LECTURE 28 8-10-09

THE MATHEMATICS OF ELECTROMAGNETISM

VECTOR CALCULUS

122 GLOBAL VIEW

321 LOCAL VIEW

THIS WEEK :	$\vec{E}$	M
	V	W
	$\vec{B}$	θ
	$\vec{A}$	F

FOUR MAXWELL EQUATIONS

$$\int_V \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\int_V \vec{B}\cdot d\vec{S} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\int \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\int \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\epsilon_0\frac{d\phi_E}{dt}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

STATICS

ELECTROSTATICS

T $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ HAS DIVERGENCE

F $\vec{\nabla} \times \vec{E} = 0$ ZERO CURL ~~CURLLESS~~

MAGNETOSTATICS

T $\vec{\nabla} \cdot \vec{B} = 0$ ZERO DIVERGENCE~~LESS~~

F $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ HAS CURL

ELECTROSTATICS	ELECTRODYNAMICS
$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2}$	
$\vec{F} = q\vec{E}$	$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
COULOMB FORCE	LORENTZ FORCE

HOW CAN WE DRAW/REPRESENT $\vec{E}(\vec{r})$

vector

every point in space

FOR 2d ~~[illegible]~~

(1) DRAW SOME SUBSET

(2) ANIMATION

(3) FIELD LINES

(4) TOPOGRAPHIC MAP
- 3d
- contours

www.falstad.com

DISTRIBUTIONS OF CHARGE

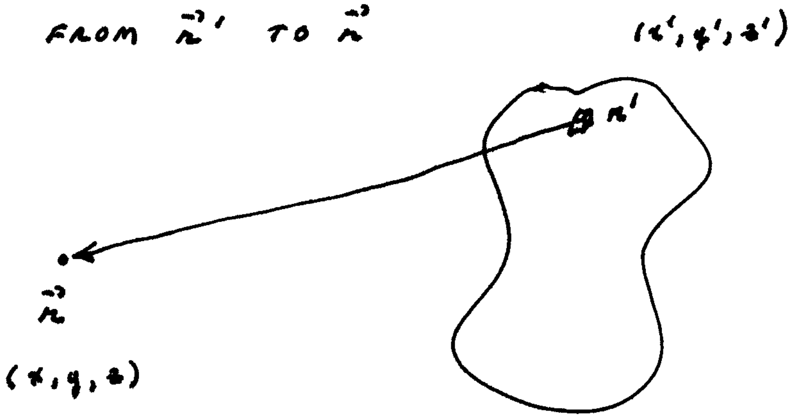
$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|^2} \hat{r}_{0i}$$

for 10^{23} electrons

$$\sum \Longrightarrow \int$$

$$\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{r^2} \hat{r} \, d^3 r'$$

