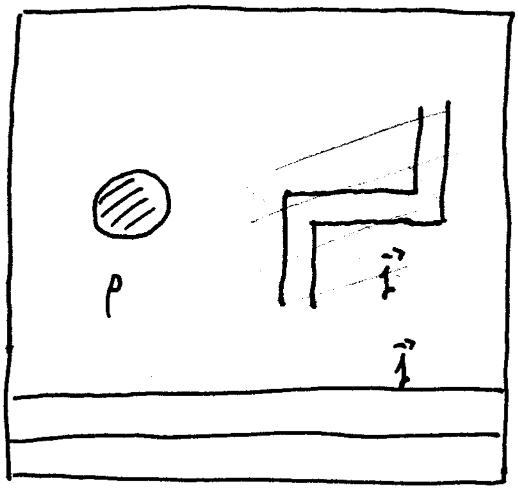
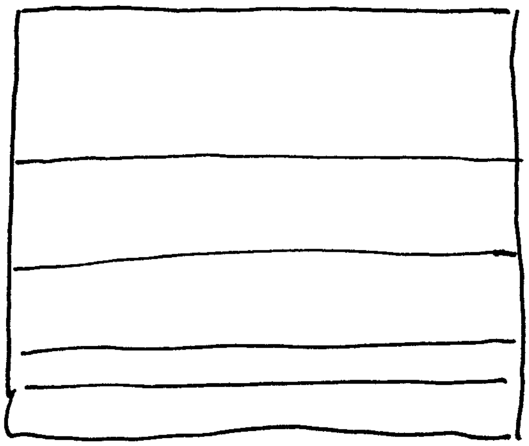
8-17-09

RELATIVITY AND EM



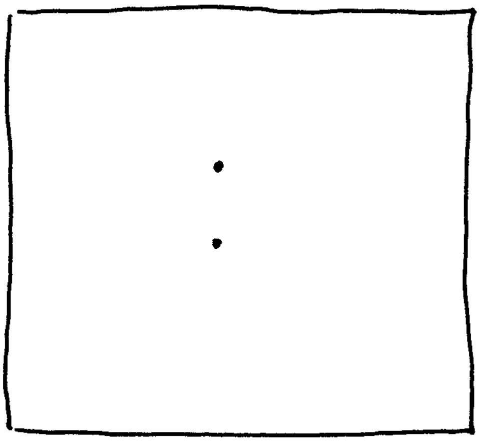


$\rho \to \vec{j}_\rho$

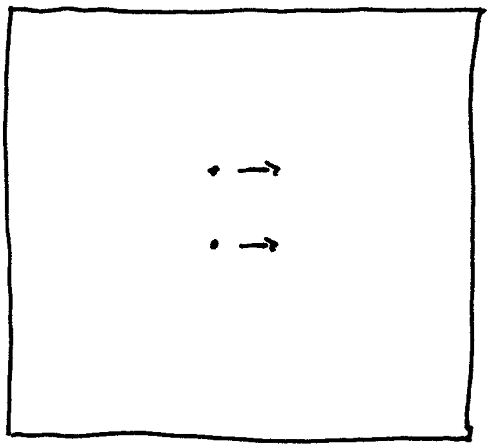
$\vec{j} \to \vec{j}' + \rho'$ in general

