LECTURE 24 8-3-09

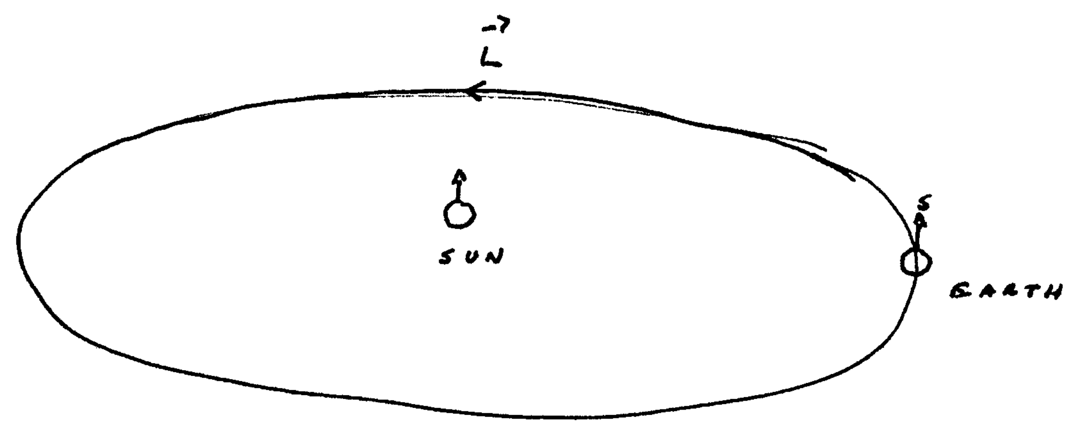
WHAT IS ANGULAR MOMENTUM?

TWO FLAVORS : ORBITAL

SPIN

BOTH ARE QUANTIZED

CLASSICAL PICTURE

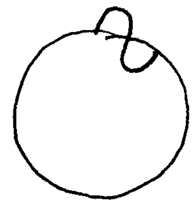


Why is orbital angular momentum quantized in integer steps?

WHY MUST ℓ BE AN INTEGER?

BOHR QUANTIZATION RULE:

$$L = n\hbar$$

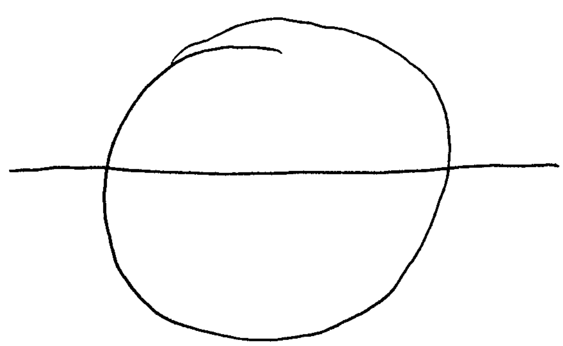


de Broglie

$$2\pi r = n\lambda = n\frac{h}{mv}$$

$$mvr = n\hbar$$

WHAT WE ARE DOING NOW IS 3d GENERALIZATION OF BOHR



field (for example, circular motion or oscillatory motion about an equilibrium point), the wave returns to its former path after a certain number of wave lengths.

Fig. 15 shows this behavior diagrammatically for a circular motion. The waves which have gone around 0, 1, 2, · · · times overlap and will, in general, destroy one another by interference (dotted waves in Fig. 15). Only in the