$\hat{r}$ FROM $\vec{r}\,'$ TO $\vec{r}$



$$E(x, y, z) = \int \frac{\rho(x', y', z')}{r^2} \hat{r} \, dx' \, dy' \, dz'$$

GAUSS' LAW

$$\int \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

GAUSS' THM

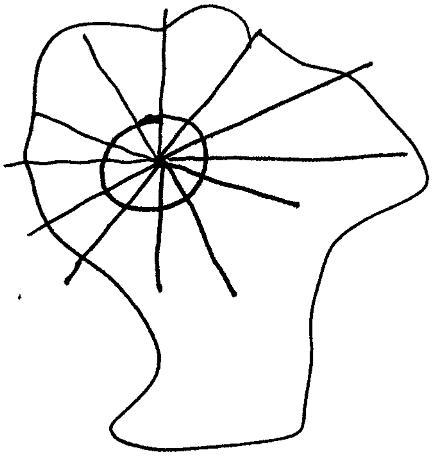
DIVERGENCE THM

$$\int_S \vec{F} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{F} \, dV$$

GAUSS' LAW $\quad E \sim \frac{1}{r^2}$

GAUSS' THM $\quad$ ANY VECTOR FIELD

TWO INTUITIVE WAYS TO THINK
BUT FIRST, ~~MORE~~ ABOUT GAUSS'S LAW



Pursell's argument

$d\Omega_1 \longrightarrow d\Omega_2$

$\vec{E}_1 \longrightarrow \vec{E}_2$

$E \sim \frac{1}{r^2} \qquad d\Omega \sim r^2$

$\vec{E}_2 \cdot d\vec{S}_2 = E_2 \, dS_2 \cos\theta$

① FARADAY'S LINES OF FORCE: START ON A CHARGE

CAN ONLY END ON A CHARGE

OR AT ∞

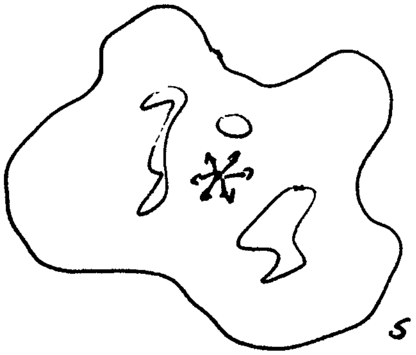
if only 1 charge

N lines near charge $\Rightarrow$ N lines at ∞

$\Rightarrow$ N lines thru any closed surface

Lines are conserved.

② WATER ~~[illegible]~~



WATER ~~[illegible]~~ THRU S

Σ WATER FROM SOURCE

SOURCES

SINKS

Water is conserved

LECTURE 2 10-1-08

USING GAUSS' LAW

$$\left.\begin{array}{c}\text{GAUSS' LAW} \\ + \\ \text{SYMMETRY}\end{array}\right\} \Rightarrow \vec{E}$$

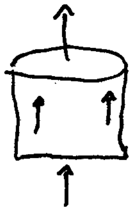
3 SYMMETRIES : PLANE
LINE
SPHERE

PLANE SHEET UNIFORM CHARGE DENSITY σ $\dfrac{[\text{COULOMBS}]}{[\text{METER}]^2}$

GAUSSIAN PILL BOX

SIDES $\perp$

FACES $\parallel$



BY SYMMETRY $\vec{E} \perp$ PLANE

$\vec{E} \perp \hat{m}$ $\vec{E} \cdot \vec{A} = EA$

$\vec{E} \cdot \vec{S} = 0$

$$\int \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

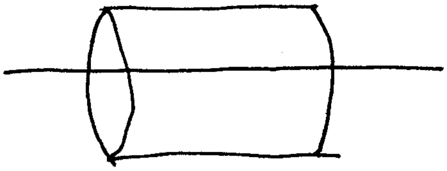
$$EA + 0 + EA = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{1}{2} \frac{\sigma}{\epsilon_0}$$

$E(r) =$ constant $\sim \dfrac{1}{r^0}$

LINE

$$\lambda = \frac{[\text{COULOMBS}]}{[\text{METER}]}$$



$$\int \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

$$0 + ES + 0 = \frac{Q}{\epsilon_0}$$

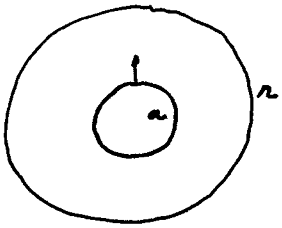
$$E\,(2\pi r)(\cancel{l}) = \frac{\lambda\,(\cancel{l})}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{r}\right)$$

$$E(r) \sim \frac{1}{r}$$

SPHERE

CHARGE DENSITY $\rho = \frac{[\text{COULOMB}]}{[\text{METER}]^3}$



$$\int \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

$\vec{E} \perp d\vec{S}$

E CONSTANT

$$E\,S = \frac{Q}{\epsilon_0}$$

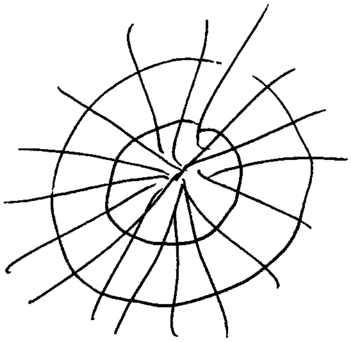
$$E\,(4\pi r^2) = \cancel{\frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\epsilon_0}} \qquad \cancel{\frac{\rho}{\frac{4}{3}\pi r^3}}$$