$\vec{j} \to \rho'$

TWO POINT CHARGES



COULOMB FORCE

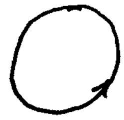


LORENTZ FORCE

$\vec{v}_1 \parallel \vec{v}_2 \Rightarrow F_{MAG}$ is ATTRACTIVE

$F = q(\vec{E} + v \times \vec{B}) \Rightarrow \vec{E}$ is larger

ULTRA RELATIVISTIC $\Rightarrow$ PANCAKE



$\vec{E}$ in the particle frame

$$\vec{E} = O \frac{Q}{r^2} \hat{r}$$

$$O = \frac{1}{4\pi\epsilon_0}$$

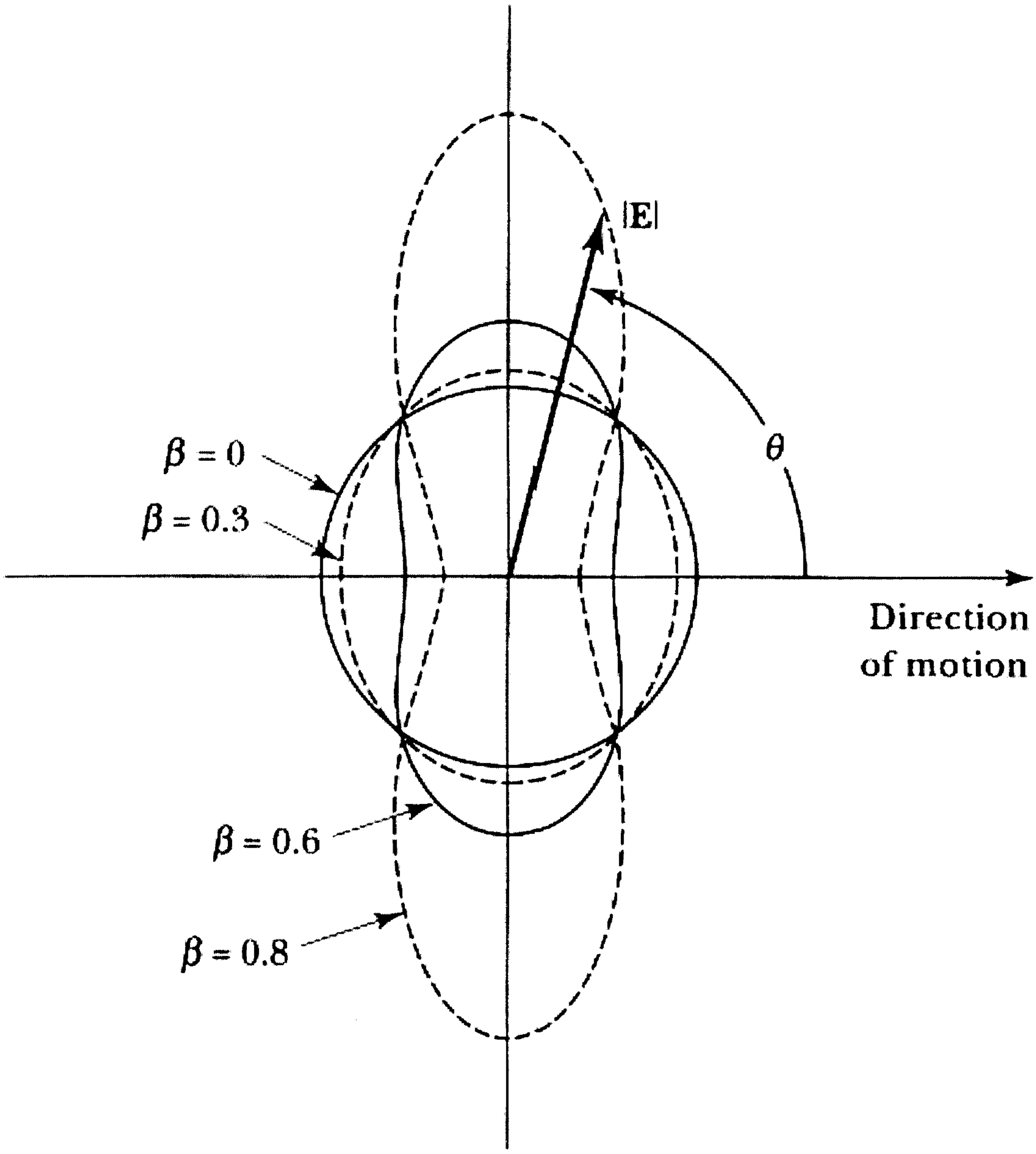
$\vec{E}$ in the lab frame

$$\vec{E} = O \frac{Q}{r^2} \hat{r} \frac{1-\beta^2}{\left[1-\beta^2 \sin^2\theta\right]^{3/2}} \hat{r}$$