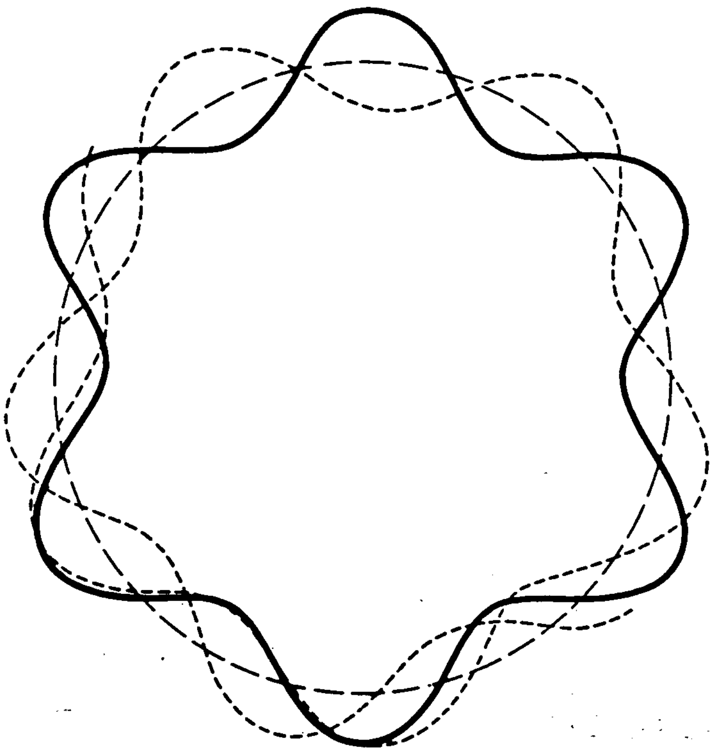


Fig. 15. De Broglie Waves for the Circular Orbits of an Electron about the Nucleus of an Atom (Qualitative). Solid line represents a stationary state (standing wave); dotted line, a quantum-theoretically impossible state (waves destroyed by interference).

special case where the frequency of the wave and, therefore, the energy of the corpuscle are such that an *integral* number of waves just circumscribe the circle (solid-line wave) do the waves which have gone around 0, 1, 2, · · · times reinforce one another so that a standing wave results. This standing wave has fixed *nodes*, and is analogous to the standing waves in a vibrating string which are possible only for certain definite frequencies, the fundamental frequency and its overtones (cf. Fig. 16). It follows, therefore, that *a stationary mode of vibration, together with a corresponding state of motion (orbit) of the corpuscle, is possible only for certain*

BOHR ATOM (CIRCULAR)

~~BOHR'S~~ ~~[illegible]~~

$$L = m v r = n \hbar$$

$$\frac{e^2}{4\pi \epsilon_0 r^2} = m \frac{v^2}{r}$$

$$r_n = \left(\frac{4\pi \epsilon_0 \hbar^2}{m e^2} \right) n^2$$

$$r_n = n^2 a_0 \qquad a_0 = 0.529 \text{ Å}$$

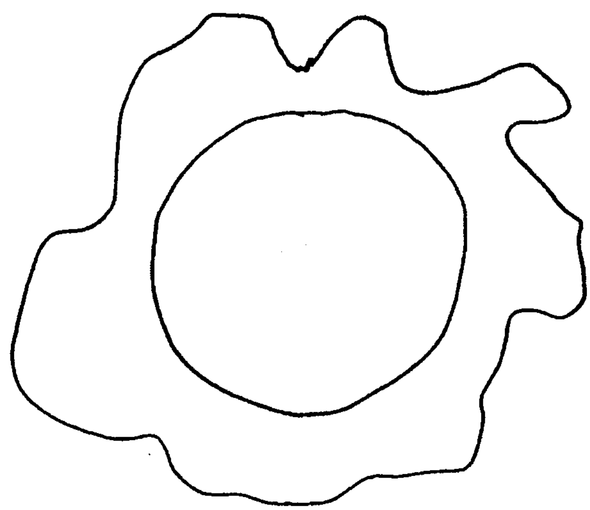
$$E_n = \cancel{\frac{e^2}{8\pi \epsilon_0}}$$

$$E_n = -\frac{m e}{2\hbar^2} \left(\frac{e^2}{4\pi \epsilon_0} \right)^2 \frac{1}{n^2}$$

$$\boxed{E_n = \frac{-R_y}{n^2}} \qquad R_y = 13.6 \text{ eV}$$

MATH : SPHERICAL HARMONICS ARE A COMPLETE ORTHONORMAL SET OF BASIS FUNCTIONS ON THE SPHERE.