$$= \frac{Q}{\epsilon_0}$$

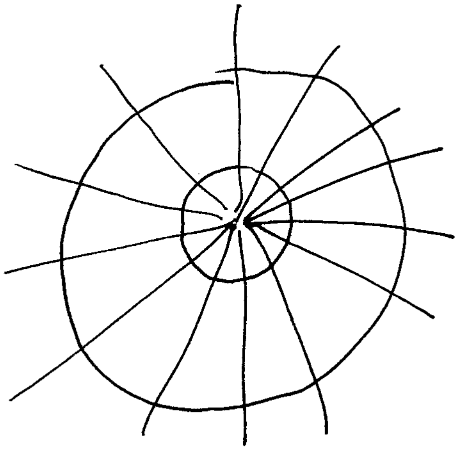
$$E = \frac{1}{4\pi\epsilon_0}\ \frac{Q}{r^2}$$

$$E(r) \sim \frac{1}{r^2}$$

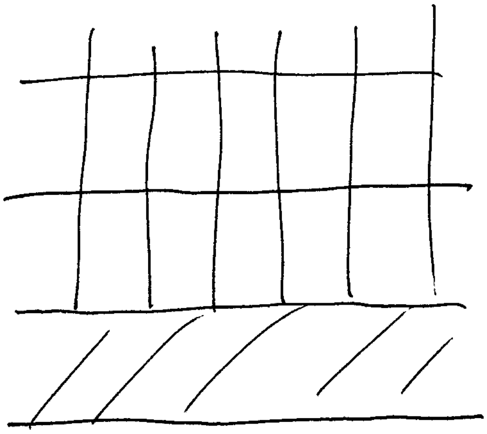
3 d	$\frac{1}{r^2}$
2 d	$\frac{1}{r}$
1 d	$\frac{1}{r^0}$
n d	$r^{-(n-1)}$



lines per area fixed



lines per circumference fixed



lines fixed

ELECTRIC FIELDS VIA INTEGRATION

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{R^2} \hat{R} \, d^3r'$$

$$\vec{R} = \vec{r} - \vec{r}\,'$$

$$R^2 = |\vec{R}|^2$$

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|}$$

SHEET
LINE
~~CYLINDER~~
~~SPHERE~~

RELATIVELY EASY (STRAIGHTFORWARD) TO INTEGRATE

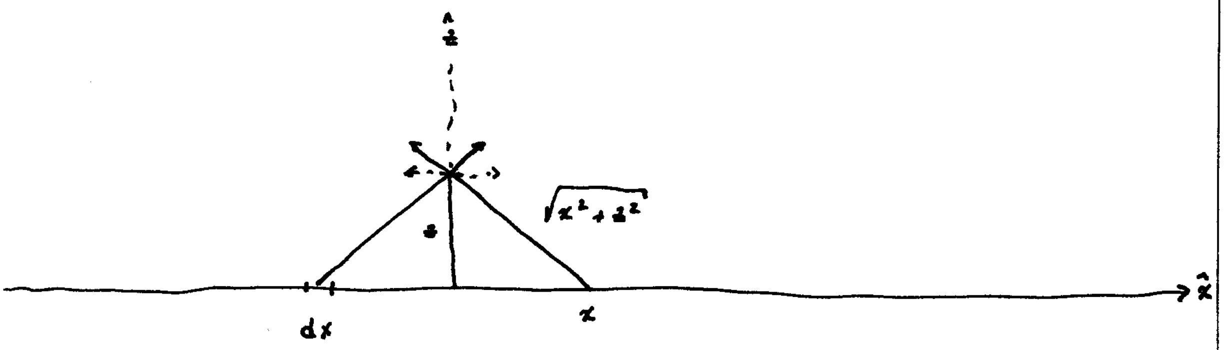
VECTOR INTEGRAL ⇒ TURN INTO 3 SCALAR INTEGRALS