AZIMUTHALLY SYMMETRIC

NOT SPHERICALLY SYMMETRIC special direction

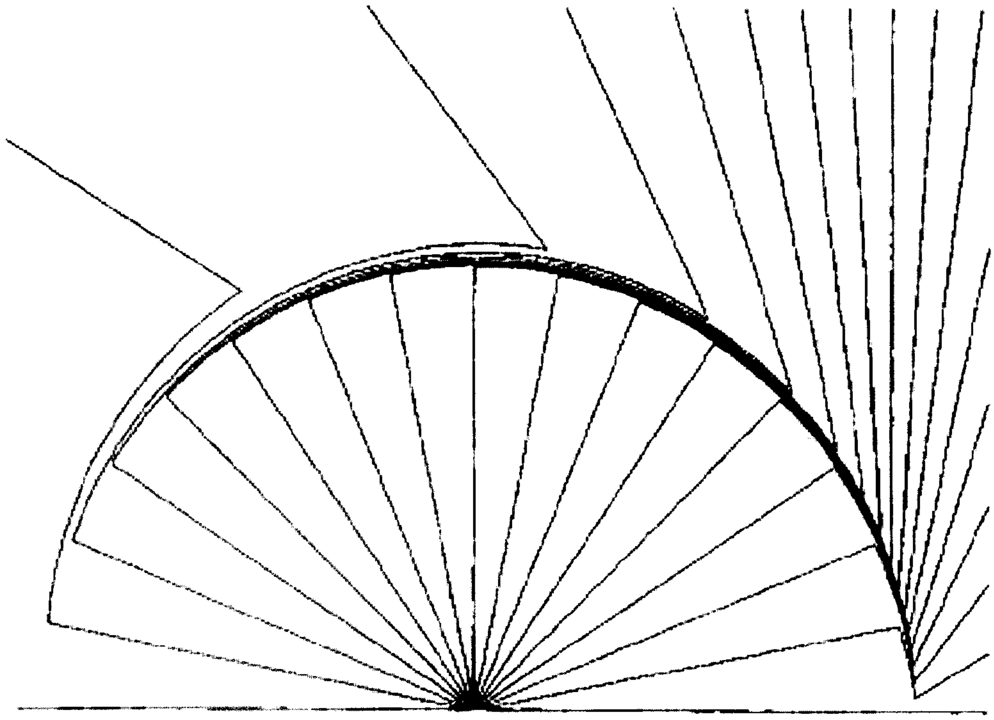
Compression of the Electric Field Lines



ß

Before $\beta = 0.95$

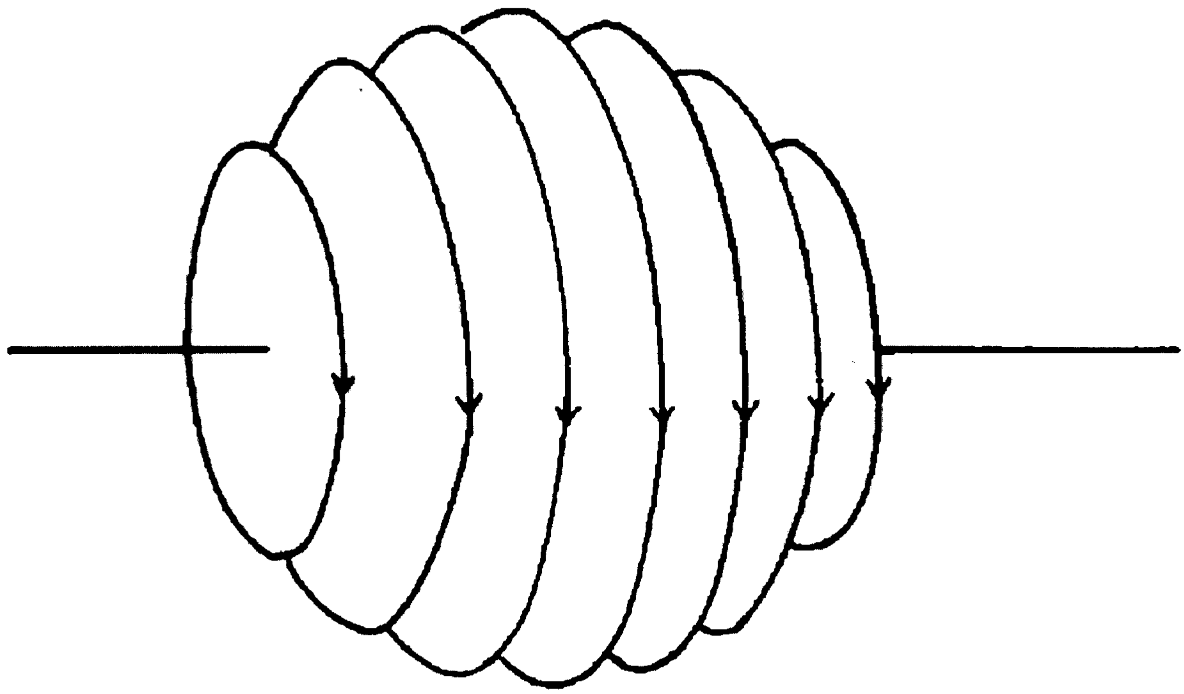
After $\beta = 0$



radial lines ~ r^{-2}

tangential lines ~ r^{-1}

Transverse
Magnetic Field



D

SPACE AND TIME $\Longrightarrow$ SPACE TIME

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \leftarrow \text{4 VECTOR}$$

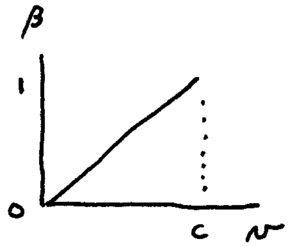
$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underset{\sim}{L} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

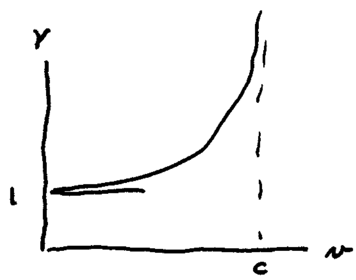
if $\vec{v} \parallel \hat{x}$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\beta = v/c$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$





SPACETIME HAS 4d

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Special Relativity in Wikipedia

$$\begin{bmatrix} c\,t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_x\,\gamma & -\beta_y\,\gamma & -\beta_z\,\gamma \\ -\beta_x\,\gamma & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\,\gamma & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\,\gamma & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix}$$