PHYSICS : SPHERICAL HARMONICS DESCRIBE THE ORBITAL ANGULAR MOMENTUM



$$\text{ANY FCN}(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=+\ell} a_{\ell m} Y_{\ell m}(\theta, \varphi)$$

$$a_{\ell m} = \int \text{ANY FCN}(\theta, \varphi)\, Y_{\ell m}(\theta, \varphi)\, d\Omega$$

$\langle \theta, \varphi |$

$$\langle \theta, \varphi | \quad | A \rangle = \sum_{\ell} \sum_{m} | \ell, m \rangle \langle \ell, m | A \rangle$$

ONE MORE COMPLETE SET OF ORTHONORMAL FCNS:

INTERVAL	FOURIER SERIES	SQUARE WELL
LINE	FOURIER TRANSFORM	FREE PARTICLE
LINE	HERMITE FUNCTIONS	SHO
HALF-LINE	LAGUERRE FCNS	RADIAL WAVEFCNS FOR H
SPHERE	SPHERICAL HARMONICS	HYDROGEN ANGULAR

TO GET THE REST

$$L_- Y_{\ell\ell} = a\, Y_{\ell\,\ell-1}$$

then normalize using

$$\int Y^*_{\ell m}\, Y_{\ell m}\, d\Omega = 1$$

$$L_- = \hbar\, e^{-i\varphi} \left(\frac{d}{d\theta} + i \cot\theta \frac{d}{d\varphi} \right)$$

SPHERICAL HARMONICS

$\ell = 0$

$$Y_{00}(\theta, \varphi) = \langle \theta, \varphi | 0, 0 \rangle = \frac{1}{\sqrt{4\pi}}$$

$$\int \left(\frac{1}{\sqrt{4\pi}} \right)^* \frac{1}{\sqrt{4\pi}}\, d\Omega = 1 \checkmark$$

$\ell = 1$

$$Y_{11}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{i\varphi}$$

note symmetry

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1-1}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{-i\varphi}$$

$\ell = 2$

$$Y_{2\pm2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} (\sin\theta)^2 \; e^{\pm 2i\varphi}$$

$$Y_{1\pm1}(\theta, \varphi) = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \; e^{\pm i\varphi}$$

$$Y_{10}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

SEMICLASSICAL VECTOR MODEL

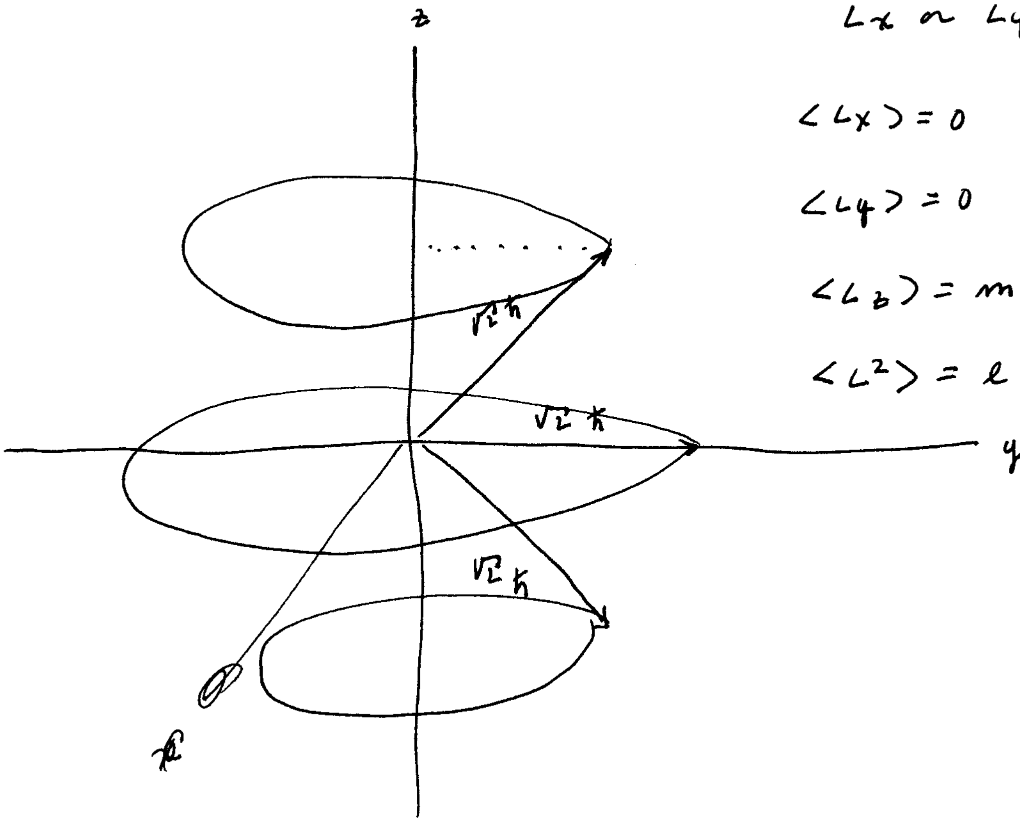
length of ang momentum vector $\sqrt{\ell(\ell+1)\hbar^2}$