E_x E_r E_ρ

E_y E_θ E_ϕ

E_z E_ϕ E_z

USE SYMMETRIES TO MAKE INTEGRALS AS SIMPLE AS POSSIBLE

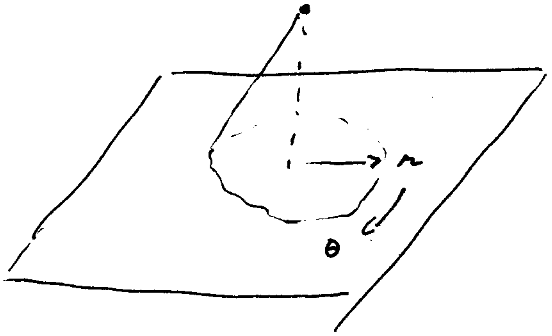


INFINITE LINE

G FINITE LINE (CENTER)

FINITE LINE
ANY WHERE

HW P QUAL



INFINITE => SAME AS GAUSS' LAW

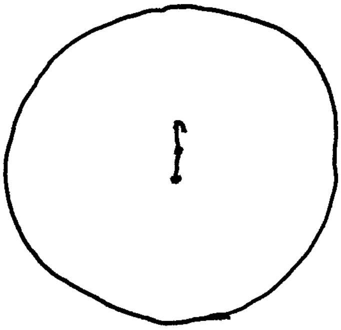
HW FINITE => DISK
(CENTER)

HW RING

GRADED OBJECTS IN 3d (GAUSSIAN SURFACES)

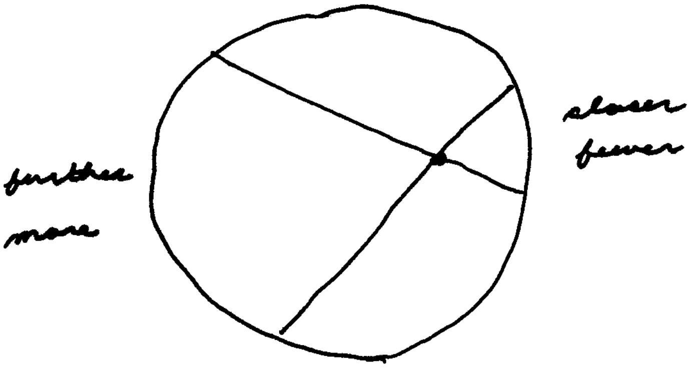
SLABS	$\rho(z)$	
CYLINDERS	$\rho(r)$	G. EXAMPLE
SPHERES	$\rho(r)$	HW G

WHAT IS THE FIELD INSIDE A SPHERICAL SHELL?



At the center, perfect symmetry

$\vec{F} = 0$



NEWTON 1686 SHOWED $\vec{F} = 0$ everywhere inside

OUTSIDE, LOOKED LIKE A POINT AT THE CENTER

DIFFERENTIAL FORM OF GAUSS' LAW

$$\int \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{Q}{\epsilon_0}$$

WHAT IS THE DIVERGENCE?

$$\vec{\nabla} \cdot \vec{E} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(E_x, E_y, E_z \right)$$

VECTOR OPERATOR

VECTOR FIELD

in cartesian coords

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

SCALAR FIELD

$\text{div}\ \vec{E} = \vec{\nabla} \cdot \vec{E}$ divergence

$$\vec{\nabla} \cdot \vec{E} = \lim_{V \to 0} \int_S \vec{E} \cdot d\vec{S}$$

S is the boundary of V

$\vec{\nabla} \cdot \vec{E}$ = "flux out of a point"

$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$ LOCAL VERSION

$\int \vec{E} \cdot d\vec{S} = Q/\epsilon_0$ GLOBAL VERSION

GAUSS' THEOREM

$$\int_S \vec{E} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{E} \, dV$$

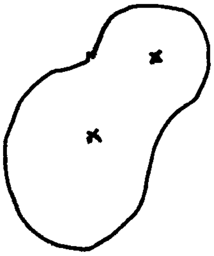
TRUE FOR ANY VECTOR FIELD!