Special Relativity in Wikipedia

PROBLEM
REASON: $\vec{E}$ AND $\vec{B}$ DO NOT FORM A 4-VECTOR!

$\vec{E}$ 3 components

$\vec{B}$ 3 components

TOGETHER 6 degrees of freedom

$6 > 4$ so no four-vector

$6 < 16$ so enough room in 4×4 matrix

too much space in " "

ANTISYMMETRIC $a_{ij} = -a_{ji}$

$$\begin{pmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{pmatrix}$$

so 6 degrees of freedom

$$\Downarrow$$

$$\underset{\approx}{F}$$

$\vec{E}$ and $\vec{B}$ are the components of the electromagnetic field tensor

$$\underset{\approx}{F}' = L \underset{\approx}{F} L^T$$

$$\vec{v}' = L \vec{v}$$

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

Electromagnetic Tensor in Wikipedia

USING THE POTENTIALS (WHEN MOTION IS NOT UNIFORM)

ρ IS SOURCE OF V

ELECTRIC POTENTIAL

$$V(\vec{r}_1, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}_2, t')}{r_{12}} d^3 r_2$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$t' = t - \frac{r}{c}$ RETARDED TIME

$t' = t + \frac{r}{c}$ ADVANCED TIME

where the sources were

$\vec{j}$ IS SOURCE OF $\vec{B}$

MAGNETIC POTENTIAL

$$\vec{A}(\vec{r}_1, t) = \mu_0 \int \frac{\vec{j}(\vec{r}_2, t')}{r_{12}} d^3 r_2$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