z projection of $\vec{L}$ $m\hbar$

$\ell = 1$ $\sqrt{\ell(\ell+1)\hbar^2} = \sqrt{2}\,\hbar$

$m\hbar = -\hbar,\ 0,\ +\hbar$

L_z does not commute with L_x or L_y $\Rightarrow$



$\langle L_x \rangle = 0$

$\langle L_y \rangle = 0$

$\langle L_z \rangle = m\hbar$

$\langle L^2 \rangle = \ell(\ell+1)\hbar^2$

ORBITAL ANGULAR MOMENTUM

$$\ell = 0, 1, 2, 3, \ldots$$

$$m = -\ell, \ldots, 0, \ldots \ell$$

$$\boxed{\begin{aligned} L^2 \,|\ell, m\rangle &= \ell(\ell+1)\hbar^2 \,|\ell, m\rangle \\ L_z \,|\ell, m\rangle &= m\hbar \,|\ell, m\rangle \\ L_\pm \,|\ell, m\rangle &= \sqrt{\ell(\ell+1) - m(m\pm 1)}\,\hbar \,|\ell, m\pm 1\rangle \end{aligned}}$$

SPECIAL CASES:

$$\ell = 0 \quad \Rightarrow m = 0$$

$$|0,0\rangle$$

$$L^2 |0,0\rangle = 0\,\hbar^2\, |0,0\rangle$$

$$L_z |0,0\rangle = 0\,\hbar\, |0,0\rangle$$

$\ell = 1 \qquad m = -1, 0, +1$

$|1, 1\rangle$

$|1, 0\rangle$

$|1, -1\rangle$

$$L^2 |1, m\rangle = 1(1+1)\hbar^2 |1, m\rangle = 2\hbar^2 |1, m\rangle$$

$$L_z |1, m\rangle = m\hbar |1, m\rangle$$

$\ell = 2 \qquad m = -2, -1, 0, +1, +2$

$|2, 2\rangle$

$|2, 1\rangle$

$|2, 0\rangle$

$|2, -1\rangle$

$|2, -2\rangle$

$$L^2 |2, m\rangle = 6\hbar^2 |2, m\rangle$$

$$L_z |2, m\rangle = m\hbar |2, m\rangle$$

OH, MY GOSH! WE ACCIDENTALLY SOLVED A MORE GENERAL PROBLEM!

ORBITAL ANGULAR MOMENTUM L^2, L_z CAN ONLY HAVE INTEGRAL L!

TOTAL ANGULAR MOMENTUM J^2, J_z CAN HAVE INTEGER OR HALF INTEGER J...

SPIN ANGULAR MOMENTUM S^2, S_z CAN HAVE INTEGER OR HALF INTEGER S

$$\vec{J} = \vec{L} + \vec{S}$$

NEXT QUARTER $\vec{J}$, and $\vec{S}$: adding angular momentum

CLEBSCH - GORDON COEFF'S

GRIFFITH'S QUOTE...

LET'S CONSIDER SOME EXAMPLES

If you think this is starting to sound like mystical numerology, I don't blame you. We will not be using the Clebsch-Gordon tables much in the rest of this book, but I wanted you to know where they fit into the scheme of things, in case you encounter them later on. In a mathematical sense this is all applied group theory---what we are talking about is the decomposition of the direct product of two irreducible representations of the rotation group into a direct sum of irreducible representations (you can quote that, to impress your friends).