FOR $\vec{E}$

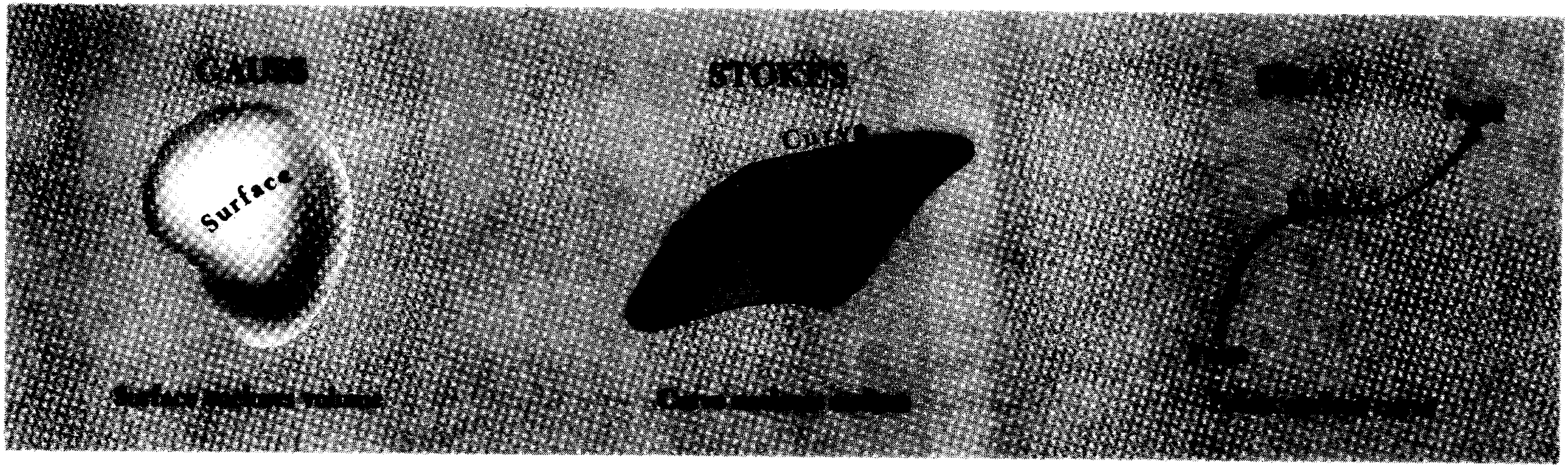
$$\int_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0} = \int_V \rho(\vec{r}) \, d^3r$$

ANOTHER VIEW

$$\int_V (\text{sources}) \, dV = \int_S \text{flow thru surface}$$



faucets	water
charges	E flux
⋮	⋮



$$\int_{\text{surface}} \mathbf{F}\cdot d\mathbf{a} = \int_{\text{volume}} \operatorname{div} \mathbf{F}\, dv \qquad \int_{\text{curve}} \mathbf{A}\cdot d\mathbf{s} = \int_{\text{surface}} \operatorname{curl} \mathbf{A}\cdot d\mathbf{a} \qquad \varphi_2 - \varphi_1 = \int_{\text{curve}} \operatorname{grad} \varphi\cdot d\mathbf{s}$$

The History of Stokes' Theorem

Let us give credit where credit is due: Theorems of Green, Gauss and Stokes appeared unheralded in earlier work.

VICTOR J. KATZ
University of the District of Columbia
Washington, D.C. 20005

Most current American textbooks in advanced calculus devote several sections to the theorems of Green, Gauss, and Stokes. Unfortunately, the theorems referred to were not original to these men. It is the purpose of this paper to present a detailed history of these results from their origins to their generalization and unification into what is today called the generalized Stokes' theorem.

Origins of the theorems

The three theorems in question each relate a k-dimensional integral to a $k-1$-dimensional integral; since the proof of each depends on the fundamental theorem of calculus, it is clear that their origins can be traced back to the late 17th century. Toward the end of the 18th century, both Lagrange and Laplace actually used the fundamental theorem and iteration to reduce k-dimensional integrals to those of one dimension less. However, the theorems as we know them today did not appear explicitly until the 19th century.