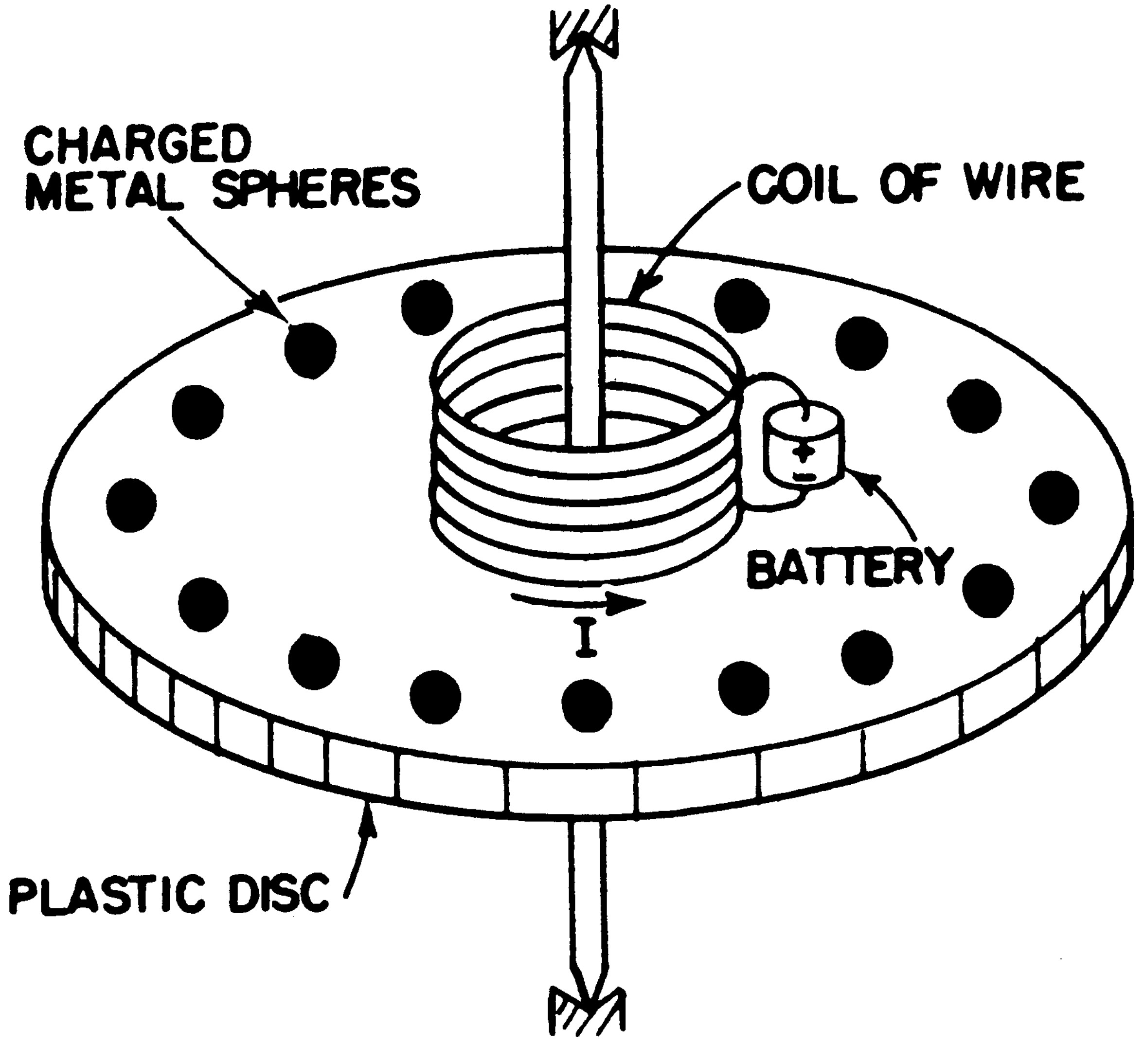
$$\begin{pmatrix} V \\ A_x \\ A_y \\ A_z \end{pmatrix} \xrightarrow{L} \begin{pmatrix} V' \\ A_x' \\ A_y' \\ A_z' \end{pmatrix}$$

USING THE FIELDS

$$\underset{\approx}{L} \begin{pmatrix} \text{EM} \\ \text{tensor} \end{pmatrix} \underset{\approx}{L}^T$$