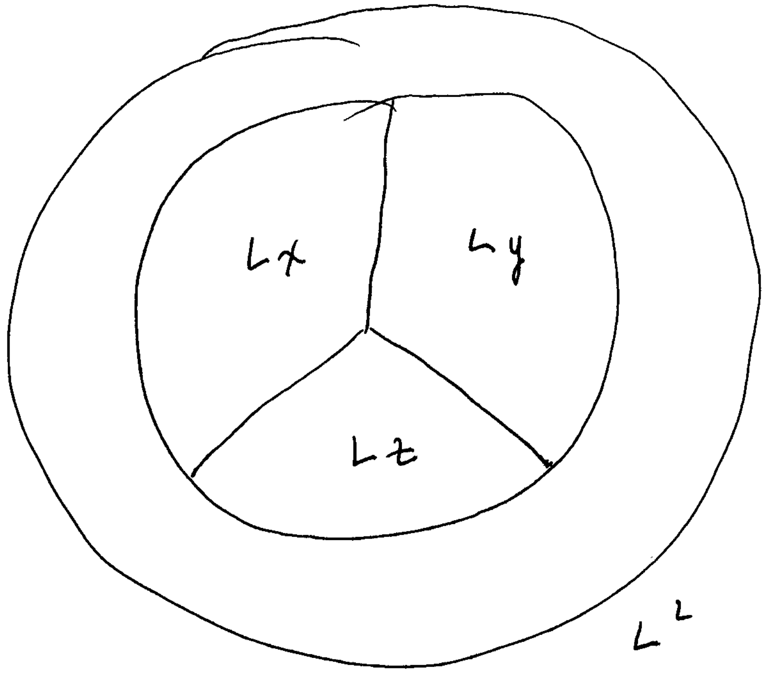
GENERALIZED UNCERTAINTY RELATION

$$(\Delta A)^2 \; (\Delta B)^2$$
$$\sigma_A^2 \; \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle [A, B] \right\rangle \right)^2$$

$$(\Delta L_x)^2 \; (\Delta L_y)^2$$
$$\sigma_{L_x}^2 \; \sigma_{L_y}^2 \geq \left(\frac{1}{2i} \left\langle i \hbar L_z \right\rangle \right)^2$$

$$\geq \frac{\hbar^2}{4} \langle L_z \rangle^2$$

$$\Delta L_x \; \Delta L_y$$
$$\sigma_{L_x} \; \sigma_{L_y} \geq \frac{\hbar}{2} \left| \langle L_z \rangle \right|$$


Lx
Ly
Lz
L2

WHAT IS SPIN?

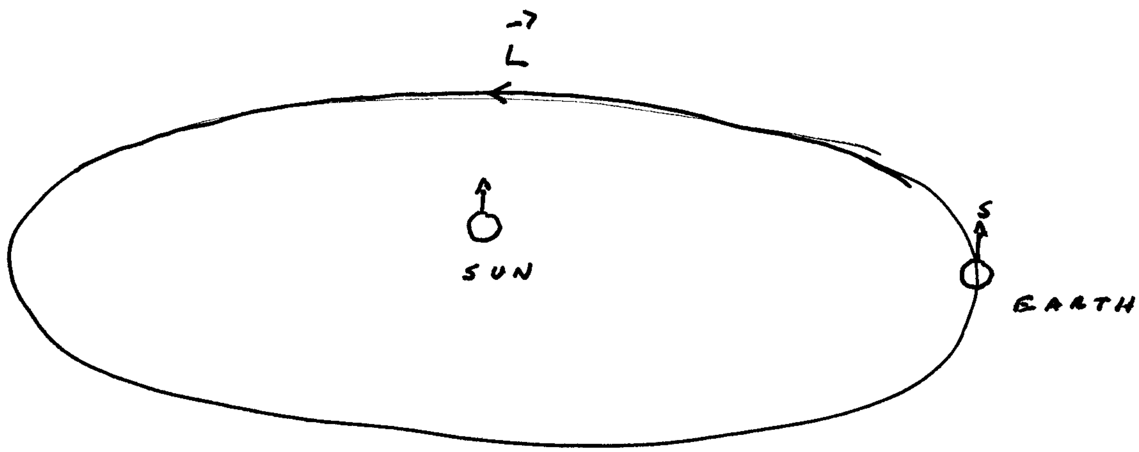
Real particles have intrinsic angular momentum.