The first of these theorems to be stated and proved in essentially its present form was the one known today as Gauss' theorem or the divergence theorem. In three special cases it occurs in an 1813 paper of Gauss **[8]**. Gauss considers a surface (superficies) in space bounding a solid body (corpus). He denotes by PQ the exterior normal vector to the surface at a point P in an infinitesimal element of surface ds and by QX, QY, QZ the angles this vector makes with the positive x-axis, y-axis, and z-axis respectively. Gauss then denotes by $d\Sigma$ an infinitesimal element of the y-z plane and erects a cylinder above it, this cylinder intersecting the surface in an even number of infinitesimal surface elements $ds_1, ds_2, \ldots, ds_{2n}$. For each j, $d\Sigma = \pm ds_j \cos QX_j$ where the positive sign is used when the angle is acute, the negative when the angle is obtuse. Since if the cylinder enters the surface where QX is obtuse, it will exit where QX is acute (see FIGURE 1), Gauss obtains $d\Sigma = -ds_1 \cos QX_1 = ds_2 \cos QX_2 = \ldots$ and concludes by summation that "The integral $\int ds \cos QX$ extended to the entire surface of the body is 0."

He notes further that if T, U, V are rational functions of only y, z, only x, z, and only x, y respectively, then "$\int (T \cos QX + U \cos QY + V \cos QZ) ds = 0$." Gauss then approximates the volume of the body by taking cylinders of length x and cross sectional area $d\Sigma$ and concludes in a similar way his next theorem: "The entire volume of the body is expressed by the integral $\int ds\, x(\cos QX)$ extended to the entire surface." We will see below how these results are special cases of the divergence theorem.

Ostrogadkssy discovered Gauss' theorem

Cauchy discovered Green's theorem

Lord Kelvin (William Thomson) discovered Stokes' theorem

Bourbaki discovered the general case
(called the generalized Stokes' Theorem)

Ostrogadkssy-Gauss Theorem
Cauchy-Green Theorem
Kelvin-Stokes Theorem

FLUX LINES lines are conserved

WATER FLOW volume is conserved

ELECTRIC FLUX flux is conserved Maxwell's math for Faraday's lines

$$\nabla \cdot \rho = 0$$

whenever there are no sources!

what happen if not Inverse Square Law?

$$\frac{C_1}{r}\,\hat{r}$$

more water per unit solid angle

central force

falls off more slowly

$$\frac{C_2}{r^3}\,\hat{r}$$

less water per $d\Omega$

more quickly

$$C_3\,\frac{e^{-\lambda r}}{r^2}\,\hat{r}$$

less water per $d\Omega$

more quickly

Only for r^{-2} are water, lines, flux conserved!

Physics 543 Electromagnetic Theory
Autumn Quarter 2008

Professor: Larry Sorensen
Office: Physics-Astronomy Building B-435
Phone: 543-0360
e-mail: seattle@u.washington.edu
Office hours: Right after class, or by appointment

Class Website: http://faculty.washington.edu/seattle

Required texts: **Electricity and Magnetism** by Edward Purcell
Introduction to Electrodynamics by David Griffiths

Purcell presents a beautiful, masterful, and extremely physical treatment of EM. I expect you to master everything in Purcell---and I will do my best to assist you.

Griffiths' book is more mathematical than Purcell's. In his preface, Griffiths writes: "Practically everything I know about electrodynamics---certainly about teaching electrodynamics---I owe to Edward Purcell."

Read Purcell for the physics. Read Griffiths for the math.

Homework: There will be 10 homework assignments, one per chapter. They will be posted on the class website on the Monday that we begin each chapter, and they will be due on the following Wednesday.
See the course schedule for details.

Exams: There will be three take home exams. You will have two weeks---including two weekends---to do each exam. See the exam schedule for details.

Grades: Your course grade will be based on your homework and exam scores.