BEAUTIFUL UNITS

$$\epsilon_0 = \mu_0 = c = 1$$


CHARGED
METAL SPHERES
COIL OF WIRE
BATTERY
I
PLASTIC DISC

ably chosen $\boldsymbol{B}$, can cause the electron to keep moving on its assumed orbit. In the betatron this transverse force causes the electron to move in a circular orbit of constant radius. We can find out what the magnetic field at the orbit must be by using again the relativistic equation of motion, but this time, for the transverse component of the force. In the betatron (see Fig. 17–4), $\boldsymbol{B}$ is at right angles to $\boldsymbol{v}$, so the transverse force is qvB. Thus the force is equal to the rate of change of the transverse component p_t of the momentum:

$$qvB = \frac{dp_t}{dt}. \tag{17.8}$$

When a particle is moving in a *circle*, the rate of change of its transverse momentum is equal to the magnitude of the total momentum times ω, the angular velocity of rotation (following the arguments of Chapter 11, Vol. I):

$$\frac{dp_t}{dt} = \omega p, \tag{17.9}$$

where, since the motion is circular,

$$\omega = \frac{v}{r}. \tag{17.10}$$

Setting the magnetic force equal to the transverse acceleration, we have

$$qvB_{\text{orbit}} = p\,\frac{v}{r}, \tag{17.11}$$

where B_{orbit} is the field at the radius r.

As the betatron operates, the momentum of the electron grows in proportion to B_{av}, according to Eq. (17.7), and if the electron is to continue to move in its proper circle, Eq. (17.11) must continue to hold as the momentum of the electron increases. The value of B_{orbit} must increase in proportion to the momentum p. Comparing Eq. (17.11) with Eq. (17.7), which determines p, we see that the following relation must hold between B_{av}, the average magnetic field *inside* the orbit at the radius r, and the magnetic field B_{orbit} at the orbit:

$$\Delta B_{\text{av}} = 2\,\Delta B_{\text{orbit}}. \tag{17.12}$$

The correct operation of a betatron requires that the average magnetic field inside the orbit increase at twice the rate of the magnetic field at the orbit itself. In these circumstances, as the energy of the particle is increased by the induced electric field the magnetic field at the orbit increases at just the rate required to keep the particle moving in a circle.

The betatron is used to accelerate electrons to energies of tens of millions of volts, or even to hundreds of millions of volts. However, it becomes impractical for the acceleration of electrons to energies much higher than a few hundred million volts for several reasons. One of them is the practical difficulty of attaining the required high average value for the magnetic field inside the orbit. Another is that Eq. (17.6) is no longer correct at very high energies because it does not include the loss of energy from the particle due to its radiation of electromagnetic energy (the so-called synchrotron radiation discussed in Chapter 36, Vol. I). For these reasons, the acceleration of electrons to the highest energies—to many billions of electron volts—is accomplished by means of a different kind of machine, called a *synchrotron*.

17–4 A paradox

We would now like to describe for you an apparent paradox. A paradox is a situation which gives one answer when analyzed one way, and a different answer when analyzed another way, so that we are left in somewhat of a quandary as to actually what should happen. Of course, in physics there are never any real paradoxes because there is only one correct answer; at least we believe that nature will act in only one way (and that is the *right way*, naturally). So in physics a paradox is only a confusion in our own understanding. Here is our paradox.

Imagine that we construct a device like that shown in Fig. 17–5. There is a thin, circular plastic disc supported on a concentric shaft with excellent bearings, so that it is quite free to rotate. On the disc is a coil of wire in the form of a short solenoid concentric with the axis of rotation. This solenoid carries a steady current I provided by a small battery, also mounted on the disc. Near the edge of the disc and spaced uniformly around its circumference are a number of small metal spheres insulated from each other and from the solenoid by the plastic material of the disc. Each of these small conducting spheres is charged with the same electrostatic charge Q. Everything is quite stationary, and the disc is at rest. Suppose now that by some accident—or by prearrangement—the current in the solenoid is interrupted, without, however, any intervention from the outside. So long as the current continued, there was a magnetic flux through the solenoid more or less parallel to the axis of the disc. When the current is interrupted, this flux must go to zero. There will, therefore, be an electric field induced which will circulate around in circles centered at the axis. The charged spheres on the perimeter of the disc will all experience an electric field tangential to the perimeter of the disc. This electric force is in the same sense for all the charges and so will result in a net torque on the disc. From these arguments we would expect that as the current in the solenoid disappears, the disc would begin to rotate. If we knew the moment of inertia of the disc, the current in the solenoid, and the charges on the small spheres, we could compute the resulting angular velocity.