The associated degree of freedom is called spin.

Fixed property of the particle; you cannot change it.

You can change $\vec{L}$.

CLASSICAL PICTURE



Okay, maybe the electron is like a little ~~spinning~~ spinning ball. How fast does it spin?

HOW BIG IS AN ELECTRON?

TWO CLASSICAL ANSWERS

(1) The classical radius r_0

$$\begin{pmatrix}\text{COULOMB}\\ \text{ENERGY}\end{pmatrix} = \begin{pmatrix}\text{REST MASS}\\ \text{ENERGY}\end{pmatrix}$$

$$\frac{e^2}{r_0} = mc^2$$

$$r_0 = \frac{e^2}{mc^2} = 2.8179 \times 10^{-15}\ m$$

$$r_0 \sim 3 \times 10^{-5}\ \text{Å} = \frac{a_0}{(137)^2}$$

MUCH SMALLER THAN AN ATOM

PHYSICAL MEANING: E & M cross section
elastic photon scattering cross section

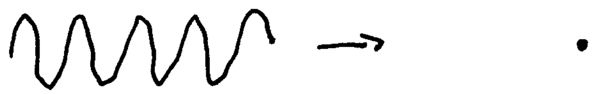
(2) The Compton radius $r_c = 3.8616 \times 10^{-13}\ m$

$$r_c = \frac{\hbar}{mc} \sim 4 \times 10^{-3}\ \text{Å}$$

$$r_c = \frac{a_0}{137} = 137\ r_0$$

PHYSICAL MEANING: inelastic electron scattering cross section

THOMPSON SCATTERING



PHOTON

FREE ELECTRON

$$\vec{E} = E_0 \hat{n} \cos(kz - \omega t)$$

$$\vec{F} = m\vec{a}$$

$$-eE_0 = m \frac{d^2 x}{dt^2}$$

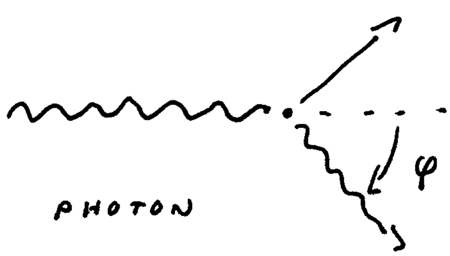
$$a = \frac{-e}{m} E_0$$

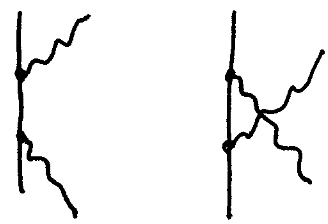
$$\frac{d\sigma}{d\Omega} = r_0^2 \left| \hat{\epsilon}_i \cdot \hat{\epsilon}_s \right|^2$$

$$= r_0^2 \sin^2 \theta$$

SEAGULL DIAGRAM

COMPTON SCATTERING





$$\Delta\lambda = \frac{h}{mc}(1 - \cos\varphi) = 0.02426 \text{ Å } (1 - \cos\varphi)$$

$$\frac{d\sigma}{d\Omega} = r_c^2 = (137)^2 r_0^2$$

HOW FAST WOULD THE ELECTRON SPIN?

$$L = \frac{1}{2}\hbar$$

$$L = I\omega = \left(\frac{2}{5} m R^2\right)\left(\frac{v}{R}\right) = \frac{2}{5} m v R$$

$$\Rightarrow v = \frac{5}{4}\,\frac{\hbar}{mR}$$

$R = r_0 \Rightarrow v_0 = 171\,c$ SUPERLUMINAL!

$R = r_c \Rightarrow v_0 = 1.25\,c$ STILL SUPERLUMINAL...

PRESENT UNDERSTANDING:

The electron is a point particle.

Nothing inside to spin!

no mass ⇒ no angular momentum

no charge ⇒ no magnetic moment

The proton has internal structure 3 quarks

intrinsic angular momentum of the quarks

orbital angular momentum of the quarks