Page contents:

Description
Homework
Applets
Lecture Notes
Reading

Physics 543, Autumn 2008
Electromagnetic Theory

http://faculty.washington.edu/seattle/

Instructor: Larry Sorensen
Email: seattle@u.washington.edu

Office: B-435 Physics-Astronomy Building
Office Hours: Right after class--or by appointment
Telephone: 543-0360

Description

Introduction to static electromagnetism:
Static electric and magnetic fields
Boundary-value problems
The electric and magnetic properties of materials
Maxwell's equations

Course Information
Exam 1
Exam 2
Exam 3

Homework

Homework Assignment 1
Purcell Problems
Example Purcell Solutions
Griffiths Problems
Purcell Solutions
Griffiths Solutions

See these solutions to learn how to solve EM problems

Homework Assignment 2
Purcell Problems
Example Purcell Solutions
Griffiths Problems
Purcell Solutions
Griffiths Solutions

Homework Assignment 3
Purcell Problems
Example Purcell Solutions
Griffiths Problems
Purcell Solutions
Griffiths Solutions

Homework Assignment 4
Purcell Problems
Purcell Solutions to Chapter 4

Homework Assignment 5
Feynman's Version
Purcell Problems
Purcell Solutions to Chapter 5

Purcell Solutions to Chapter 6

Purcell Solutions to Chapter 7

Purcell Solutions to Chapter 9

Purcell Solutions to Chapter 10

Purcell Solutions to Chapter 11

PURCELL PROBLEMS

1.5 SEMICIRCLE

1.16 SPHERE WITH SPHERE REMOVED

1.19 PLANAR SHEET next to PLANAR SLAB

1.24 ROD ON and OFF AXIS (OFF wrt CENTER)

1.29 SPHERICAL SHELL MISSING DISK

PURCELL EXAMPLES

SPHERE, SPHERICAL SHELL

LINE

SHEET

GRIFFITHS

EXAMPLES

2.1 line segment (INTEGRAL AT MIDPOINT)

2.2 outside uniform sphere (GAUSSIAN SURFACE)

2.3 radially varying cylinder (GAUSSIAN SURFACE)

2.4 infinite plane (using GAUSSIAN SURFACE)

2.5 capacitor 2 infinite parallel planes GAUSSIAN SURFACE)

GRIFFITHS PROBLEMS

2.1 polygon

2.5 ring

2.6 disc

2.12 inside uniform sphere

2.15 radially graded spherical shell inside, in shell, outside

DISTRIBUTIONS OF CHARGE

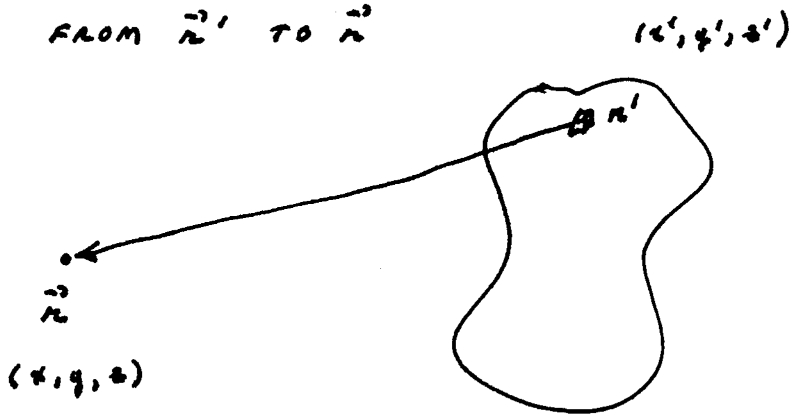
$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|^2}\,\hat{r}_{oi}$$