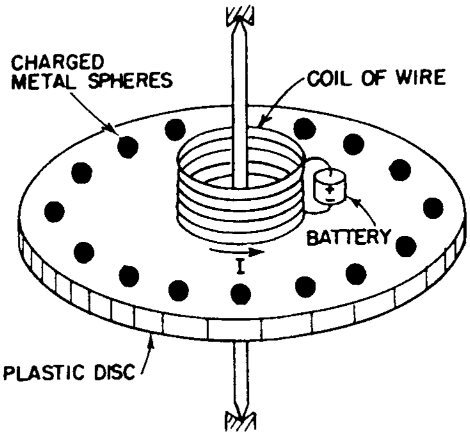


Fig. 17–5. Will the disc rotate if the current I is stopped?

But we could also make a different argument. Using the principle of the conservation of angular momentum, we could say that the angular momentum of the disc with all its equipment is initially zero, and so the angular momentum of the assembly should remain zero. There should be no rotation when the current is stopped. Which argument is correct? Will the disc rotate or will it not? We will leave this question for you to think about.

We should warn you that the correct answer does not depend on any nonessential feature, such as the asymmetric position of a battery, for example. In fact, you can imagine an ideal situation such as the following: The solenoid is made of superconducting wire through which there is a current. After the disc has been carefully placed at rest, the temperature of the solenoid is allowed to rise slowly. When the temperature of the wire reaches the transition temperature between superconductivity and normal conductivity, the current in the solenoid will be brought to zero by the resistance of the wire. The flux will, as before, fall to zero, and there will be an electric field around the axis. We should also warn you that the solution is not easy, nor is it a trick. When you figure it out, you will have discovered an important principle of electromagnetism.

17–5 Alternating-current generator

In the remainder of this chapter we apply the principles of Section 17–1 to analyze a number of the phenomena discussed in Chapter 16. We first look in more detail at the alternating-current generator. Such a generator consists basically of a coil of wire rotating in a uniform magnetic field. The same result can also be achieved by a fixed coil in a magnetic field whose direction rotates in the manner described in the last chapter. We will consider only the former case. Suppose we have a circular coil of wire which can be turned on an axis along one of its diameters. Let this coil be located in a uniform magnetic field perpendicular to the axis of rotation, as in Fig. 17–6. We also imagine that the two ends of the coil are brought to external connections through some kind of sliding contacts.

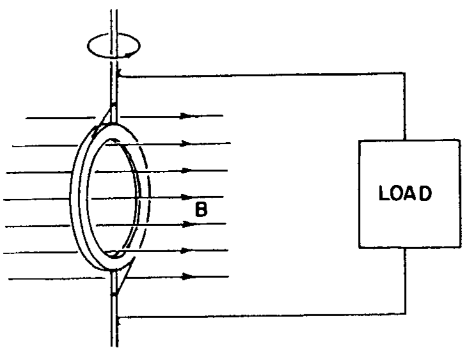


Fig. 17–6. A coil of wire rotating in a uniform magnetic field—the basic idea of the ac generator.

Due to the rotation of the coil, the magnetic flux through it will be changing. The circuit of the coil will therefore have an emf in it. Let S be the area of the coil and θ the angle between the magnetic field and the normal to the plane of the coil.*

* Now that we are using the letter A for the vector potential, we prefer to let S stand for a Surface area.

EM FIELDS HAVE ENERGY
MOMENTUM

$(E, \vec{p})$

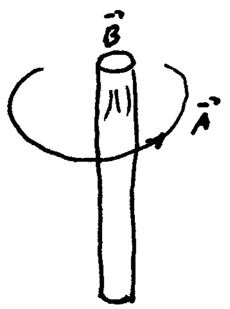
$(V, \vec{A})$

ANGULAR MOMENTUM ~~IMPARTED~~
IN FIELD

$\Downarrow$

MECHANICAL ANG MOM.

WHAT DOES $\vec{A}$ LOOK LIKE?



LINES OF $\vec{A}$ WRAP AROUND LINES OF $\vec{B}$
LIKE
LINES OF $\vec{B}$ WRAP AROUND LINES OF $\vec{j}$

particle momentum → $\vec{p}$; field momentum → $e\vec{A}$

$$\frac{p^2}{2m} \Rightarrow \frac{\pi^2}{2m} = \frac{(\vec{p} - e\vec{A})^2}{2m}$$

The Aharonov-Bohm Experiment