for 10^{23} electrons

$$\sum \Longrightarrow \int$$

$$\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{r^2}\,\hat{r}\;d^3r'$$

$\hat{r}$ FROM $\vec{r}\,'$ TO $\vec{r}$



$$E(x,y,z) = \int \frac{\rho(x',y',z')}{r^2}\,\hat{r}\;dx'\,dy'\,dz'$$

ELECTRIC FIELDS VIA INTEGRATION

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{R^2}\,\hat{r}\; d^3r'$$

$$\vec{R} = \vec{r} - \vec{r}\,'$$

$$R^2 = |\vec{R}|^2$$

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|}$$

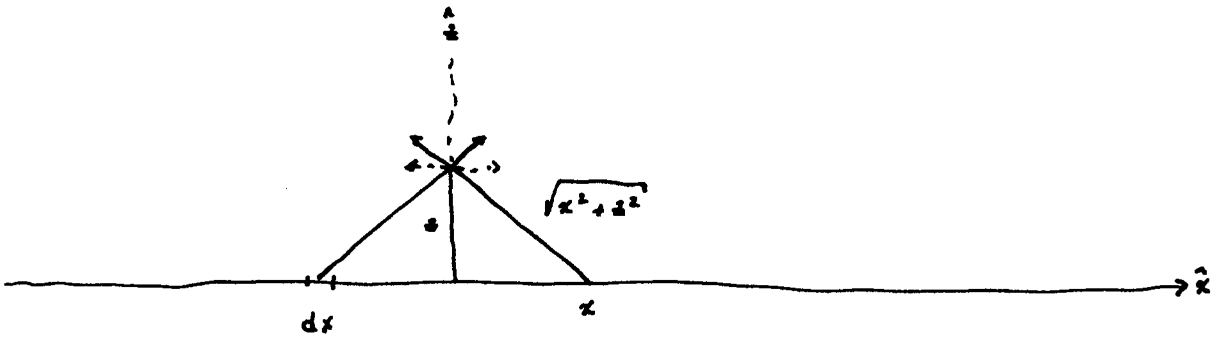
SHEET
LINE
~~CYLINDER~~
~~SPHERE~~

RELATIVELY EASY (STRAIGHTFORWARD) TO INTEGRATE

VECTOR INTEGRAL ⇒ TURN INTO 3 SCALAR INTEGRALS

E_x	E_r	E_ρ
E_y	E_θ	E_ϕ
E_z	E_φ	E_z

USE SYMMETRIES TO MAKE INTEGRALS AS SIMPLE AS POSSIBLE

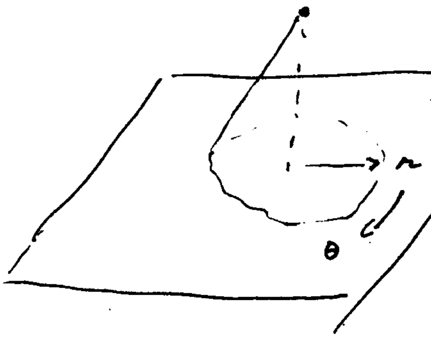


INFINITE LINE

G FINITE LINE (CENTER

FINITE LINE
ANY WHERE

HW P QUAL



INFINITE => SAME AS GAUSS' LAW

HW FINITE => DISK
(CENTER)

HW RING

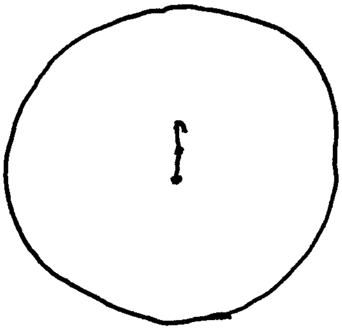
GRADED OBJECTS IN 3d (GAUSSIAN SURFACES)

SLABS $\rho(z)$

CYLINDERS $\rho(r)$ G. EXAMPLE

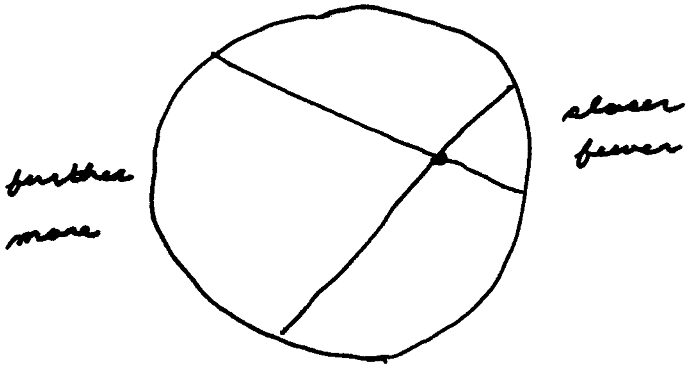
SPHERES $\rho(r)$ HW G

WHAT IS THE FIELD INSIDE A SPHERICAL SHELL?



At the center, perfect symmetry

$\vec{F} = 0$



NEWTON 1686 SHOWED $\vec{F} = 0$ everywhere inside

OUTSIDE, LOOKED LIKE A POINT AT THE CENTER